Applied Mathematics E-Notes, 16(2016), 133-143 © ISSN 1607-2510
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Sequential Pareto Subdifferential Sum Rule And Sequential Efficiency*

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Received 15 October 2015

Abstract

The aim of this note is to establish sequential formulas for the Pareto subdifferential (weak and proper) of the sum of two convex vector valued mappings. An application is given dealing with sequential efficiency optimality conditions for constrained convex vector optimization problem.

1 Introduction

Sequential convex subdifferential calculus have received a great deal of interest from the scientific community in recent years (see [1, 2, 6, 7, 11]). One knows that a qualification condition is required for any exact subdifferential calculus rule and also for deriving optimality conditions related to a constrained optimization problem. In the absence of constraint qualification, sequential calculus constitutes an alternative way for establishing formulas for the subdifferential as the sum or the composition of convex functions in terms of the subdifferential of data functions at nearby points. In this note, our main object is to establish formulas for the sequential Pareto subdifferential (weak and proper) of the sum of two convex vector mappings. This enables us to derive sequential efficiency optimality conditions in terms of subgradients and normal cones, which characterize weak (resp. proper) efficient solution of constrained convex vector optimization problem.

The rest of the work is written as follows. In section 2, we present some basic definitions and preliminary material. In section 3, we establish the sequential formulas for the weak and proper Pareto subdifferential of the sum of two convex vector valued mappings. In section 4, we derive sequential efficiency optimality conditions of constrained convex vector optimization problem.

*Mathematics Subject Classifications: 90C48, 90C25.

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2 Notations, Definitions and Preliminaries

Let X and Y be two Hausdorff topological vector spaces, X^* and Y^* be their respective topological duals paired in duality by $\langle , \rangle$ and Y_+ be a nonempty convex cone of Y with nonempty interior (int $\mathrm{Y}_+ \neq \varnothing$). Let $l(Y_+) = Y_+ \cap -Y_+$ be the lineality of Y_+, when it is nul, the cone Y_+ is said to be pointed. In what follows, the convex cone Y_+ is not supposed to be a linear subspace so that it cannot coincide with its lineality. For any $y_1, y_2 \in Y$, we define the following ordering relations

$$\begin{aligned} y_1 \leq_{Y_+} \quad & y_2 \iff y_2 - y_1 \in Y_+, \\ y_1 <_{Y_+} \quad & y_2 \iff y_2 - y_1 \in \text{int } Y_+, \\ y_1 \lneq_{Y_+} \quad & y_2 \iff y_2 - y_1 \in Y_+ \backslash l(Y_+). \end{aligned}$$

We adjoint to Y an artificial element $+\infty$ such that $y \leq_{Y_+} +\infty$, for any $y \in Y$ and $y + (+\infty) = +\infty$ for any $y \in Y \cup \{+\infty\}$ and $\alpha.(+\infty) = +\infty$ for any $\alpha \in \mathbb{R}_+$.

The polar cone Y_+^* of Y_+ is the set of $y^* \in Y^*$ such that $y^*(Y_+) \sqsubseteq \mathbb{R}_+$, while the strict polar cone $(Y_+^*)^\circ$ of Y_+ is the set of $y^* \in Y^*$ such that

$$y^*(Y_+ \diagdown l(Y_+)) \sqsubseteq \mathbb{R}_+ \diagdown \{0\}.$$

Since convexity and lower semi-continuity play an important role in the sequel, let us recall the concept of cone convex mappings and the concept of lower semi-continuity in the sense of Penot-Théra.

DEFINITION 2.1.

1) A mapping $F : X \longrightarrow Y \cup \{+\infty\}$ is said to be Y_+- convex if for every $\lambda \in [0, 1]$ and $x_1, x_2 \in X$,

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_{Y_+} \lambda F(x_1) + (1 - \lambda)F(x_2)$$

2) Following [3, 9,10], one says that a mapping $F : X \longrightarrow Y \cup \{+\infty\}$ is lower semi-continuous (l.s.c) at $\bar{x} \in$ dom F $:= \{x \in X : F(x) \in Y\}$ if for any neighborhood V of $F(\bar{x})$ in Y, there exists a neighborhood U of $\bar{x}$ such that

$$F(U) \subseteq (V + Y_+) \cup \{+\infty\}; \tag{1}$$

When $F(\bar{x}) = +\infty$, F is said to be lower semi-continuous (l.s.c) at $\bar{x}$ if for any $y \in Y$, any neighborhood V of y, there exists a neighborhood U of $\bar{x}$ such that (1) is satisfied.

REMARK 2.1.

1) In the case when $Y = \mathbb{R},\ Y_+ = [0, +\infty[$ we recover the notion of scalar semi-continuity.

2) Let us note that it was shown in [10] that if $F : X \longrightarrow Y \cup \{+\infty\}$ is l.s.c, then the epigraph
$$\text{EpiF} := \{(x, y) \in X \times Y : F(x) \leq_{Y_+} y\}$$
of F is closed in $X \times Y$, but the converse is false in general (see a counterexample given in [10]).

3) A sequential characterization of the vector lower semi-continuity was studied in [4].

In what follows, we will need a stability result linked to the composition operation of mappings. Given a function $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ and define the composite function $g \circ F : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ as follows (see [9])

$$(g \circ F)(x) := \begin{cases} g(F(x)) & \text{if } x \in \text{dom F}, \\ \sup_{y \in Y} g(y) & \text{otherwise.} \end{cases} \tag{2}$$

Let us recall that $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be Y_+-nondecreasing if for any $y_1, y_2 \in Y$, we have
$$y_1 \leq_{Y_+} y_2 \Longrightarrow g(y_1) \leq g(y_2).$$

PROPOSITION 2.2 ([4, 9]). Let us assume that the mapping $F : X \longrightarrow Y \cup \{+\infty\}$ is lower semi-continuous and that the function $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is Y_+-nondecreasing and lower semi-continuous. Then the function $g \circ F$ is lower semi-continuous.

REMARK 2.2.

1) Proposition 2.2 fails to hold if the lower semi-continuity is replaced by the weaker assumption that EpiF is closed (see a counterexample given in [8]).

2) It is more usual to set $(g \circ F)(x) = +\infty$ for $x \in X \backslash \text{dom F}$, but, with such a definition, Proposition 2.2 is no longer true (see a counterexample given in [4, 9]).

3) In the sequel, we shall need the lower semi-continuity of the mapping $y^* \circ F$ for any $y^* \in Y_+^*$. For this, let us observe that for any $y^* \in Y_+^*$, y^* is Y_+-nondecreasing and lower semi-continuous and according to (2), $y^* \circ F = 0$ for $y^* = 0$, and $(y^* \circ F)(x) = +\infty$ for $y^* \neq 0$ and $F(x) = +\infty$. Now, if $F : X \longrightarrow Y \cup \{+\infty\}$ is lower semi-continuous and by using Proposition 2.2, we easily obtain that $y^* \circ F$ is lower semi-continuous.

Consider now the vector optimization problem

$$(P) \quad \min_{x \in C} F(x),$$

where $F : X \longrightarrow Y \cup \{+\infty\}$ is a given mapping and C is a nonempty subset of X. A point $\bar{x} \in \mathrm{dom\ F} \cap \mathrm{C}$ is said to be

► efficient solution of problem (P) if $\nexists x \in C,\ F(x) \lneq_{Y_+} F(\bar{x})$.

► weak efficient solution (w-efficient solution) of problem (P) if $\nexists x \in C,\ F(x) <_{Y_+} F(\bar{x})$.

► proper efficient solution (p-efficient solution) of problem (P) if

$$\exists \hat{Y}_+ \subsetneq Y \text{ convex cone such that } Y_+ \setminus l(Y_+) \sqsubseteq \mathrm{int}\hat{Y}_+,\ \nexists x \in C,\ F(x) \lneq_{\hat{Y}_+} F(\bar{x}).$$

The sets of efficient points, weakly and properly efficient points will be denoted respectively by $\mathrm{E}_e(F,C),\ \mathrm{E}_w(F,C)$ and $\mathrm{E}_p(F,C)$.

REMARK 2.3.

1) We have $E_p(F,C) \sqsubseteq E_e(F,C) \sqsubseteq E_w(F,C)$.
2) It is easy to see that if $\bar{x} \in \mathrm{dom\ F} \cap \mathrm{C}$ is a strong solution of the problem (P) i.e. $F(\bar{x}) \leq F(x)$ for any $x \in \mathrm{dom\ F} \cap \mathrm{C}$ then every efficient solution of the problem (P) is a strong solution i.e. $E_e(F,C)$ is exactly the set of strong solutions of problem (P).

These notions enable us to define the weak and proper subdifferential of a vector valued mapping $F : X \longrightarrow Y \cup \{+\infty\}$ at $\bar{x} \in \mathrm{dom\ F}$ as follows

- weak subdifferential

$$\partial^w F(\bar{x}) := \{A \in \mathrm{L}(X,Y) : \nexists x \in X,\ F(x) - F(\bar{x}) <_{Y_+} A(x - \bar{x})\}.$$

- proper subdifferential

$$\partial^p F(\bar{x}) := \{A \in \mathrm{L}(X,Y) : \exists \hat{Y}_+ \subsetneq Y \text{ convex cone such that } Y_+ \setminus l(Y_+) \sqsubseteq \mathrm{int}\hat{Y}_+,\ \nexists x \in X,\ F(x) - F(\bar{x}) \lneq_{\hat{Y}_+} A(x - \bar{x})\}.$$

where $\mathrm{L}(X,Y)$ is the space of linear continuous operators from X to Y.

In what follows, for simplicity we shall regroup in one notation σ-efficient solution, $\mathrm{E}_\sigma(F,C),\ \partial^\sigma F(\bar{x})$ for $\sigma \in \{w,p\}$ and

$$Y_+^\sigma := \begin{cases} Y_+^* \setminus \{0\} & \text{if } \sigma = w, \\ (Y_+^*)^\circ & \text{if } \sigma = p. \end{cases}$$

An important property follows immediately from the above definitions

$$\bar{x} \in E_\sigma(F,X) \Longleftrightarrow 0 \in \partial^\sigma F(\bar{x}) \quad (\sigma \in \{w,p\}).$$

In the sequel, we will need the concept of the strong subdifferential of F at $\bar{x} \in \mathrm{dom\ F}$ given by

$$\partial^s F(\bar{x}) := \{A \in L(X,Y) : A(x - \bar{x}) \leq_{Y_+} F(x) - F(\bar{x}),\ \forall x \in X\}.$$

When F is a function, $\partial^s F(\bar{x})$ reduces to the well-known scalar Fenchel subdifferential

$$\partial F(\bar{x}) := \{x^* \in X^* : F(x) - F(\bar{x}) \geq \langle x^*, x - \bar{x}\rangle, \ \forall x \in X\}.$$

A vector mapping $F : X \longrightarrow Y \cup \{+\infty\}$ is said to be σ-regular subdifferentiable ($\sigma \in \{w, p\}$), if

$$\partial(y^* \circ F)(\bar{x}) = y^* \circ \partial^s F(\bar{x}), \ \forall y^* \in Y_+^\sigma, \ \forall \bar{x} \in \text{dom F}$$

where it is understood that the set $y^* \circ \partial^s F(\bar{x}) := \{y^* \circ A : \ A \in \partial^s F(\bar{x})\}$.

REMARK 2.4. One can easily show that $(Y_+^*)^\circ \sqsubseteq Y_+^* \setminus \{0\}$. Hence we have F is w-regular subdifferentiable implies F is p-regular subdifferentiable.

The following scalarization result will be needed.

THEOREM 2.3 ([5]). Let X and Y be two Hausdorff topological vector spaces. Let $F : X \longrightarrow Y \cup \{+\infty\}$ be Y_+-convex vector valued mapping. For $\sigma \in \{w, p\}$ and $\bar{x} \in X$, we have

$$\partial^\sigma F(\bar{x}) = \bigcup_{y^* \in Y_+^\sigma} \{A \in L(X, Y) : y^* \circ A \in \partial(y^* \circ F)(\bar{x})\}.$$

When $\sigma = p$, we assume that Y_+ is pointed.

In the sequel, $(X, \|.\|_X)$ is a reflexive Banach space and $(X^*, \|.\|_{X^*})$ its topological dual. Let $(x_n^*)_{n\in\mathbb{N}}$ be a sequence in X^* (resp. $(x_n)_{n\in\mathbb{N}}$ be a sequence in X) and $x^* \in X^*$ (resp. $x \in X$). We write $x_n^* \overset{\|\cdot\|_{X^*}}{\longrightarrow} x^*$ (resp. $x_n \overset{\|\cdot\|_X}{\longrightarrow} x$) if $\|x_n^* - x^*\|_{X^*} \longrightarrow 0$ when $n \longrightarrow +\infty$ (resp. $\|x_n - x\|_X \longrightarrow 0$ when $n \longrightarrow +\infty$).

For establishing our main result, let us recall a sequential general formula without qualification conditions for the subdifferential of the sum of two proper lower semi-continuous convex functions defined on a reflexive Banach space.

THEOREM 2.4 ([11]). Let X be a reflexive Banach space. Let $f_1, f_2 : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be two proper convex and lower semi-continuous functions. Then for any $\bar{x} \in \text{dom f}_1 \cap \text{dom f}_2$ we have $x^* \in \partial(f_1 + f_2)(\bar{x})$ if and only if there exist $(x_{1,n}, x_{2,n}) \in \text{dom f}_1 \times \text{dom f}_2$, $x_{1,n}^* \in \partial f_1(x_{1,n})$ and $x_{2,n}^* \in \partial f_2(x_{2,n})$ such that

$$x_{1,n}^* + x_{2,n}^* \overset{\|\cdot\|_{X^*}}{\longrightarrow} x^*, \quad x_{1,n} \overset{\|\cdot\|_X}{\longrightarrow} \bar{x}, \quad x_{2,n} \overset{\|\cdot\|_X}{\longrightarrow} \bar{x} \quad (n \longrightarrow +\infty),$$

$$f_1(x_{1,n}) - \langle x_{1,n}^*, x_{1,n} - \bar{x}\rangle - f_1(\bar{x}) \longrightarrow 0 \quad (n \longrightarrow +\infty),$$

and

$$f_2(x_{2,n}) - \langle x_{2,n}^*, x_{2,n} - \bar{x}\rangle - f_2(\bar{x}) \longrightarrow 0 \quad (n \longrightarrow +\infty).$$

REMARK 2.5. The above theorem holds if we take the convergence of the sequence $x_{1,n}^* + x_{2,n}^* \overset{\|.\|_{X^*}}{\longrightarrow} x^*$ with respect to the weak star convergence $\sigma(X^*, X)$ instead of the norm convergence $\|.\|_{X^*}$ (see [11]).

3 Sequential Formula for Pareto Subdifferential of the Sum of Two Convex Vector Valued Mappings

In this section, we state the sequential Pareto subdifferential (weak and proper) of the sum of two convex vector valued mappings.

THEOREM 3.1. Let X be a reflexive Banach space, Y be a normed space and Y_+ be a nonempty convex cone of Y. Let $F_1, F_2 : X \longrightarrow Y \cup \{+\infty\}$ be two proper, Y_+-convex and lower semi-continuous vector valued mappings. Suppose, in addition, that Y_+ is pointed as $\sigma = p$. Then for $\bar{x} \in \mathrm{dom\ F_1} \cap \mathrm{dom\ F_2}$, $A \in \partial^\sigma(F_1 + F_2)(\bar{x})$ if and only if there exist $y^* \in Y_+^\sigma$, $(x_{i,n}, x_{i,n}^*) \in \mathrm{dom\ F_i} \times X^* \quad (i = 1, 2)$ satisfying

$$x_{i,n} \overset{\|\cdot\|_X}{\longrightarrow} \bar{x} \quad \text{and} \quad x_{i,n}^* \in \partial(y^* \circ F_i)(x_{i,n})$$

such that

$$x_{1,n}^* + x_{2,n}^* \overset{\|\cdot\|_{X^*}}{\longrightarrow} y^* \circ A$$

and

$$(y^* \circ F_i)(x_{i,n}) - \langle x_{i,n}^*, x_{i,n} - \bar{x} \rangle \longrightarrow (y^* \circ F_i)(\bar{x}) \quad (n \longrightarrow +\infty).$$

PROOF. Let $A \in \partial^\sigma(F_1 + F_2)(\bar{x})$. By applying the scalarization Theorem 2.3, there exists some $y^* \in Y_+^\sigma$ such that

$$y^* \circ A \in \partial(y^* \circ F_1 + y^* \circ F_2)(\bar{x}).$$

Since F_1 and F_2 are proper, convex and lower semi-continuous, it follows according to Remark 2.2, that the scalar functions $y^* \circ F_1$ and $y^* \circ F_2$ are proper, convex and lower semi-continuous and since $\mathrm{dom\ F_i} = \mathrm{dom\ (y^* \circ F_i)} \quad (i = 1, 2)$, we have $\bar{x} \in \mathrm{dom\ (y^* \circ F_1)} \cap \mathrm{dom\ (y^* \circ F_2)}$ and hence by virtue of Theorem 2.4, there exist $(x_{i,n}, x_{i,n}^*) \in \mathrm{dom\ F_i} \times X^* \quad (i = 1, 2)$ satisfying $x_{i,n} \overset{\|\cdot\|_X}{\longrightarrow} \bar{x}$ and $x_{i,n}^* \in \partial(y^* \circ F_i)(x_{i,n})$ such that

$$x_{1,n}^* + x_{2,n}^* \overset{\|\cdot\|_{X^*}}{\longrightarrow} y^* \circ A$$

and

$$(y^* \circ F_i)(x_{i,n}) - \langle x_{i,n}^*, x_{i,n} - \bar{x} \rangle \longrightarrow (y^* \circ F_i)(\bar{x}) \quad (n \longrightarrow +\infty).$$

THEOREM 3.2. Let X be a reflexive Banach space, Y be a normed space and Y_+ be a nonempty convex cone of Y. Let $F_1, F_2 : X \longrightarrow Y \cup \{+\infty\}$ be two proper, Y_+-convex and lower semi-continuous vector valued mappings. Suppose that F_1 and F_2 are σ-regular subdifferentiables on X and, in addition, that Y_+ is pointed as $\sigma = p$. Then for $\bar{x} \in \mathrm{dom\ F_1} \cap \mathrm{dom\ F_2}$, $A \in \partial^\sigma(F_1 + F_2)(\bar{x})$ if and only if there exist $y^* \in Y_+^\sigma$, $(x_{i,n}, x_{i,n}^*) \in X \times X^* \quad (i = 1, 2)$, $A_{i,n} \in \partial^s F_i(x_{i,n})$ satisfying $x_{i,n} \overset{\|\cdot\|_X}{\longrightarrow} \bar{x}$ and $x_{i,n}^* = y^* \circ A_{i,n}$ such that

$$y^* \circ A_{1,n} + y^* \circ A_{2,n} \overset{\|\cdot\|_{X^*}}{\longrightarrow} y^* \circ A$$

and

$$(y^* \circ F_i)(x_{i,n}) - < y^* \circ A_{i,n}, x_{i,n} - \bar{x} > \longrightarrow (y^* \circ F_i)(\bar{x}) \quad (n \longrightarrow +\infty).$$

PROOF. Following the proof of Theorem 3.1 and by using the σ-regular subdifferentiability of F_i $(i = 1, 2)$, we have $\partial(y^* \circ F_i)(x_{i,n}) = y^* \circ \partial^s F_i(x_{i,n})$ for any $y^* \in Y_+^{\sigma}$. So, there exists some $A_{i,n} \in \partial^s F_i(\bar{x})$ $(i = 1, 2)$ such that $x_{i,n}^* = y^* \circ A_{i,n}$ and hence we obtain

$$y^* \circ A_{1,n} + y^* \circ A_{2,n} \overset{\|.\|_{X^*}}{\longrightarrow} y^* \circ A$$

and

$$(y^* \circ F_i)(x_{i,n}) - \langle y^* \circ A_{i,n}, x_{i,n} - \bar{x} \rangle \longrightarrow (y^* \circ F_i)(\bar{x}) \quad (n \longrightarrow +\infty).$$

REMARK 3.1. Let us note that Theorem 3.2 may be used if one of the vector mapping F_1 or F_2 is σ-regular subdifferentiable.

4 Sequential Efficiency for Constrained Vector Optimization

In this section we are concerned with the vector optimization problem

$$(P) \quad \min_{x \in C} F(x)$$

where $F : X \longrightarrow Y \cup \{+\infty\}$ is a proper, Y_+-convex and lower semi-continuous mapping and C is a nonempty convex closed subset of X.

By introducing the vector indicator mapping of the nonempty subset $C \subseteq X$,

$$\delta_C^v : X \longrightarrow Y \cup \{+\infty\},$$

$$\delta_C^v(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{otherwise,} \end{cases}$$

the problem (P) becomes equivalent to the unconstrained vector minimization problem

$$\min_{x \in X} (F(x) + \delta_C^v(x))$$

in the following sense.

LEMMA 4.1.

1) Let $F : X \longrightarrow Y \cup \{+\infty\}$, then $E_\sigma(F, C) = E_\sigma(F + \delta_C^v, X)$ $(\sigma \in \{w, p\})$.

2) If C is closed then δ_C^v is lower semi-continuous.

PROOF. The proof of statement 1) can be found in [5]. 2) If $\bar{x} \in C$, let V be any neighborhood of $\delta_C^v(\bar{x}) = 0$ in Y. By choosing $U := X$ as a neighborhood of $\bar{x}$ in X and by distinguishing the cases $u \in C$, $u \in X \setminus C$ $(u \in U)$, the inclusion $\delta_C^v(U) \subset (V + Y_+) \cup \{+\infty\}$ follows immediately. Now if $\bar{x} \in X \setminus C$, let $y \in Y$ and V any neighborhood of y. By putting $U := X \setminus C$ which is an open neighborhood of $\bar{x}$ (since C is closed), one can easily check $\delta_C^v(U) \subset (V + Y_+) \cup \{+\infty\}$.

The vector indicator mapping appears to possess properties like the scalar one. By considering the normal cone of C at $\bar{x} \in C$ in a vector sense

$$N_C^v(\bar{x}) := \{A \in L(X,Y) : A(x - \bar{x}) \leq_{Y_+} 0, \quad \forall x \in C\},$$

it is easy to see that $N_C^v(\bar{x}) = \partial^s \delta_C^v(\bar{x})$. If $Y = \mathbb{R}$, the vector normal cone reduces to the usual normal cone

$$N_C(\bar{x}) := \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq 0, \quad \forall x \in C\}.$$

In fact, the presence of the vector indicator mapping enables us to state the following sequential optimality conditions characterizing a σ-efficient solution ($\sigma \in \{w, p\}$) of the problem (P).

THEOREM 4.2. Let X be a reflexive Banach space, Y be a normed space and Y_+ be a nonempty convex cone of Y. Let $F : X \longrightarrow Y \cup \{+\infty\}$ be a proper, Y_+-convex and lower semi-continuous mapping and C be a nonempty convex closed subset of X. Suppose, in addition, that Y_+ is pointed as $\sigma = p$. Then, $\bar{x} \in \mathrm{dom\ F} \cap C$ is a σ-efficient solution of problem (P) if and only if there exist $y^* \in Y_+^\sigma$, $(x_{1,n}, x_{1,n}^*) \in \mathrm{dom\ F} \times X^*$ and $(x_{2,n}, x_{2,n}^*) \in C \times N_C(x_{2,n})$ satisfying

$$x_{i,n} \overset{\|\cdot\|_X}{\longrightarrow} \bar{x} \ (i = 1, 2) \quad \text{ and } \quad x_{1,n}^* \in \partial(y^* \circ F)(x_{1,n})$$

such that

$$x_{1,n}^* + x_{2,n}^* \overset{\|\cdot\|_{X^*}}{\longrightarrow} 0,$$
$$(y^* \circ F)(x_{1,n}) - \langle x_{1,n}^*, x_{1,n} - \bar{x}\rangle \longrightarrow (y^* \circ F)(\bar{x}) \quad (n \longrightarrow +\infty),$$
$$\langle x_{2,n}^*, x_{2,n} - \bar{x}\rangle \longrightarrow 0 \quad (n \longrightarrow +\infty).$$

PROOF. By virtue of Lemma 4.1, we can write

$$\bar{x} \in E_\sigma(F, C) \Longleftrightarrow \bar{x} \in E_\sigma(F + \delta_C^v, X) \Longleftrightarrow 0 \in \partial^\sigma(F + \delta_C^v)(\bar{x})$$

and also we have δ_C^v is lower semi-continuous. The vector mappings F and δ_C^v satisfy together all the hypotheses of Theorem 3.1, thus we deduce $0 \in \partial^\sigma(F + \delta_C^v)(\bar{x})$ if and only if there exist $y^* \in Y_+^\sigma$, $(x_{1,n}, x_{1,n}^*) \in \mathrm{dom\ F} \times X^*$ and $(x_{2,n}, x_{2,n}^*) \in \mathrm{dom}\ \delta_{\mathrm{C}}^{\mathrm{v}} \times \mathrm{X}^* = C \times X^*$, satisfying $x_{i,n} \overset{\|\cdot\|_X}{\longrightarrow} \bar{x}$ $(i = 1, 2)$ and

$$x_{1,n}^* \in \partial(y^* \circ F)(x_{1,n}) \quad \text{and} \quad x_{2,n}^* \in \partial(y^* \circ \delta_C^v)(x_{2,n}) \tag{3}$$

such that

$$x_{1,n}^* + x_{2,n}^* \overset{\|\cdot\|_{X^*}}{\longrightarrow} 0,$$

$$(y^* \circ F)(x_{1,n}) - \langle x_{1n}^*, x_{1,n} - \bar{x} \rangle \longrightarrow (y^* \circ F)(\bar{x}) \quad (n \longrightarrow +\infty),$$

and

$$(y^* \circ \delta_C^v)(x_{2,n}) - \langle x_{2,n}^*, x_{2,n} - \bar{x} \rangle \longrightarrow (y^* \circ \delta_C^v)(\bar{x}) \quad (n \longrightarrow +\infty). \tag{4}$$

As $y^* \circ \delta_C^v = \delta_C$ for any $y^* \in Y_+^\sigma$, it follows from (3) that

$$x_{2,n} \in C \quad \text{and} \quad x_{2,n}^* \in \partial \delta_C(x_{2,n}) = N_C(x_{2,n})$$

and therefore (4) becomes equivalent to

$$\langle x_{2,n}^*, x_{2,n} - \bar{x} \rangle \longrightarrow 0 \quad (n \longrightarrow +\infty)$$

which completes the proof.

By using the same arguments as in the proof of Theorem 4.2 and the equality $N_C(\bar{x}) = y^* \circ N_C^v(\bar{x})$ obtained from the fact that the vector indicator mapping δ_C^v is w-regular subdifferentiable (see [5]), we can easily derive the following result.

THEOREM 4.3. Let X be a reflexive Banach space, Y be a normed space and Y_+ be a nonempty convex cone of Y. Let $F : X \longrightarrow Y \cup \{+\infty\}$ be proper, Y_+-convex and lower semi-continuous and C be a nonempty convex closed subset of X. Suppose, in addition, that Y_+ is pointed as $\sigma = p$. Then, $\bar{x} \in \text{dom F} \cap \text{C}$ is a σ-efficient solution of problem (P) if and only if there exist $y^* \in Y_+^\sigma$, $(x_{1,n}, x_{1,n}^*) \in \text{dom F} \times X^*$, $(x_{2,n}, x_{2,n}^*) \in C \times N_C(x_{2,n})$ and $A_n \in N_C^v(x_{2,n})$ satisfying

$$x_{i,n} \overset{\|\cdot\|_X}{\longrightarrow} \bar{x} \ (i = 1, 2), \quad x_{2,n}^* = y^* \circ A_n \quad \text{and} \quad x_{1,n}^* \in \partial (y^* \circ F)(x_{1,n})$$

such that

$$x_{1,n}^* + y^* \circ A_n \overset{\|\cdot\|_{X^*}}{\longrightarrow} 0,$$

$$(y^* \circ F)(x_{1,n}) - \langle x_{1,n}^*, x_{1,n} - \bar{x} \rangle \longrightarrow (y^* \circ F)(\bar{x}) \quad (n \longrightarrow +\infty),$$

and

$$\langle y^* \circ A_n, x_{2,n} - \bar{x} \rangle \longrightarrow 0 \quad (n \longrightarrow +\infty).$$

If F is furthermore σ-regular subdifferentiable, it follows from Theorem 4.2 the next result.

THEOREM 4.4 Let X be a reflexive Banach space, Y be a normed space and Y_+ be a nonempty convex cone of Y. Let $F : X \longrightarrow Y \cup \{+\infty\}$ be proper, Y_+-convex, lower semi-continuous and σ-regular subdifferentiable. Let C be a nonempty convex closed subset of X. Suppose, in addition, that Y_+ is pointed as $\sigma = p$. Then, $\bar{x} \in \text{dom F} \cap \text{C}$ is a σ-efficient solution of problem (P) if and only if there exist

$y^* \in Y_+^\sigma$, $(x_{1,n}, x_{1,n}^*) \in \text{dom F} \times X^*$, $(x_{2,n}, x_{2,n}^*) \in C \times N_C(x_{2,n})$, $A_{1,n} \in \partial^s F(x_{1,n})$ and $A_{2,n} \in N_C^v(x_{2,n})$ satisfying

$$x_{1,n}^* = y^* \circ A_{1,n}, \quad x_{2,n}^* = y^* \circ A_{2,n},$$

$$x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x} \ (i = 1, 2) \ \text{ and } \ x_{1,n}^* \in \partial(y^* \circ F)(x_{1,n})$$

such that

$$y^* \circ A_{1,n} + y^* \circ A_{2,n} \xrightarrow{\|\cdot\|_{X^*}} 0,$$

$$(y^* \circ F)(x_{1,n}) - \langle y^* \circ A_{1,n}, x_{1,n} - \bar{x} \rangle \longrightarrow (y^* \circ F)(\bar{x}) \quad (n \longrightarrow +\infty),$$

and

$$\langle y^* \circ A_{2,n}, x_{2,n} - \bar{x} \rangle \longrightarrow 0 \quad (n \longrightarrow +\infty).$$

REMARK 4.1. The study of the sequential Pareto subdifferential (weak and proper) of a convex composed operator will appear in a forthcoming paper.

Acknowledgment. The authors are grateful to the referee for his valuable comments and suggestions.

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