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Euler Savary's Formula On Complex Plane $\mathbb{C}$ *

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Abstract

In this article, we consider a base curve, a rolling curve and a roulette on complex plane. We investigate the third one when any two of the base curve, the rolling curve, and a roulette are known.We obtain Euler Savary's formula, which gives the relation between the curvatures of these three curves.

1 Introduction

Complex numbers have important role in mathematics. It has caused new numbers such as dual numbers, hyperbolic numbers, quaternions and octonions, etc. Complex numbers are not only used in mathematics but also have essential concrete applications in a variety of scientific and related areas such as physics, chemistry, biology, economics, electrical engineering, statistics, signal processing, control theory, electromagnetism, fluid dynamics, quantum mechanics, cartography, and vibration analysis.

On the Euclidean plane $\mathbb{E}^2$, let us consider two curves: a *base curve* and *rolling curve* which are denoted by (B) and (R), respectively. Assume that X is a point which is relative to a rolling curve (R). Suppose that the rolling curve (R) rolls without splitting along the base curve (B). Then, the locus of the point X makes a curve which is called *roulette* and denoted by (X). For instance, if (B) is a line, (R) is a circle and X is a point on (R), then (X) is cycloid.

Euler Savary's formula is a very famous theorem which gives relation between curvatures of the roulette and these base curve and rolling curve. It is used on quite serious fields of mathematics and engineering. It is worked by Alexander and Maddocks, [3], Buckley and Whitfield, [4], Dooner and Griffis, [5], Ito and Takahaski, [6], Pennock and Raje, [7], Wang at all, [8].

However, in 1956, Müller, [9], obtained Euler Savary's formula for one parameter motion in Euclidean plane $\mathbb{E}^2$. In 2003, T. Ikawa, [11], examined Euler Savary's formula in Minkowskian geometry and also showed a new way for a generalization of the Euler Savary's formula in the Euclidean plane in this article. In 2010, Masal at al., [10], expressed Euler Savary's formula for one parameter motion in the complex plane $\mathbb{C}$. In this paper, we research a generalization of the Euler Savary's formula in the complex plane $\mathbb{C}$ by a different way from [10]. Also, we study on a base curve, a rolling curve and a roulette in the complex plane $\mathbb{C}$.

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2 Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{C}$ from an open interval I into $\mathbb{C}$ be a planar curve with arc length parameter s. Let the curve α be defined by $\alpha(s) = \alpha_1(s) + i\alpha_2(s)$, [1]. Then, the unit tangent vector of the curve α at the point $\alpha(s)$ is defined by

$$\mathbf{T}(s) = \alpha_1'(s) + i\alpha_2'(s).$$

Frenet formulas of the curve α have the following equations

$$\begin{aligned} \mathbf{T}' &= \kappa\mathbf{N}, \\ \mathbf{N}' &= -\kappa\mathbf{T}, \end{aligned}$$

where $\mathbf{N}$ is the unit normal vector of the curve α and κ is curvature of α in [1, 2]. However, we can define a new curve

$$\alpha_A(s) = \alpha(s) + x(s)\ \mathbf{T}(s) + y(s)\mathbf{N}(s),$$

which is called *associated curve,* where $x(s),\ y(s) \in \mathbb{R}$. If we calculate the velocity vector of $\alpha_A(s)$, then we get

$$\frac{d(\alpha_A(s))}{ds} = \left(1 - \kappa(s)y(s) + \frac{dx(s)}{ds}\right)\mathbf{T}(s) + \left(\kappa(s)x(s) + \frac{dy(s)}{ds}\right)\mathbf{N}(s).$$

Besides, we can write the associated curve as

$$\alpha_A(s) = x(s) + iy(s),$$

with respect to the frame $\{\alpha(s); \mathbf{T}(s), \mathbf{N}(s)\}$. Also, according to the frame

$$\{\alpha(s);\ \mathbf{T}(s),\ \mathbf{N}(s)\},$$

the velocity vector of the associated curve is calculated as

$$\frac{d(\alpha_A(s))}{ds} = v_1(s) + iv_2(s),$$

where

$$v_1(s) + iv_2(s) := 1 - \kappa(s)y(s) + \frac{dx(s)}{ds} + i\left(\kappa(s)x(s) + \frac{dy(s)}{ds}\right). \tag{1}$$

Moreover, assume that s_A is the arc length parameter of α_A, then the Frenet frame $\{\mathbf{T}_A, \mathbf{N}_A\}$ of α_A holds

$$\begin{aligned} \mathbf{T}_A'(s_A) &= \kappa_A(s_A)\mathbf{N}_A(s_A), \\ \mathbf{N}_A'(s_A) &= -\kappa_A(s_A)\mathbf{T}_A(s_A), \end{aligned}$$

where κ_A is the curvature of α_A. Let θ be a slope angle of α and ω a slope angle of α_A. Then, we can write

$$\kappa_A(s_A) = \frac{d\omega}{ds_A} = \frac{d\omega}{ds}\frac{ds}{ds_A} = \left(\kappa + \frac{d\phi}{ds}\right)\frac{1}{\|v_1(s) + iv_2(s)\|},$$

where $\phi = \omega - \theta$.

3 Euler Savary's Formula

In this section, we investigate the Euler Savary's formula which gives the relation between the curvatures of a base curve, a rolling curve, and a roulette. In addition to this, we investigate the third one when any two of the base curve, the rolling curve, and the roulette are known on the complex plane.

CASE 1. *Let the base curve and the rolling curve be given.*

Let (B) be the base curve with curvature κ_B. Assume that X is a point relative to the rolling curve (R) and the roulette of the locus of this point X is denoted by (X). Then, we can consider that (X) is an the associated curve of (B). So the relative coordinate $\{x, y\}$ of (X) with respect to the curve (B) satisfies the following equation

$$v_1(s) + iv_2(s) = 1 - \kappa(s)\, y(s) + \frac{dx(s)}{ds} + i\left(\kappa(s)\, x(s) + \frac{dy(s)}{ds}\right),$$

from the equation (1).

Moreover, we know that when the rolling curve rolls without splitting along the base curve (B) at each point of contact, the relative coordinate $\{x, y\}$ is also a relative coordinate of (X) with respect to the curve (R) for a suitable parameter s_R. In this situation, the associated curve is a point X and the following is provided:

$$v_1(s_R) + iv_2(s_R) = 1 - \kappa(s_R)\, y(s_R) + \frac{dx(s_R)}{ds_R} + i\left(\kappa(s_R)\, x(s_R) + \frac{dy(s_R)}{ds_R}\right) = 0.$$

Then, we have

$$\frac{dx(s_R)}{ds_R} + i\frac{dy(s_R)}{ds_R} = -1 + \kappa_R(s_R)\, y(s_R) - i\kappa_R(s_R)\, x(s_R). \tag{2}$$

If we substitute these equations into (1), we get

$$v_1(s_R) + iv_2(s_R) = (\kappa_R - \kappa_B)\, y + i(\kappa_B - \kappa_R)\, x.$$

However, we can write the associated curve on the polar coordinate with respect to $\{(B)(s); x, y\}$ as follows:

$$(X) = re^{i\phi(s)},$$

where r is the distance from the origin point $(B)(s)$ to the point X. Then, from the equation (2), we calculate

$$\begin{aligned} \frac{d(X)}{ds_R} &= \frac{dr}{ds_R}e^{i\phi(s)} + ire^{i\phi(s_R)}\frac{d\phi(s_R)}{ds_R} \\ &= \kappa_R r\sin\phi - 1 + i(-\kappa_R r\cos\phi). \end{aligned}$$

If we solve this equation with respect to $r\frac{d\phi}{ds_R}$, then we find

$$r\frac{d\phi}{ds_R} = -\kappa_R r + \mathrm{Im}(\mathrm{e}^{\mathrm{i}\phi}). \tag{3}$$

Furthermore, we know that

$$\begin{aligned}\kappa_{(X)} &= \left(\kappa_B+\frac{d\phi}{ds}\right)\frac{1}{\left|\kappa_B-\kappa_R\right|\sqrt{x^2+y^2}}\\ &= \left(\kappa_B+\frac{d\phi}{ds}\right)\frac{1}{\left|\kappa_B-\kappa_R\right| r}.\end{aligned}\tag{4}$$

So, we get

$$r\kappa_{(X)}=\frac{\kappa_B-\kappa_R}{\left|\kappa_B-\kappa_R\right|}+\frac{\mathrm{Im}(\mathrm{e}^{\mathrm{i}\phi})}{r\left|\kappa_B-\kappa_R\right|},\tag{5}$$

from the equations (3) and (4).

So, we may give the following theorem:

THEOREM 1. Assume that a curve (R) rolls without splitting along a curve (B) on the complex plane $\mathbb{C}$. Let (X) be a locus of a point that is relative to (R). Let Q be a point on (X) and R be a point of contact of (B) and (R) corresponds to Q relative to the rolling relation. By (r,ϕ), we denote a polar coordinate of Q with respect to the origin R and the base line $(B)'|_R$. Then, the curvatures κ_B, κ_R and $\kappa_{(X)}$ of the curves (B), (R), and (X), respectively, satisfy

$$r\kappa_{(X)}=\pm1+\frac{\mathrm{Im}(\mathrm{e}^{\mathrm{i}\phi})}{r\left|\kappa_B-\kappa_R\right|}.$$

CASE 2. *Assume that the base curve and the roulette are given.*

Suppose that $(B)(s_B)=u(s_B)+iv(s_B)$ is a base curve with the arc length parameter s_B. Let us draw the normal to the roulette (X) for a point Q of the curve (B) and let the point $R=x(s_B)+iy(s_B)$ be the foot of this normal. Then the length of the normal **QR** is

$$d(Q,R)=\sqrt{\left(x(s_B)-u(s_B)\right)^2+\left(y(s_B)-v(s_B)\right)^2}.\tag{6}$$

However, by considering the equation (6) on the rolling curve (R), this equation represents the length of the point X relative to (R) and a point of (R). So, the orthogonal coordinate $f(s_B)+ig(s_B)$ of (R) is given by the following equations:

$$\begin{aligned}\left\|f(s_B)+ig(s_B)\right\| &= \left\|\left(x(s_B)-u(s_B)\right)+i\left(y(s_B)-v(s_B)\right)\right\|,\\ \left\|\left(\frac{df}{ds_B}\right)+i\left(\frac{dg}{ds_B}\right)\right\| &= 1.\end{aligned}$$

CASE 3. *Now, assume that the rolling curve (R) and the roulette (X) are given.*

Suppose that $(X)(s_A)=x(s_A)+iy(s_A)$ is the roulette with arc length parameter s_A and $(R)(s_R)$ is given by the polar coordinate $r(s_R)$ with the arc length parameter s_R. Because of the normal of (X) is $\mathbf{N}(s_A)=-y'(s_A)+ix'(s_A)$, a point $u(s_B)+iv(s_B)$ of the point curve (B) is given by

$$u(s_B)+iv(s_B)=x(s_A)+iy(s_A)\pm r(s_R)\mathbf{N}$$

or

$$u(s_B)+iv(s_B)=x(s_A)\mp r(s_R)\,y'(s)+i\,(y(s_A)\pm r(s_R)\,x'(s)). \tag{7}$$

So, we can write

$$\frac{du}{ds_R}+i\frac{dv}{ds_R}=\frac{dx}{ds_A}\frac{ds_A}{ds_R}+i\frac{dy}{ds_A}\frac{ds_A}{ds_R}\pm\frac{dr}{ds_R}\mathbf{N}\pm r\frac{d\mathbf{N}}{ds_A}\frac{ds_A}{ds_R}$$

or

$$\begin{aligned}\frac{du}{ds_R}+i\frac{dv}{ds_R} &= \frac{dx}{ds_A}\frac{ds_A}{ds_R}+i\frac{dy}{ds_A}\frac{ds_A}{ds_R}\pm\frac{dr}{ds_R}\left(-\frac{dy}{ds_A}+i\frac{dx}{ds_A}\right)\\ &\quad \mp r\left(\kappa_{(X)}\left(\frac{dx}{ds_A}+i\frac{dy}{ds_A}\right)\right)\frac{ds_A}{ds_R}.\end{aligned}$$

Then, we find

$$\frac{du}{ds_R}+i\frac{dv}{ds_R}=\frac{dx}{ds_A}\left(1\mp r\kappa_{(X)}\right)\frac{ds_A}{ds_R}\mp\frac{dr}{ds_R}\frac{dy}{ds_A}+i\left(\frac{dy}{ds_A}\left(1\mp r\kappa_{(X)}\right)\frac{ds_A}{ds_R}\pm\frac{dr}{ds_R}\frac{dx}{ds_A}\right),$$

where $\kappa_{(X)}$ is the curvature of (X). Since s_R is also the arc length of (B), we have

$$\left\|\left(\frac{du}{ds_R}\right)+i\left(\frac{dv}{ds_R}\right)\right\|=1,$$

and

$$\left(\frac{ds_A}{ds_R}\right)^2\left(1\mp r\kappa_{(X)}\right)^2+\left(\frac{dr}{ds_R}\right)^2=1.$$

From this differential equation, we can solve $s_A=s_A(s_R)$. By substituting this equation into equation (7), we can have the orthogonal coordinate of (B).

EXAMPLE 1. Let's find the curvature of the trajectory curve of a point on the circle (R) and circles (B) and (R) at the moment $t=15$ when circle $(R)=\cos\theta+i\sin\theta$ roles without splitting along $(B)=2\cos\theta+i2\sin\theta$ where $\theta\in[0,2\pi]$. Here t shows the moment when circle (R) rolls splitting along circle (B).

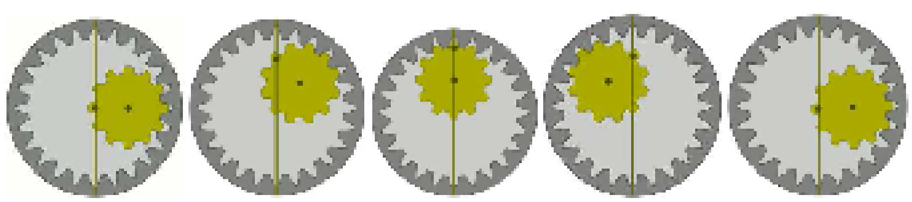

Figure 1: The rolling at the moment $t=0$, $t=15$, $t=30$, $t=45$, $t=119$, respectively.

SOLUTION. When a circle (R) roles without splitting along a circle (B), a trajectory curve of a point on (R) is a line, (see, Figure 1). So, we know that the curvature of this trajectory curve is zero.

If we want to solve from Euler Savary's formula, since $r = \sqrt{2}$ and $\phi = \frac{\pi}{4}$ at the moment $t = 15$ and $\kappa_B = \frac{1}{2}$ and $\kappa_R = 1$, we find

$$\sqrt{2}\kappa_{(X)} = -1 + \frac{\frac{\sqrt{2}}{2}}{\sqrt{2}\frac{1}{2}} = 0.$$

So, $\kappa_{(X)} = 0$.

DISCUSSION. If $\kappa_B - \kappa_R > 0$, then we have

$$\left(\frac{1}{r'} - \frac{1}{r}\right) \mathrm{Im}(\mathrm{e}^{\mathrm{i}\phi}) = \kappa_{\mathrm{B}} - \kappa_{\mathrm{R}},$$

from the equation (5), where $r' = r - \frac{1}{\kappa_{(X)}}$ is the distance from origin point to the curvature center of (X). In addition, if $\kappa_B - \kappa_R < 0$, then we get

$$\left(\frac{1}{r'} - \frac{1}{r}\right) \mathrm{Im}(\mathrm{e}^{\mathrm{i}\phi}) = \kappa_{\mathrm{B}} - \kappa_{\mathrm{R}},$$

from the equation (5), where $r' = r + \frac{1}{\kappa_{(X)}}$ is the distance between origin and the curvature center of (X). Consequently, the Euler Savary's formula in the complex plane $\mathbb{C}$ coincides with the Euler Savary's formula in the Euclidean plane $\mathbb{E}^2$ in [9]. But, in [10], the Euler Savary formula in the complex plane $\mathbb{C}$ is found as

$$\left(\frac{1}{r'} - \frac{1}{r}\right) i \left(e^{-i\phi}\right) = \kappa_B - \kappa_R.$$

It is different from Euler Savary's formula in the Euclidean plane. Note that the left hand-side of the equation is complex number and right hand-side of the equation is real number. So, it is provided when $\phi = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$. But, sometimes ϕ may not equal to $\frac{\pi}{2} + k\pi$.

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