Applied Mathematics E-Notes, 16(2016), 154-160 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Representations Of The Constant e By Way Of Products And Symmetric Polynomials*

Martin Griffiths†, Gareth Wilkes‡

Received 2 November 2015

Abstract

We provide here a new representation of the exponential constant e, and show how this leads to a connection between e and the elementary symmetric polynomials.

1 Introduction

The exponential constant e is generally defined by way of the following infinite sum:

$$e=\sum_{k=0}^{\infty}\frac{1}{k!}.$$

It is well-known that e may alternatively be expressed as the limit given by

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}. \tag{1}$$

In fact, e is sometimes even defined by way of this limit. It is reasonably straightforward to prove the equality of the two representations above by applying the binomial theorem and then examining the behavior of the resultant series as n tend to infinity. It is probably not quite so well-known, however, that the exponential constant may also be expressed as the following infinite product [2]:

$$e=2\left(\frac{2}{1}\right)^{1/2}\left(\frac{2\cdot 4}{3\cdot 3}\right)^{1/4}\left(\frac{4\cdot 6\cdot 6\cdot 8}{5\cdot 5\cdot 7\cdot 7}\right)^{1/8}\cdots.$$

There are in fact many other known ways of representing e via infinite sums or products. However, following an extensive literature search, it would appear that the representation given in this note is new. We state and prove our main theorem, a generalization of (1), in Section 2, making use of analytic techniques. This result is then utilized to demonstrate a particular connection between the exponential constant and elementary symmetric polynomials.

*Mathematics Subject Classifications: 05E05, 33B10, 40A20, 40A30.

†Department of Mathematical Sciences, University of Essex, Colchester CO4 3SQ, United Kingdom
‡St John's College, University of Oxford, Oxford OX1 3JP, United Kingdom

2 The Representation of e

Our purpose in the current section is to prove the following result:

THEOREM 1. Assume that $m > 0$ is a fixed real number. Then

$$e^x = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{mk^{m-1}}{n^m}\, x\right) \text{ for all } x \in \mathbb{R}. \tag{2}$$

PROOF. First we define $f_{n,m}(x)$ by

$$f_{n,m}(x) = \prod_{k=1}^{n} \left(1 + \frac{mk^{m-1}}{n^m}\, x\right) \tag{3}$$

and let $l_{n,m}(x) = \log f_{n,m}(x)$. From these definitions it follows that

$$l'_{n,m}(x) = \sum_{k=1}^{n} \frac{mk^{m-1}}{n^m} \left(1 + \frac{mk^{m-1}}{n^m}\, x\right)^{-1}. \tag{4}$$

We now carry out the proof of Theorem 1 in two parts. The first part covers the situation in which $m \geq 1$, while the second deals with the case $0 < m < 1$. Suppose first that r and m be any fixed positive real numbers, where $m \geq 1$. Then, for any $n \in \mathbb{N}$ such that $n > 2mr$, $k \in \mathbb{N}$ such that $1 \leq k \leq n$, and $x \in [-r, r]$, it is the case that

$$\left|\frac{mk^{m-1}}{n^m}\, x\right| \leq \left|\frac{mx}{n}\right| \leq \frac{mr}{n} < \frac{1}{2},$$

and we may thus expand

$$\left(1 + \frac{mk^{m-1}}{n^m}\, x\right)^{-1}$$

by way of the geometric series expansion. So, on setting

$$A = mx \left(\frac{k}{n}\right)^{m-1},$$

we have

$$\begin{aligned}
\left(1 + \frac{mk^{m-1}}{n^m}\, x\right)^{-1} &= \left(1 + \frac{A}{n}\right)^{-1} \\
&= 1 - \frac{A}{n} + \left(\frac{A}{n}\right)^2 - \left(\frac{A}{n}\right)^3 + \cdots \\
&= 1 - \frac{A}{n} + \frac{\left(\frac{A}{n}\right)^2}{1 + \frac{A}{n}} \\
&= 1 - \frac{A}{n} + \frac{A^2}{n(n+A)} \\
&= 1 - \frac{1}{n}\left(A - \frac{A^2}{n+A}\right).
\end{aligned}$$

From the definition of A it follows, when $n > 2mr$, that

$$\left|A - \frac{A^2}{n+A}\right| \le |A| + \left|\frac{A^2}{n+A}\right| \le mr + \frac{(mr)^2}{2mr} = \frac{3mr}{2},$$

leading to the result

$$\left(1 + \frac{mk^{m-1}}{n^m}\,x\right)^{-1} = 1 + O\left(\frac{1}{n}\right). \tag{5}$$

By (4) and (5), we see that

$$\begin{aligned}
l'_{n,m}(x) &= \sum_{k=1}^{n} \frac{mk^{m-1}}{n^m}\left(1 + O\left(\frac{1}{n}\right)\right) \\
&= \frac{m}{n^m}\left(1 + O\left(\frac{1}{n}\right)\right)\sum_{k=1}^{n} k^{m-1} \\
&= \frac{m}{n^m}\left(1 + O\left(\frac{1}{n}\right)\right)\left(\frac{n^m}{m} + O\left(n^{m-1}\right)\right) \\
&= 1 + O\left(\frac{1}{n}\right),
\end{aligned}$$

where we have used the result

$$\sum_{k=1}^{n} k^s = \frac{n^{s+1}}{s+1} + O(n^s).$$

It is thus the case that

$$\lim_{n\to\infty} l'_{n,m}(x) = 1 \text{ for } m \ge 1.$$

Suppose now that r and m be any fixed positive real numbers, where $0 < m < 1$. Then, for any $n \in \mathbb{N}$ such that $n > (2mr)^{1/m}$, $k \in \mathbb{N}$ such that $1 \le k \le n$, and $x \in [-r, r]$, we have

$$\left|\frac{mk^{m-1}}{n^m}\,x\right| \le \left|\frac{mx}{n^m}\right| \le \frac{mr}{n^m} < \frac{1}{2},$$

noting, in particular, that the left-hand inequality above is true since $0 < m < 1$ so that $k^{m-1} \le 1$. Thus, as with the previous case, we may expand

$$\left(1 + \frac{mk^{m-1}}{n^m}\,x\right)^{-1}$$

by way of the geometric series expansion. This time, with

$$B = \frac{mk^{m-1}}{n^{m/2}}\,x,$$

we obtain

$$\begin{aligned}\left(1+\frac{mk^{m-1}}{n^m}\,x\right)^{-1} &= \left(1+\frac{B}{n^{m/2}}\right)^{-1}\\ &= 1-\frac{B}{n^{m/2}}+\left(\frac{B}{n^{m/2}}\right)^2-\left(\frac{B}{n^{m/2}}\right)^3+\cdots\\ &= 1-\frac{1}{n^{m/2}}\left(B-\frac{B^2}{n^{m/2}+B}\right),\end{aligned}$$

which implies that

$$\left(1+\frac{mk^{m-1}}{n^m}\,x\right)^{-1} = 1+O\left(\frac{1}{n^{m/2}}\right).$$

In this case, we have that

$$\begin{aligned}l'_{n,m}(x) &= \sum_{k=1}^{n}\frac{mk^{m-1}}{n^m}\left(1+O\left(\frac{1}{n^{m/2}}\right)\right)\\ &= \frac{m}{n^m}\left(1+O\left(\frac{1}{n^{m/2}}\right)\right)\sum_{k=1}^{n}k^{m-1}\\ &= \frac{m}{n^m}\left(1+O\left(\frac{1}{n^{m/2}}\right)\right)\left(\frac{n^m}{m}+O\left(n^{m-1}\right)\right)\\ &= 1+O\left(\frac{1}{n^{m/2}}\right),\end{aligned}$$

from which we see that

$$\lim_{n\to\infty} l'_{n,m}(x) = 1 \text{ when } 0<m<1.$$

Next, note that $l'_{n,m}(x)$ is monotone decreasing on the interval $[-r,r]$, so that

$$l'_{n,m}(r) \le l'_{n,m}(x) \le l'_{n,m}(-r) \text{ for all } x\in[-r,r].$$

Therefore

$$\left|l'_{n,m}(x)-1\right| \le \max\left\{\left|l'_{n,m}(-r)-1\right|,\left|l'_{n,m}(r)-1\right|\right\} \text{ for all } x\in[-r,r].$$

From this we may infer that

$$\left\|l'_{n,m}-1\right\|_\infty \le \max\left\{\left|l'_{n,m}(-r)-1\right|,\left|l'_{n,m}(r)-1\right|\right\},$$

which in turn implies that

$$\lim_{n\to\infty}\left\|l'_{n,m}-1\right\|_\infty = 0.$$

The convergence is thus uniform over the interval $[-r,r]$.

Since $l'_{n,m}(x)$ converges uniformly to 1 on $[-r,r]$, and the sequence $\{l_{n,m}(0)\}$ converges (all its terms are in fact equal to 0), we know from Theorem 9.13 in [1] that

$l'_{n,m}(x)$ tends to a function $L(x)$ uniformly on $[-r, r]$ where $L'(x) = 1$. Furthermore, as $l_{n,m}(0) = 0$ for all n, it follows that $L(x) = x$. The continuity of the exponential function then implies that

$$\lim_{n\to\infty} f_{n,m}(x) = e^x \tag{6}$$

on $[-r, r]$. However, since the choice of r was arbitrary, we see that (6) is actually true for all $x \in \mathbb{R}$, as required.

Note that specializing (2) to $x = 1$ and then $m = x = 1$ gives rise to

$$e = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{mk^{m-1}}{n^m}\right)$$

and (1), respectively. It is worth pointing out here that the limit given by (2) is in fact independent of m. Furthermore, it is possible to obtain the representation as given below.

COROLLARY 1. For any fixed $m \in \mathbb{N}$, it is the case that

$$e^x = \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{mg_{m-1}(k)}{n^m}\, x\right),$$

where $g_{m-1}(k)$ is a fixed, though arbitrary, monic polynomial of degree $m - 1$.

PROOF. This proceeds along similar lines to the proof of Theorem 1, so we provide merely an outline here. Let r be a fixed positive real number and m be a fixed positive integer. First, since g_{m-1} is an arbitrary monic polynomial of degree $m - 1$, it follows that for any given $\epsilon > 0$ there exists some positive integer $N(\epsilon, g_{m-1})$, a function of both ϵ and g_{m-1}, such that for any fixed $n > N(\epsilon, g_{m-1})$ it is true that

$$0 < \left|\frac{g_{m-1}(k)}{n^m}\right| < \epsilon$$

for each $k \in \mathbb{N}$ satisfying $1 \le k \le n$. Therefore, for any $x \in [-r, r]$, it is the case that

$$\left|\frac{mg_{m-1}(k)}{n^m}\, x\right| \le \left|\frac{mrg_{m-1}(k)}{n^m}\right| < mr\epsilon$$

for each $k \in \mathbb{N}$ satisfying $1 \le k \le n$ when $n > N(\epsilon, g_{m-1})$. So, when

$$n > N\left(\frac{1}{2mr}, g_{m-1}\right),$$

it follows that

$$\left|\frac{mg_{m-1}(k)}{n^m}\, x\right| < \frac{1}{2},$$

allowing us to expand

$$\left(1+\frac{mg_{m-1}(k)}{n^m}\,x\right)^{-1}$$

by way of the geometric series expansion. Then, on setting

$$C=\frac{mg_{m-1}(k)}{n^{m-1}}\,x,$$

we have

$$\left(1+\frac{mg_{m-1}(k)}{n^m}\,x\right)^{-1}=\left(1+\frac{C}{n}\right)^{-1},$$

which can be shown to be equal to $1+O(1/n)$. We omit the remainder of the proof since it differs only in minor details to the proof of the corresponding part of Theorem 1.

We now employ Theorem 1 to establish a link between the exponential constant and a limit involving the elementary symmetric polynomials. A polynomial in the variables $x_1, x_2, \ldots, x_n$ is called symmetric if it is left unchanged by any permutation of these variables. In particular, the elementary symmetric polynomial $E_{k,n}$ is defined to be the sum of all possible products of k distinct elements from the set $\{x_1, x_2, \ldots, x_n\}$, noting that if $k > n$ then $E_{k,n}$ is defined to be zero.

COROLLARY 2. The exponential constant is related to the elementary symmetric polynomials by way of the following:

$$e=\lim_{n\to\infty}\left(1+\sum_{k=1}^{n}E_{k,n}\left(\frac{m}{n^m}\right)^k\right).$$

PROOF. From (3), we have

$$\begin{aligned}
f_{n,m}(1)&=\prod_{k=1}^{n}\left(1+\frac{mk^{m-1}}{n^m}\right)\\
&=\left(1+\frac{m}{n^m}\right)\left(1+\frac{2^{m-1}m}{n^m}\right)\left(1+\frac{3^{m-1}m}{n^m}\right)\cdots\left(1+\frac{n^{m-1}m}{n^m}\right)\\
&=1+\left(\frac{m}{n^m}\right)\left(1+2^{m-1}+3^{m-1}+\cdots+n^{m-1}\right)\\
&\qquad+\left(\frac{m}{n^m}\right)^2\left(1\cdot 2^{m-1}+1\cdot 3^{m-1}+\cdots+(n-1)^{m-1}\cdot n^{m-1}\right)\\
&\qquad\qquad\vdots\\
&\qquad+\left(\frac{m}{n^m}\right)^n\left(1\cdot 2^{m-1}\cdot 3^{m-1}\cdots n^{m-1}\right)\\
&=1+\sum_{k=1}^{n}E_{k,n}\left(\frac{m}{n^m}\right)^k,
\end{aligned}$$

where the variables $x_1, x_2, \ldots, x_n$ have been specialized by setting $x_k = k^{m-1}$, $k = 1, 2, \ldots, n$, and m is any fixed positive integer. The result follows from Theorem 1.

We have thus shown that e may be expressed as the limit of a sequence of finite sums where, for each of these sums, the elementary symmetric polynomials play, in some sense, the role of coefficients. For the particular case in which $m \in \mathbb{N}$, $f_{n,m}(1)$ gives rise to a sequence of rational approximations to e. On setting $m = 1$, we have the sequence of rational approximations given by (1). With $m = 3$, for example, the sequence

$$4, \frac{55}{16}, \frac{260}{81}, \frac{810901}{262144}, \frac{3689013248}{1220703125}, \ldots$$

is obtained. For the latter case the convergence to e is somewhat slow; indeed, it is not until $n = 51$ that the approximations are correct to within one decimal place.

Acknowledgment. The authors would like to thank an anonymous referee for several suggestions that have improved the overall clarity of this paper.

References

[1] T. M. Apostol, Mathematical Analysis: A Modern Approach to Advanced Calculus, 2^{nd} ed., Pearson, London, 1974.

[2] N. Pippenger, Mathematical Notes: An Infinite Product for e, Amer. Math. Monthly, 87(1980), 391.