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Certain Multiple Factorial Series And Their Asymptotic Properties *

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Abstract

In this paper we give a closed expression for the series

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{n_1 \cdots n_k}{(n_1 + \cdots + n_k)!},$$

for all $k = 1, 2, 3, \ldots$, solving Open Problem 3.137 in the recent book [5, Chapt. 3.7, problem 3.137] by Furdui. The method is based on properties of divided differences. It applies also to similar series and certain generalizations. Furthermore, we study the asymptotic behaviour of these series as k tends to infinity.

1 Introduction

In his recent book [5, Chapt. 3.7, Problem 3.137] Ovidiu Furdui states the open problem to give a closed expression for the multiple factorial series

$$S_k := \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{n_1 \cdots n_k}{(n_1 + \cdots + n_k)!},$$

for all integers $k \geq 4$. Moreover, he conjectured that S_k is, for all integers $k \in \mathbb{N}$, a rational multiple of e, i.e., $S_k = a_k e$ with $a_k \in \mathbb{Q}$. It is easy to see that $S_1 = e$. Using the Beta function technique Furdui [5, Problem 3.114 and 3.118, respectively] shows that $a_2 = 2/3$ and $a_3 = 31/120$.

More generally, Furdui considers the series

$$S_{k,0} := \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{(n_1 + \cdots + n_k)!},$$

$$S_{k,j} := \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{n_1 \cdots n_j}{(n_1 + \cdots + n_k)!} \qquad (1 \leq j \leq k).$$

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Obviously, we have $S_k = S_{k,k}$. Furdui [5, Problems 3.117 and 3.120, respectively] determines the exact values $S_{k,1} = (k!)^{-1} e$ and $S_{3,2} = (5/24)\, e$. Also an expression for $S_{k,0}$ is given [5, Problem 3.119]:

$$S_{k,0} = (-1)^k \left(1 - e \sum_{j=0}^{k-1} \frac{(-1)^j}{j!}\right). \tag{1}$$

More generally, one defines, for real numbers $x_1, \ldots, x_k$, the function

$$S_k (x_1, \ldots, x_k) := \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{x_1^{n_1} \cdots x_k^{n_k}}{(n_1 + \cdots + n_k)!}. \tag{2}$$

Closed expressions for $S_k (x_1, \ldots, x_k)$ in the special case $k = 2$ can be found in [5, Problem 3.115 (see also Problem 3.116)].

In this note we give an affirmative answer to Furdui's conjecture $e^{-1} S_k = a_k \in \mathbb{Q}$ and provide an explicit representation of a_k in the form

$$a_k = \frac{1}{(2k-1)!} \left[\left(\frac{d}{dx} \right)^{2k-1} \left(x^{k-1} e^x \right) \right] \Bigg|_{x=1}.$$

Moreover, we derive similar expressions for $S_{k,j}$. Our main result considers even more general sums. Finally, we represent $S_k (x_1, \ldots, x_k)$ as a finite sum, for all $k \in \mathbb{N}$.

The proofs are based on divided differences. For pairwise different real or complex numbers $x_0, \ldots, x_k$, in most textbooks, the divided differences of a function f are defined recursively: $[x_0; f] = f(x_0)$, ...,

$$[x_0, \ldots, x_k; f] = \frac{[x_1, \ldots, x_k; f] - [x_0, \ldots, x_{k-1}; f]}{x_k - x_0}.$$

2 Main Results

Let

$$g(z) = \sum_{n=0}^{\infty} g_n z^n,$$

be a power series converging for $|z| < R$ with $R > 1$. For integers $\ell \geq 0$, let

$$g_\ell (z) = \sum_{n=0}^{\infty} g_{n+\ell} z^n.$$

Hence $g_0 = g$ and, for $\ell \geq 1$,

$$z^\ell g_\ell (z) = g(z) - \sum_{n=0}^{\ell-1} g_n z^n.$$

For $k \in \mathbb{N}$ and $t \in \mathbb{R}$, we define

$$G_{k,\ell}\left(x_1,\ldots,x_k;t\right)=\sum_{n_1=0}^{\infty}\cdots\sum_{n_k=0}^{\infty}g_{n_1+\cdots+n_k+\ell}\cdot x_1^{n_1}\cdots x_k^{n_k}\cdot t^{n_1+\cdots+n_k}. \qquad (3)$$

Throughout the paper we assume that $|tx_j|<R$, for $j\in\{1,\ldots,k\}$.

Our main results are presented in the following theorems.

THEOREM 1. With the above notation, for all $k\in\mathbb{N}$, integers $\ell\geq 0$, and $t\in\mathbb{R}$, such that $|tx_j|<R$ $(1\leq j\leq k)$,

$$G_{k,\ell}\left(x_1,\ldots,x_k;t\right)=\left[x_1,\ldots,x_k;z^{\ell-1}g_\ell\left(tz\right)\right]_z,$$

where the index z indicates that the divided difference is taken with respect to the variable z.

THEOREM 2. Let k,j be integers such that $1\leq j\leq k$ and let $i_1,\ldots,i_j\in\{1,\ldots,k\}$ be pairwise different integers. Then, for all $t\in\mathbb{R}$, such that $|tx_j|<R$ $(1\leq j\leq k)$,

$$\lim_{x_1,\ldots,x_k\to x}\frac{\partial^j}{\partial x_{i_1}\cdots\partial x_{i_j}}G_{k,\ell}\left(x_1,\ldots,x_k;t\right)=\frac{1}{(k+j-1)!}\left[\left(\frac{d}{dz}\right)^{k+j-1}z^{k-1}g_\ell\left(tz\right)\right]\Bigg|_{z=x}.$$

For convenience, we define, for $k,\ell\in\mathbb{N}$ and real numbers $x_1,\ldots,x_k$,

$$f_{k,\ell}\left(x_1,\ldots,x_k;t\right):=\sum_{n_1=0}^{\infty}\cdots\sum_{n_k=0}^{\infty}\frac{x_1^{n_1}\cdots x_k^{n_k}}{(n_1+\cdots+n_k+\ell)!}t^{n_1+\cdots+n_k}. \qquad (4)$$

In the special case of the exponential function $g=\exp$, Theorem 1 provides the representation

$$f_{k,\ell}\left(x_1,\ldots,x_k;t\right)=\left[x_1,\ldots,x_k;x^{\ell-1}\exp_\ell\left(tx\right)\right]_x. \qquad (5)$$

With regard to the series $S_{k,j}$ as defined in Section 1 it follows that

$$\begin{aligned}S_{k,j} &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_j=0}^{\infty}\sum_{n_{j+1}=1}^{\infty}\cdots\sum_{n_k=1}^{\infty}\frac{n_1\cdots n_j}{(n_1+\cdots+n_k)!}\\ &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_k=0}^{\infty}\frac{n_1\cdots n_j}{(n_1+\cdots+n_k+k-j)!}\\ &= \frac{\partial^j f_{k,k-j}}{\partial x_1\cdots\partial x_j}\left(1,\ldots,1;1\right).\end{aligned}$$

Hence, Theorem 1 implies the following theorem as an immediate corollary.

THEOREM 3. Let k,j be integers such that $0\leq j\leq k$. Then the series $S_{k,j}$ has the representation

$$S_{k,j}=\frac{1}{(k+j-1)!}\left[\left(\frac{d}{dz}\right)^{k+j-1}z^{k-1}\exp_{k-j}\left(z\right)\right]\Bigg|_{z=1}.$$

In the special case $j = 0$, we obtain

$$S_{k,0} \equiv \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{(n_1 + \cdots + n_k)!} = \frac{1}{(k-1)!} \left[\left(\frac{d}{dz} \right)^{k-1} \frac{e^z - 1}{z} \right] \Bigg|_{z=1}$$

and an application of the Leibniz rule immediately leads to formula (1). In the cases $1 \le j \le k$ the formula of Theorem 3 simplifies to

$$S_{k,j} = \frac{1}{(k+j-1)!} \left[\left(\frac{d}{dz} \right)^{k+j-1} z^{j-1} e^z \right] \Bigg|_{z=1}.$$

An application of the Leibniz rule yields the explicit formula

$$S_{k,j} = e \sum_{i=0}^{j-1} \binom{j-1}{i} \frac{1}{(k+i)!}. \tag{6}$$

Hence, the series $S_{k,j}$ are rational multiples of e for $j = 1, \ldots, k$. We list some initial values:

$k \backslash j$	0	1	2	3	4	5
1	$e - 1$	1				
2	1	1/2	2/3			
3	$e/2 - 1$	1/6	5/24	31/120		
4	$1 - e/3$	1/24	1/20	43/720	179/2520	
5	$3e/8 - 1$	1/120	7/720	19/1680	529/40320	787/51840

We close with the special case $j = k$:

$$S_k \equiv S_{k,k} = e \sum_{i=0}^{k-1} \binom{k-1}{i} \frac{1}{(k+i)!}.$$

We mention that this finite sum can be expressed in terms of the first of Kummer's functions (a confluent hypergeometric function; see [1, Eq. (13.1.2)])

$$S_k = \frac{e}{k!} M\left(1-k, k+1, -1\right)$$

or by virtue of the Kummer transformation ([1, Eq. (13.1.27)])

$$S_k = \frac{1}{k!} M\left(2k, k+1, 1\right).$$

For the convenience of the reader we list some exact and numerical values of $a_k = e^{-1} S_k$:

k	a_k	
1	1	$= 1.000000$
2	2/3	≈ 0.666667
3	31/120	≈ 0.258333
4	179/2520	≈ 0.0710317
5	787/51840	≈ 0.0151813
10		$5.912338752837942 \cdot 10^{-7}$
100		$2.829019570367539 \cdot 10^{-158}$

Finally, we mention that the series $S_k(x_1, \ldots, x_k)$ as defined in (2) is connected to the function $f_{k,\ell}$ as defined in (4) by the relation

$$S_k(x_1, \ldots, x_k) = x_1 \cdots x_k \cdot f_{k,k}(x_1, \ldots, x_k; 1).$$

Hence, by Eq. (5), we have the new approach

$$S_k(x_1, \ldots, x_k) = x_1 \cdots x_k \cdot \left[x_1, \ldots, x_k; x^{k-1} \exp_k(x)\right].$$

Experiments with different functions g may be subject of further studies.

In [5, Chapt. 3.7, Problem 3.137] Furdui arose the question of studying the properties of the sequence $\left(e^{-1} S_k\right)_{k \in \mathbb{N}} = (a_k)_{k \in \mathbb{N}}$. We study the asymptotic behaviour of this sequence as k tends to infinity.

THEOREM 4. The sequence $(S_k)_{k \in \mathbb{N}}$ has the asymptotic expansion

$$S_k \sim \frac{e^2}{k!} \left(1 - \frac{3}{k} + \frac{27}{2k^2} - \frac{218}{3k^3} + \cdots\right)$$

as $k \to \infty$.

More generally, the proof shows that (S_k) has a complete asymptotic expansion

$$S_k \sim \frac{e^2}{k!} \sum_{\nu=0}^{\infty} c_\nu k^{-\nu} \qquad (k \to \infty).$$

Using our method it is possible to compute arbitrarily many coefficients c_ν explicitly.

REMARK 1. After preparation of the paper the author learned by personal communication that Huizeng Qin and Ovidiu Furdui recently found the expressions (6) by a completely different approach. Their main result (see [8, Theorem 1.7, p. 735]) reads

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{z^{n_1+\cdots+n_k}}{(n_1+\cdots+n_k)!} = (-1)^k + e^z \sum_{i=0}^{k-1} (-1)^{k-1-i} \frac{z^i}{i!},$$

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{n_1 \cdots n_j}{(n_1+\cdots+n_k)!} z^{n_1+\cdots+n_k} = e^z \sum_{i=0}^{j-1} \binom{j-1}{i} \frac{z^{k+i}}{(k+i)!} \qquad (1 \le j \le k).$$

This generalization is a corollary of Theorem 2.

3 Auxiliary Results

Let $x_0, \ldots, x_k$ be pairwise different real or complex numbers. In most textbooks, the divided differences of a function f are defined recursively: $[x_0; f] = f(x_0)$, ...,

$$[x_0, \ldots, x_k; f] = \frac{[x_1, \ldots, x_k; f] - [x_0, \ldots, x_{k-1}; f]}{x_k - x_0}$$

In this paper we make use of some properties of divided differences which are considered in the following lemmas.

LEMMA 1. The divided differences have the integral representation

$$\begin{aligned} & [x_0, \ldots, x_k; f] \\ = & \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-1}} f^{(k)}\left(x_0 + (x_1 - x_0)\, t_1 + \cdots + (x_k - x_{k-1})\, t_k\right) dt_k \cdots dt_2 dt_1, \end{aligned}$$

provided that $f^{(k-1)}$ is absolutely continuous.

This can be proved by mathematical induction on k (see [2, Chapt. 4, §7, Eq. (7.12) and below]).

LEMMA 2. Let $1 \le j \le k$ and let $i_1, \ldots, i_j \in \{1, \ldots, k\}$ be pairwise different integers. Then, for each function f having a derivative of order $k+j-1$,

$$\lim_{x_1,\ldots,x_k \to x} \frac{\partial^j [x_1, \ldots, x_k; f]}{\partial x_{i_1} \cdots \partial x_{i_j}} = \frac{1}{(k+j-1)!} f^{(k+j-1)}(x).$$

PROOF. Because the divided differences are invariant with respect to the order of knots we can restrict ourselves to the case $i_\nu = \nu$ $(\nu = 1, \ldots, j)$. By Lemma 1, we have

$$\begin{aligned} & \frac{\partial^j [x_1, \ldots, x_k; f]}{\partial x_1 \cdots \partial x_j} \\ = & \frac{\partial^j}{\partial x_1 \cdots \partial x_j} \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-2}} f^{(k-1)}(x_1 + (x_2 - x_1)\, t_1 \\ & + \cdots + (x_k - x_{k-1})\, t_{k-1}) dt_{k-1} \cdots dt_2 dt_1 \\ = & \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-2}} f^{(k+j-1)}\left(x_1 (1 - t_1) + x_2 (t_1 - t_2) + \cdots + x_k (t_{k-1} - t_k)\right) \\ & \times (1 - t_1)(t_1 - t_2) \cdots (t_{j-1} - t_j)\, dt_{k-1} \cdots dt_2 dt_1, \end{aligned}$$

where we put $t_k = 0$. Taking the limit we obtain

$$\begin{aligned} & \lim_{x_1,\ldots,x_k \to x} \frac{\partial^j [x_1, \ldots, x_k; f]}{\partial x_1 \cdots \partial x_j} \\ = & f^{(k+j-1)}(x) \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{k-2}} (1 - t_1)(t_1 - t_2) \cdots (t_{j-1} - t_j)\, dt_{k-1} \cdots dt_2 dt_1. \end{aligned}$$

An inductive argument shows that the multiple integral has the value $1/(k+j-1)!$ which completes the proof of Lemma 2.

Popoviciu [7] proved the following formula for monomials.

LEMMA 3. For each integer $r \ge 0$,

$$\left[x_0, \ldots, x_k; z^{k+r}\right] = \sum x_0^{n_0} \cdots x_k^{n_k},$$

where the sum runs over all nonnegative integers $n_0, \ldots, n_k$ satisfying $n_0 + \cdots + n_k = r$.

4 Proofs of the Theorems

PROOF OF THEOREM 1. By Eq. (3) and Lemma 3 we have

$$\begin{aligned} G_{k,\ell}(x_1,\ldots,x_k;t) &= \sum_{n=0}^{\infty} g_{n+\ell} t^n \sum_{n_1+\cdots+n_k=n} x_1^{n_1}\cdots x_k^{n_k} \\ &= \sum_{n=0}^{\infty} g_{n+\ell} t^n \left[x_1,\ldots,x_k; z^{k-1+n}\right]_z = \left[x_1,\ldots,x_k; z^{k-1} g_\ell(tz)\right]_z \end{aligned}$$

which completes the proof.

PROOF OF THEOREM 2. By Theorem 1, we have

$$G_{k,\ell}(x_1,\ldots,x_k) = \left[x_1,\ldots,x_k; z^{k-1} g_\ell(z)\right]$$

and Theorem 2 is a consequence of Lemma 1.

PROOF OF THEOREM 4. For convenience, we consider a_{k+1}. By Theorem 3, we obtain

$$a_{k+1} = \sum_{i=0}^{k} \binom{k}{i} \frac{1}{(2k+1-i)!} = \frac{1}{(2k+1)!} \left[\left(\frac{d}{dz}\right)^k \left(z^{2k+1} e^{z-1}\right) \right]\Bigg|_{z=1}$$

and an application of the Cauchy integral formula yields

$$(2k+1)! a_{k+1} = \frac{k!}{2\pi i} \int_W \frac{z^{2k+1} e^{z-1}}{(z-1)^{k+1}} dz,$$

where the integration path $W = \{z : |z-1| = 1\}$ encircles $z = 1$ counterclockwise. With $z = 1 + e^{it}$ we have

$$(2k+1)! a_{k+1} = \frac{k!}{2\pi} \int_0^{2\pi} \frac{\left(1+e^{it}\right)^{2k+1}}{e^{ikt}} e^{e^{it}} dt = \frac{k!}{2\pi} \int_0^{2\pi} (2+2\cos t)^k f(t)\, dt,$$

with $f(t) = \left(1+e^{it}\right) e^{e^{it}}$. Because of the symmetries $\operatorname{Re} f(t) = \operatorname{Re} f(2\pi - t)$ and $\operatorname{Im} f(t) = -\operatorname{Im} f(2\pi - t)$ we obtain

$$(2k+1)! a_{k+1} = \frac{k!}{\pi} \int_0^{\pi} (2+2\cos t)^k \operatorname{Re} f(t) dt.$$

This can be rewritten as a Laplace-type integral

$$b_k := \frac{\pi (2k+1)!}{4^k k!} a_{k+1} = \int_0^{\pi} e^{kh(t)} g(t)\, dt, \tag{7}$$

where

$$h(t) = \log((1+\cos t)/2) = -\frac{t^2}{4} - \frac{t^4}{96} - \frac{t^6}{1440} - \frac{17t^8}{322560} - \cdots$$

and

$$\begin{aligned} g(t) &= \mathrm{Re} f(t) = e^{\cos t}\left[(1+\cos t)\cos(\sin t) - \sin t \sin(\sin t)\right] \\ &= 2e - \frac{7et^2}{2} + \frac{67et^4}{24} - \frac{3et^6}{2} + \frac{8429et^8}{13440} + \cdots . \end{aligned}$$

Thus it is well known (e.g., the integral meets the assumptions of [3, Theorem 1, Chapt. 3, §5]), that it has the complete asymptotic expansion

$$b_k \sim \frac{1}{2}\sum_{\nu=0}^{\infty} c_\nu \frac{\Gamma((\nu+1)/2)}{k^{(\nu+1)/2}} \qquad (k \to \infty)$$

(note that $h(0) = 0$) with coefficients

$$c_\nu = \frac{1}{\nu!}\left\{\frac{d^\nu}{dt^\nu}\left[g(t)\left(t/\sqrt{-h(t)}\right)^{\nu+1}\right]\right\}\Big|_{t=0}.$$

Because g and h are even functions, it follows that $c_1 = c_3 = \cdots = 0$. By direct calculation, we find that

$$c_0 = 4e, \quad c_2 = -29e, \quad c_4 = \frac{2425}{24}e, \quad c_6 = \frac{354053}{1440}e, \quad c_8 = \frac{77089969}{161280}e.$$

Hence,

$$b_k \sim \frac{2e\sqrt{\pi}}{k^{1/2}} - \frac{29e\sqrt{\pi}}{4k^{3/2}} + \frac{2425e\sqrt{\pi}}{64k^{5/2}} - \frac{354053e\sqrt{\pi}}{1536k^{7/2}} + \frac{77089969e\sqrt{\pi}}{161280k^{9/2}} + \cdots \qquad (k \to \infty)$$

and, by (7),

$$\begin{aligned} &(k+1)!a_{k+1} \\ =\;& \frac{4^k k!\,(k+1)!}{\pi(2k+1)!} b_k = \frac{4^k}{\pi(2k+1)C_k} b_k \\ \sim\;& \frac{4^k e}{(2k+1)\sqrt{\pi}C_k}\left(\frac{2}{k^{1/2}} - \frac{29}{4k^{3/2}} + \frac{2425}{64k^{5/2}} - \frac{354053}{1536k^{7/2}} + \frac{77089969}{161280k^{9/2}} + \cdots\right), \end{aligned}$$

where C_k are Catalan numbers defined by $C_k = (k+1)^{-1}\binom{2k}{k}$. It is well-known that $C_k \sim 4^k/\sqrt{\pi k^3}$ as $k \to \infty$ (see [4, Eq. (33)]). More precisely, the Catalan numbers have the asymptotic expansion

$$C_k \sim \frac{4^k}{\sqrt{\pi k^3}}\left(1 - \frac{9}{8k} + \frac{145}{128k^2} - \frac{1155}{1024k^3} + \frac{36939}{32768k^4} - \frac{295911}{262144k^5} + \cdots\right) \qquad (k \to \infty).$$

(see [4, Page 384]). A proof can be given by an application of the Stirling formula for factorials or directly by the generating function of the Catalan numbers. Hence, we have

$$\begin{aligned} (k+1)!a_{k+1} &\sim \frac{e}{2k+1}\cdot\frac{2k - \frac{29}{4} + \frac{2425}{64k} - \frac{354053}{1536k^2} + \frac{77089969}{161280k^3} + \cdots}{1 - \frac{9}{8k} + \frac{145}{128k^2} - \frac{1155}{1024k^3} + \cdots} \\ &= e\left(1 - \frac{3}{k} + \frac{33}{2k^2} - \frac{308}{3k^3} + \cdots\right) \end{aligned}$$

which implies that

$$k!a_k \sim e\left(1 - \frac{3}{k} + \frac{27}{2k^2} - \frac{218}{3k^3} + \cdots\right).$$

Because $S_k = a_k e$ the proof is completed.

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