Applied Mathematics E-Notes, 16(2016), 99-104 © ISSN 1607-2510
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Approximate Wave Solutions Of Delay Diffusive Models Using A Differential Transform Method*

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Received 03 October 2015

Abstract

A method is proposed to approximate the wave solutions of local delayed reaction-diffusion models of single species populations. Using an extended differential transform method, it is shown that the boundary value problem associated with the wave equation and logistic birth function can be transformed into a nonlinear system of algebraic equations. The solution of the truncated nonlinear system may represent the approximate wave solution of the model.

1 Introduction

In the present work we consider the following scalar delayed Reaction-Diffusion model

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} - du(x,t) + b(u(x,t-\tau)), \tag{1}$$

where $u(x,t)$ is the population density of a single species at time t and position x; $\tau \geq 0$ is a delay term representing the maturation time of individuals; $b(w)$ is the nonlinear birth function; D is the diffusion coefficient; and d is the death rate. Model (1) has been extensively investigated including the asymptotic behavior of solutions [15], Hopf bifurcation [15, 16], solutions of the corresponding Dirichlet problem [17] and the traveling wave solutions [11, 15, 18]. In the absence of diffusion, the work by Gurney et al. [9] considers the specific birth function $b(u) = pue^{-au}$, where a Hopf bifurcation point is obtained for τ. Furthermore, the local and global stability of (1) has been discussed in various studies (see [8, 14] and the references therein). Model (1) has also been extended to include nonlocality [19] and various two-dimensional spatial domains [6, 13, 20], where the existence and behavior of traveling wave solutions [2, 5, 19] has been investigated. A solution $u(x,t)$ of (1) is a traveling wave solution, if it is in the form of

$$u(x,t) = \phi(x+ct) = \phi(z), z = x + ct, \tag{2}$$

where c is the speed of propagation and z is the wave variable. Then substituting $\phi(z)$ into (1) and replacing z with t, the wave equation corresponding to (1) is given by

$$D\phi''(t) - c\phi'(t) - d\phi(t) + b(\phi(t-c\tau)) = 0. \tag{3}$$

*Mathematics Subject Classifications: 34K99, 35K55, 92B99.

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To be biologically meaningful, the wave solution $\phi(t)$ must be bounded and nonnegative for all $t \in \mathbb{R}$. Also, a wave solution of (1) is a solution of (3) which satisfies the boundary conditions $\lim_{t\to-\infty} \phi(t) = \phi_1$ and $\lim_{t\to\infty} \phi(t) = \phi_2$, where ϕ_1 and ϕ_2 are the equilibria of (3). Wave solutions of delay diffusive population models have been center of attention for several decades [3, 19, 20]. Although the existence and uniqueness of the wave solutions have been extensively studied, less efforts have been made towards wave approximations. By approximating wave solutions we will be able to investigate impacts of model parameters on the behavior of the wave solutions. For instance, it has been numerically shown that the wave solution may become humped-shaped when the monotonicity condition is violated [12]. An approximate wave solution may provide a better understanding of the behavior of the wave solutions in the spatial domain [3, 4]. The main goal of this short note is to apply a differential transform method to approximate the wave equation (3) with the above-mentioned boundary conditions. In particular, the Differential Transform Method (DTM) [21] has been recently extended for solving delay differential equations [10]. Thus, the wave solution can be approximated to any desired degree of exactness by transforming the wave equation to an algebraic system of nonlinear equations.

2 Extended Differential Transform Method

DEFINITION 1. The differential transform of a function $\psi(z)$ at a point z_0 is defined by

$$\Psi(k) = \frac{1}{k!}\left[\frac{d^k}{dz}\psi(z)\right]_{z=z_0}, \tag{4}$$

where ψ is analytic at z_0.

DEFINITION 2. The inverse of the differential transform $\Psi(k)$ is defined by

$$\psi(z) = \sum_{k=0}^{\infty} \Psi(k)(z-z_0)^k. \tag{5}$$

Let the small and capital letters represent the original and transformed functions, respectively. Then using (4) and (5), the differential transforms have the following properties (see [1, 10, 21] for the proofs).

1. If $\phi(t) = f(t) \pm g(t)$, then $\Phi(k) = F(k) \pm G(k)$.
2. If $\phi(t) = \gamma f(t)$, then $\Phi(k) = \gamma F(k)$, where γ is a constant.
3. If $\phi(t) = d^n f(t)/dt^n$, then $\Phi(k) = [(k+n)!/k!]\, F(k+n)$.
4. If $\phi(t) = f(t)g(t)$, then $\Phi(k) = \sum_{k_1=0}^{k} F(k_1)G(k-k_1)$.

5. If $\phi(t) = f(t + a)$, then

$$\Phi(k) = \sum_{h_1=k}^{\infty} \binom{h_1}{k} a^{h_1-k} F(h_1) \quad \text{where} \quad \binom{h_1}{k} = \frac{h_1!}{k!(h_1 - k)!}.$$

For $M > 0$ sufficiently large, we may replace the original boundary conditions with their approximations. Specifically, the boundary value problem corresponding to wave equation (3) is given by

$$\begin{cases} D\phi^{''}(t) - c\phi^{'}(t) - d\phi(t) + b(\phi(t - c\tau)) = 0, \\ \phi(-M) = \phi_1 \text{ and } \phi(M) = \phi_2, \end{cases} \tag{6}$$

where ϕ_1 and ϕ_2 are the steady states of the wave equation (3). A major benefit of the extended DTM is that the method does not require the history function and it only requires the boundary values at the two ends. By rescaling the problem, $-M$ and M are transformed to 0 and 1. In particular, let $z = t/2M + 1/2$ and $\psi(z) = \phi(t)$, then the wave equation (6) is transformed to

$$\begin{cases} D\psi^{''}(z) - 2cM\psi^{'}(z) - 4dM^2\psi(z) + 4M^2 b(\psi(z - \frac{c\tau}{2M})) = 0, \\ \psi(0) = \phi_1 \text{ and } \psi(1) = \phi_2. \end{cases} \tag{7}$$

To apply the extended DTM we consider the well-known logistic birth function given by $b(\psi) = p\psi(1 - \frac{\psi}{k_c})$, where p is the growth rate and the k_c is the carrying capacity.

THEOREM 1. The differential transform of $\phi(t) = f_1(t + a)f_2(t + b)$ with $a, b \in \mathbb{R}$ is given by

$$\Phi(k) = \lim \sum_{k_1=0}^{k} \sum_{h_1=k}^{N} \sum_{h_2=k-k_1}^{N} \binom{h_1}{k_1} \binom{h_2}{k - k_1} a^{h_1-k_1} b^{h_2-k+k_1} F_1(h_1)F_2(h_2),$$

for $N \to \infty$.

PROOF. Let the differential transforms of $f_1(t), f_2(t), f(t) = f_1(t + a)$ and $g(t) = f_2(t + b)$ at $t = t_0$ be $F_1(k), F_2(k), F(k)$ and $G(k)$, respectively. Using the property (4) the differential transform of $\phi(t)$ is given by

$$\Phi(k) = \sum_{k_1=0}^{k} F(k_1)G(k - k_1). \tag{8}$$

From property (5) we get

$$F(k_1) = \sum_{h_1=k_1}^{N} \binom{h_1}{k_1} a^{h_1-k_1} F_1(h_1) \text{ for } N \to \infty. \tag{9}$$

Similarly,

$$G(k-k_1)=\sum_{h_2=k-k_1}^{N}\left(\begin{array}{c}h_2\\ k-k_1\end{array}\right)b^{h_2-k+k_1}F_2(h_2) \text{ for } N\to\infty. \tag{10}$$

The proof is completed by substituting (9) and (10) into (8).

Using the properties (1)–(5) and Theorem 1, the differential transform of problem (7) with the logistic birth function is given by

$$\begin{aligned}
&D(k+2)(k+1)\Psi(k+2)-2cM(k+1)\Psi(k+1)\\
&-4dM^2\Psi(k)+4pM^2\sum_{h_1=k}^{N}\left(\begin{array}{c}h_1\\ k\end{array}\right)\left(\frac{-c\tau}{2M}\right)^{h_1-k}\Psi(h_1)\\
&-\frac{4pM^2}{k_c}\sum_{k_1=0}^{k}\sum_{h_1=k}^{N}\sum_{h_2=k-k_1}^{N}\left(\begin{array}{c}h_1\\ k_1\end{array}\right)\left(\begin{array}{c}h_2\\ k-k_1\end{array}\right)\left(\frac{-c\tau}{2M}\right)^{h_1+h_2}\Psi(h_1)\Psi(h_2)\\
&=0,
\end{aligned}\tag{11}$$

for $N\to\infty$ and subject to the boundary conditions $\psi(0)=\phi_1$ and $\psi(1)=\phi_2$. We may truncate the infinite sums in (11) by letting $N=M$ and $k=0,1,\ldots,M-2$, where M is the positive constant chosen in (6) and (7). Using the boundary conditions, equation (11) corresponds to a homogeneous nonlinear system of $M-1$ equations with $M-1$ unknowns (i.e. $\Psi(k)$ for $k=1,\ldots,M-1$). Note that $\Psi(0)$ and $\Psi(M)$ are known through the given boundary conditions. Then the nonlinear system can be symbolically solved using Matlab or Maple software. Each solution set $\{\Psi(k)\}_{k=0}^{M}$ is plugged into equation (5) with $z_0=0$ and the desired approximated wave solution can be found. In a broad context, the above mentioned approach provides a basis to implement techniques of solving nonlinear homogeneous systems for finding approximations of the wave solutions.

Remarks by the Editor in Chief: It appeasrs that by assuming analytic solutions of (3), one may arrive at (11) also. See [7].

Acknowledgment. The author would like to thank the anonymous reviewers for their valuable suggestions and corrections. This work was partially supported by the University of Missouri Research Board grant (ID: KZ016055).

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