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Jacobi-Dunkl Dini Lipschitz Functions In The Space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$*

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Abstract

In this paper, we obtain an analog of Younis Theorem 5.2 in [7] for the Jacobi-Dunkl transform on the real line for functions satisfying the (η,γ)-Jacobi-Dunkl Lipschitz condition in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$ where $\alpha \geq \beta \geq \frac{-1}{2}$ and $\alpha \neq \frac{-1}{2}$.

1 Introduction

Younis Theorem 5.2 [7] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Dini Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have the following Theorem 1.1.

THEOREM 1.1 ([7, Theorem 5.2]). Let $f \in L^2(\mathbb{R})$. Then the following statements are equivalents

(a) $\|f(x+h) - f(x)\| = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\gamma}\right)$ as $h \to 0$, where $0 < \eta < 1$ and $\gamma \geq 0$.

(b) $\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\eta}}{(\log r)^{2\gamma}}\right)$ as $r \to \infty$, where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we obtain an analog of Theorem 1.1 for the Jacobi-Dunkl transform on the real line. For this purpose, we use a generalized translation operator. In this section, we recapitulate from [1, 2, 3, 5] some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$. The Jacobi-Dunkl function with parameters (α,β), $\alpha \geq \beta \geq \frac{-1}{2}$, and $\alpha \neq \frac{-1}{2}$, defined by the formula

$$\forall x \in \mathbb{R}, \quad \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda}\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) & , \text{if } \lambda \in \mathbb{C}\backslash\{0\}, \\ 1 & , \text{if } \lambda = 0. \end{cases}$$

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where $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$, F is the Gausse hypergeometric function (see [1,6]), and $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by

$$\varphi_\mu^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}; \frac{\rho - i\mu}{2}; \alpha + 1; -(\sinh(x))^2\right).$$

Then $\psi_\lambda^{\alpha,\beta}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}\mathcal{U} = i\lambda\mathcal{U} \quad & \lambda \in \mathbb{C}, \\ \mathcal{U}(0) = 1, \end{cases}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha + 1)\coth x + (2\beta + 1)\tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}.$$

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator D given by

$$D\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'(x)}{A(x)} \times \left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right),$$

where $A(x) = |x|^{2\alpha+1}B(x)$, and B a function of class C^∞ on $\mathbb{R}$, even and positive. The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A(x) = A_{\alpha,\beta}(x) = 2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2 + \rho^2}{4(\alpha + 1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form above (see [2])

$$\psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{4(\alpha + 1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x), \quad x \in \mathbb{R}.$$

Denote $L^p_{\alpha,\beta}(\mathbb{R}) = L^p_{\alpha,\beta}(\mathbb{R}, A_{\alpha,\beta}(x)dx)$, $1 < p \le 2$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha,\beta} = \left(\int_{\mathbb{R}} |f(x)|^p A_{\alpha,\beta}(x)dx\right)^{1/p} < +\infty.$$

Using the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform of a function $f \in L^p_{\alpha,\beta}(\mathbb{R})$ by

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = \int_{\mathbb{R}} f(x)\psi_\lambda^{\alpha,\beta}(x)A_{\alpha,\beta}(x)dx, \quad \lambda \in \mathbb{R},$$

and the inversion formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}f(\lambda)\psi_{-\lambda}^{\alpha,\beta}(x)d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|} 1_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}, \quad \mu \in \mathbb{C}\backslash(i\mathbb{N}),$$

and $1_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$. The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2(\mathbb{R}, d\sigma(\lambda))$, i.e.,

$$\|f\|_{2,\alpha,\beta} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2(\mathbb{R},d\sigma(\lambda))}. \tag{1}$$

Plancherel's theorem (1) and the Marcinkiewics interpolation theorem (see [8]) we get for $f \in L^p_{\alpha,\beta}(\mathbb{R})$ with $1 < p \leq 2$ and q such that $\frac{1}{p}+\frac{1}{q}=1$,

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^q(\mathbb{R},d\sigma(\lambda))} \leq K\|f\|_{p,\alpha,\beta}, \tag{2}$$

where K is a positive constant (see [9]).

The operator of Jacobi-Dunkl translation is defined by

$$T_x f(y) = \int_{\mathbb{R}} f(z)d\nu^{\alpha,\beta}_{x,y}(z), \quad \forall x, y \in \mathbb{R},$$

where $\nu^{\alpha,\beta}_{x,y}(z), \quad x, y \in \mathbb{R}$ are the signed measures given by

$$d\nu^{\alpha,\beta}_{x,y}(z) = \begin{cases} K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{, if } x,y \in \mathbb{R}^*, \\ \delta_x & \text{, if } y = 0, \\ \delta_y & \text{, if } x = 0. \end{cases}$$

Here, δ_x is the Dirac measure at x. And,

$$\begin{aligned} K_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh|x|\sinh|y|\sinh|z|)^{-2\alpha}1_{I_{x,y}} \times \int_0^\pi \rho_\theta(x,y,z) \\ &\quad \times (g_\theta(x,y,z))_+^{\alpha-\beta-1}\sin^{2\beta}\theta d\theta, \end{aligned}$$

$$I_{x,y} = [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|],$$

$$\rho_\theta(x,y,z) = 1 - \sigma^\theta_{x,y,z} + \sigma^\theta_{z,x,y} + \sigma^\theta_{z,y,x},$$

$$\forall z \in \mathbb{R}, \theta \in [0,\pi], \sigma^\theta_{x,y,z} = \begin{cases} \frac{\cosh x + \cosh y - \cosh z \cos\theta}{\sinh x \sinh y} & \text{, if } xy \neq 0, \\ 0 & \text{, if } xy = 0, \end{cases}$$

$$g_\theta(x,y,z) = 1 - \cosh^2 x - \cosh^2 y - \cosh^2 z + 2\cosh x \cosh y \cosh z \cos\theta,$$

$$t_+ = \begin{cases} t & \text{, if } t > 0, \\ 0 & \text{, if } t \leq 0, \end{cases}$$

and

$$M_{\alpha,\beta} = \begin{cases} \frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & , \text{ if } \alpha > \beta, \\ 0 & , \text{ if } \alpha = \beta. \end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_h f)(\lambda) = \psi_\lambda^{\alpha,\beta}(h)\mathcal{F}_{\alpha,\beta}(f)(\lambda) \text{ and } \mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta} f)(\lambda) = i\lambda\mathcal{F}_{\alpha,\beta}(f)(\lambda). \quad (3)$$

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind defined by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n(\frac{x}{2})^{2n}}{n!\Gamma(n+\alpha+1)}, \quad x \in \mathbb{R}.$$

Moreover, we see that

$$\lim_{x\to 0} \frac{j_\alpha(x)-1}{x^2} \neq 0, \quad (4)$$

by consequence, there exists $C_1 > 0$ and $\nu > 0$ satisfying

$$|j_\alpha(x) - 1| \geq C_1|x|^2 \text{ for } |x| \leq \nu. \quad (5)$$

By Lemma 9 in [4], we obtain the following Lemma 1.2.

LEMMA 1.2. Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$. Then for $|v| \leq \rho$, there exists a positive constant C_2 such that

$$\left|1 - \varphi_{\mu+i\nu}^{\alpha,\beta}(x)\right| \geq C_2|1 - j_\alpha(\mu x)|.$$

Denote by $L_p^m(\Lambda_{\alpha,\beta}), 1 < p \leq 2, m = 0, 1, 2, ...,$ the class of functions $f \in L_{\alpha,\beta}^p(\mathbb{R})$ that have generalized derivatives satisfying $\Lambda_{\alpha,\beta}^m f \in L_{\alpha,\beta}^p(\mathbb{R})$, i.e.,

$$L_p^m(\Lambda_{\alpha,\beta}) = \left\{f \in L_{\alpha,\beta}^p(\mathbb{R}) : \Lambda_{\alpha,\beta}^m f \in L_{\alpha,\beta}^p(\mathbb{R})\right\},$$

where $\Lambda_{\alpha,\beta}^0 f = f$ and $\Lambda_{\alpha,\beta}^m f = \Lambda_{\alpha,\beta}(\Lambda_{\alpha,\beta}^{m-1} f)$ for $m = 1, 2, ...$

2 Dini Lipschitz Condition

Denote N_h by

$$N_h = T_h + T_{-h} - 2I,$$

where I is the unit operator in the space $L_{\alpha,\beta}^p(\mathbb{R})$.

DEFINITION 2.1. Let $f \in L_p^m(\Lambda_{\alpha,\beta})$, and define

$$\|N_h\Lambda_{\alpha,\beta}^m f(x)\|_{p,\alpha,\beta} \leq C\frac{h^\eta}{(\log\frac{1}{h})^\gamma}, \quad \gamma \geq 0 \text{ and } m = 0, 1, 2, ...,$$

i.e.,

$$\|N_h\Lambda^m_{\alpha,\beta}f(x)\|_{p,\alpha,\beta} = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\gamma}\right),$$

for all x in $\mathbb{R}$ and for all sufficiently small h, C being a positive constant. Then we say that f satisfies a Jacobi-Dunkl Dini Lipschitz of order η, or f belongs to $Lip(\eta,\gamma,p)$.

DEFINITION 2.2. If

$$\frac{\|N_h\Lambda^m_{\alpha,\beta}f(x)\|_{p,\alpha,\beta}}{\frac{h^\eta}{(\log\frac{1}{h})^\gamma}} \to 0, \quad \text{as} \quad h \to 0,$$

i.e.,

$$\|N_h\Lambda^m_{\alpha,\beta}f(x)\|_{p,\alpha,\beta} = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\gamma}\right),$$

then f is said to belong to the little Jacobi-Dunkl Dini Lipschitz class $lip(\eta,\gamma,p)$.

REMARK**.** It follows immediately from these definitions that

$$lip(\eta,\gamma,p) \subset Lip(\eta,\gamma,p).$$

THEOREM 2.3. Let $\eta > 1$. If $f \in Lip(\eta,\gamma,p)$, then $f \in lip(1,\gamma,p)$.

PROOF. For $x \in \mathbb{R}$, h small and $f \in Lip(\eta,\gamma,p)$, we have

$$\|N_h\Lambda^m_{\alpha,\beta}f(x)\|_{p,\alpha,\beta} \le C\frac{h^\eta}{(\log\frac{1}{h})^\gamma}.$$

Then

$$(\log\frac{1}{h})^\gamma\|N_h\Lambda^m_{\alpha,\beta}f(x)\|_{p,\alpha,\beta} \le Ch^\eta.$$

Therefore,

$$\frac{(\log\frac{1}{h})^\gamma}{h}\|N_h\Lambda^m_{\alpha,\beta}f(x)\|_{p,\alpha,\beta} \le Ch^{\eta-1},$$

which tends to zero with $h \to 0$. Thus

$$\frac{(\log\frac{1}{h})^\gamma}{h}\|N_h\Lambda^m_{\alpha,\beta}f(x)\|_{p,\alpha,\beta} \to 0, \quad h \to 0.$$

Then $f \in lip(1,\gamma,p)$.

THEOREM 2.4**.** If $\eta < \nu$, then $Lip(\eta,0,p) \supset Lip(\nu,0,p)$ and $lip(\eta,0,p) \supset lip(\nu,0,p)$.

PROOF. We have $0 \le h \le 1$ and $\eta < \nu$, then $h^\nu \le h^\eta$. Then the proof of the theorem is immediate.

3 New Results on Dini Lipschitz Class

LEMMA 3.1. For $f \in L_p^m(\Lambda_{\alpha,\beta})$, then

$$\left(\int_{\mathbb{R}} 2^q \lambda^{qm} |\varphi_\mu^{\alpha,\beta}(h) - 1|^q |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{1}{q}} \leq K \|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, ...$

PROOF. From (3), we have

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}^m f)(\lambda) = i^m \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad m = 0, 1, 2, ... \tag{6}$$

By using formulas (3) and (6), we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda_{\alpha,\beta}^m f)(\lambda) = i^m (\psi_\lambda^{(\alpha,\beta)}(h) + \psi_\lambda^{(\alpha,\beta)}(-h) - 2) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

Since

$$\psi_\lambda^{(\alpha,\beta)}(h) = \varphi_\mu^{\alpha,\beta}(h) + i \frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi_\mu^{\alpha+1,\beta+1}(h),$$

$$\psi_\lambda^{(\alpha,\beta)}(-h) = \varphi_\mu^{\alpha,\beta}(-h) - i \frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi_\mu^{\alpha+1,\beta+1}(-h),$$

and $\varphi_\mu^{\alpha,\beta}$ is even, we see that

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda_{\alpha,\beta}^m f)(\lambda) = 2i^m (\varphi_\mu^{\alpha,\beta}(h) - 1) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

Now by formula (2), we have the result.

THEOREM 3.2. Let $\eta > 2$. If f belongs to the Jacobi-Dunkl Dini Lipschitz class, i.e.,

$$f \in Lip(\eta, \gamma, p), \quad \eta > 2, \gamma \geq 0.$$

Then f is null almost everywhere on $\mathbb{R}$.

PROOF. Assume that $f \in Lip(\eta, \gamma, p)$. Then we have

$$\|N_h \Lambda_{\alpha,\beta}^m f(x)\|_{p,\alpha,\beta} \leq C \frac{h^\eta}{(\log \frac{1}{h})^\gamma}, \quad \gamma \geq 0.$$

From Lemma 3.1, we get

$$\int_{\mathbb{R}} \lambda^{qm} |\varphi_\mu^{\alpha,\beta}(h) - 1|^q |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) \leq \frac{K^q C^q}{2^q} \frac{h^{q\eta}}{(\log \frac{1}{h})^{q\gamma}}.$$

In view of Lemma 1.3, we conclude that

$$\int_{\mathbb{R}} |1 - j_\alpha(\mu h)|^q \lambda^{qm} |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) \leq \frac{K^q C^q}{2^q C_2^q} \frac{h^{q\eta}}{(\log \frac{1}{h})^{q\gamma}}.$$

Then
$$\frac{\int_{\mathbb{R}}|1-j_\alpha(\mu h)|^q\lambda^{qm}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)}{h^{2q}} \leq \frac{K^qC^q}{2^qC_2^q}\frac{h^{q\eta-2q}}{(\log\frac{1}{h})^{q\gamma}}.$$

Since $\eta > 2$, we have
$$\lim_{h\to 0}\frac{h^{q\eta-2q}}{(\log\frac{1}{h})^{q\gamma}} = 0.$$

Thus
$$\lim_{h\to 0}\int_{\mathbb{R}}\left(\frac{|1-j_\alpha(\mu h)|}{\mu^2h^2}\right)^q\mu^{2q}\lambda^{qm}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) = 0,$$

and also from the formula (4) and Fatou theorem, we obtain
$$\int_{\mathbb{R}}\mu^{2q}\lambda^{qm}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) = 0.$$

Hence $\mu^2\lambda^m\mathcal{F}_{\alpha,\beta}f(\lambda) = 0$ for all $\lambda\in\mathbb{R}$, and so $f(x)$ is the null function.

Analog of the Theorem 3.2, we obtain this theorem.

THEOREM 3.3**.** Let $f\in L_p^m(\Lambda_{\alpha,\beta})$. If f belong to $lip(2,0,p)$, i.e.,
$$\|N_h\Lambda_{\alpha,\beta}^m f(x)\|_{p,\alpha,\beta} = O(h^2), \quad \text{as} \quad h\to 0.$$
Then f is null almost everywhere on $\mathbb{R}$.

THEOREM 3.4**.** Let f belong to $Lip(\eta,\gamma,p)$. Then
$$\int_{|\lambda|\geq r}\lambda^{qm}|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) = O\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}}\right), \quad \text{as} \quad r\to\infty,$$
where $\frac{1}{p}+\frac{1}{q} = 1$ and $m = 0,1,2,....$

PROOF**.** Let $f\in Lip(\eta,\gamma,p)$. Then
$$\|N_h\Lambda_{\alpha,\beta}^m f(x)\|_{p,\alpha,\beta} = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\gamma}\right), \quad \text{as} \quad h\to 0.$$

From Lemma 3.1, we have
$$\int_{\mathbb{R}}\lambda^{qm}|\varphi_\mu^{\alpha,\beta}(h)-1|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) \leq \frac{K^q}{2^q}\|N_h\Lambda_{\alpha,\beta}^m f(x)\|_{p,\alpha,\beta}^q.$$

By (5) and Lemma 1.2, we get
$$\begin{aligned}&\int_{\frac{\nu}{2h}\leq|\lambda|\leq\frac{\nu}{h}}\lambda^{qm}|1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda)\\ \geq\ & C_1^qC_2^q\int_{\frac{\nu}{2h}\leq|\lambda|\leq\frac{\nu}{h}}\lambda^{qm}|\mu h|^{2q}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).\end{aligned}$$

From $\frac{\nu}{2h} \leq |\lambda| \leq \frac{\nu}{h}$, we have

$$\left(\frac{\nu}{2h}\right)^2 - \rho^2 \leq \mu^2 \leq \left(\frac{\nu}{h}\right)^2 - \rho^2,$$

which implies that

$$\mu^2 h^2 \geq \frac{\nu^2}{4} - \rho^2 h^2.$$

Take $h \leq \frac{\nu}{3\rho}$, then we have $\mu^2 h^2 \geq C_3 = C_3(\nu)$. So

$$\begin{aligned} & \int_{\frac{\nu}{2h} \leq |\lambda| \leq \frac{\nu}{h}} \lambda^{qm} |1 - \varphi_\mu^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ \geq \quad & C_1^q C_2^q C_3^q \int_{\frac{\nu}{2h} \leq |\lambda| \leq \frac{\nu}{h}} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda). \end{aligned}$$

Note that there exists then a positive constant C_4 such that

$$\begin{aligned} \int_{\frac{\nu}{2h} \leq |\lambda| \leq \frac{\nu}{h}} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) & \leq C_4 \int_{\mathbb{R}} \lambda^{qm} |1 - \varphi_\mu^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ & = O\left(\frac{h^{q\eta}}{(\log \frac{1}{h})^{q\gamma}}\right). \end{aligned}$$

So we obtain

$$\int_{r \leq |\lambda| \leq 2r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \leq C_5 \frac{r^{-q\eta}}{(\log r)^{q\gamma}},$$

where C_5 is a positive constant. Now, we have

$$\begin{aligned} \int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) & = \sum_{i=0}^{\infty} \int_{2^i r \leq |\lambda| \leq 2^{i+1} r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ & \leq C_5 \left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(2r)^{-q\eta}}{(\log 2r)^{q\gamma}} + \frac{(4r)^{-q\eta}}{(\log 4r)^{q\gamma}} + \cdots \right) \\ & \leq C_5 \left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(2r)^{-q\eta}}{(\log r)^{q\gamma}} + \frac{(4r)^{-q\eta}}{(\log r)^{q\gamma}} + \cdots \right) \\ & \leq C_5 \frac{r^{-q\eta}}{(\log r)^{q\gamma}} \left(1 + 2^{-q\eta} + (2^{-q\eta})^2 + (2^{-q\eta})^3 + \cdots \right) \\ & \leq K_\eta \frac{r^{-q\eta}}{(\log r)^{q\gamma}}, \end{aligned}$$

where $K_\eta = C_5(1 - 2^{-q\eta})^{-1}$ since $2^{-q\eta} < 1$. Consequently

$$\int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) = O\left(\frac{r^{-q\eta}}{(\log r)^{q\gamma}}\right), \quad \text{as} \quad r \to \infty.$$

Thus, the proof is finished.

COROLLARY 3.5. Let $f \in L_p^m(\Lambda_{\alpha,\beta})$. If

$$\|N_h \Lambda^m_{\alpha,\beta} f\|_{p,\alpha,\beta} = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\gamma}\right), \quad \text{as} \quad h \to 0,$$

then

$$\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O\left(\frac{r^{-qm-q\eta}}{(\log r)^{q\gamma}}\right), \quad \text{as} \quad r \to \infty.$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, ...$

DEFINITION 3.6. A function $f \in L_p^m(\Lambda_{\alpha,\beta})$ is said to be in the (ψ, γ)-Jacobi-Dunkl Dini Lipschitz class, denoted by $Lip(\psi, \gamma, p)$, if

$$\|N_h \Lambda^m_{\alpha,\beta} f(x)\|_{p,\alpha,\beta} = O\left(\frac{\psi(h)}{(\log\frac{1}{h})^\gamma}\right), \quad \text{as} \quad h \to 0, \gamma \geq 0,$$

where

(1) $\psi(t)$ a continuous increasing function on $[0, \infty)$,

(2) $\psi(0) = 0$,

(3) $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$.

THEOREM 3.7. Let f belong to $Lip(\psi, \gamma, p)$. Then

$$\int_{|\lambda|\geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) = O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right), \quad \text{as} \quad r \to \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, ...$

PROOF. Let $f \in Lip(\psi, \gamma, p)$. Then

$$\|N_h \Lambda^m_{\alpha,\beta} f(x)\|_{p,\alpha,\beta} = O\left(\frac{\psi(h)}{(\log\frac{1}{h})^\gamma}\right), \quad \text{as} \quad h \to 0.$$

From Lemma 3.1, we have

$$\int_{\mathbb{R}} \lambda^{qm} |\varphi_\mu^{\alpha,\beta}(h) - 1|^q |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) \leq \frac{K^q}{2^q} \|N_h \Lambda^m_{\alpha,\beta} f(x)\|^q_{p,\alpha,\beta}.$$

By (5) and Lemma 1.2, we get

$$\begin{aligned} &\int_{\frac{\nu}{2h}\leq|\lambda|\leq\frac{\nu}{h}} \lambda^{qm} |1 - \varphi_\mu^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ \geq\quad & C_1^q C_2^q \int_{\frac{\nu}{2h}\leq|\lambda|\leq\frac{\nu}{h}} |\mu h|^{2q} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda). \end{aligned}$$

From $\frac{\nu}{2h} \leq |\lambda| \leq \frac{\nu}{h}$ we have

$$\left(\frac{\nu}{2h}\right)^2 - \rho^2 \leq \mu^2 \leq \left(\frac{\nu}{h}\right)^2 - \rho^2,$$

which implies that

$$\mu^2 h^2 \geq \frac{\nu^2}{4} - \rho^2 h^2.$$

Take $h \leq \frac{\nu}{3\rho}$, then we have $\mu^2 h^2 \geq C_3 = C_3(\nu)$. So

$$\begin{aligned} &\int_{\frac{\nu}{2h} \leq |\lambda| \leq \frac{\nu}{h}} \lambda^{qm} |1 - \varphi_\mu^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ \geq \quad & C_1^q C_2^q C_3^q \int_{\frac{\nu}{2h} \leq |\lambda| \leq \frac{\nu}{h}} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda). \end{aligned}$$

Note that there exists then a positive constant C_4 such that

$$\begin{aligned} \int_{\frac{\nu}{2h} \leq |\lambda| \leq \frac{\nu}{h}} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \quad &\leq \quad C_4 \int_{\mathbb{R}} \lambda^{qm} |1 - \varphi_\mu^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ &= \quad O\left(\frac{(\psi(h))^q}{(\log \frac{1}{h})^{q\gamma}}\right) = O\left(\frac{\psi(h^q)}{(\log \frac{1}{h})^{q\gamma}}\right). \end{aligned}$$

So we obtain

$$\int_{r \leq |\lambda| \leq 2r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \leq C_6 \frac{\psi(r^{-q})}{(\log r)^{q\gamma}},$$

where C_6 is a positive constant. Now, we have

$$\begin{aligned} \int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \quad &= \quad \sum_{i=0}^{\infty} \int_{2^i r \leq |\lambda| \leq 2^{i+1} r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ &\leq \quad C_6 \left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}} + \frac{\psi((2r)^{-q})}{(\log 2r)^{q\gamma}} + \frac{\psi((4r)^{-q})}{(\log 4r)^{q\gamma}} + \cdots \right) \\ &\leq \quad C_6 \left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}} + \frac{\psi((2r)^{-q})}{(\log r)^{q\gamma}} + \frac{\psi((4r)^{-q})}{(\log r)^{q\gamma}} + \cdots \right) \\ &\leq \quad C_6 \frac{\psi(r^{-q})}{(\log r)^{q\gamma}} \left(1 + \psi(2^{-q}) + \psi^2(2^{-q}) + \psi^3(2^{-q}) + \cdots\right) \\ &\leq \quad C_\eta \frac{\psi(r^{-q})}{(\log r)^{q\gamma}}, \end{aligned}$$

where $C_\eta = C_6(1 - \psi(2^{-q}))^{-1}$ since $\psi(2^{-q}) < 1$. Consequently,

$$\int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) = O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right), \quad \text{as} \quad r \to \infty,$$

and this ends the proof.

COROLLARY 3.8. Let $f \in L_p^m(\Lambda_{\alpha,\beta})$. If

$$\|N_h\Lambda^m_{\alpha,\beta}f\|_{p,\alpha,\beta} = O\left(\frac{h^\eta}{(\log\frac{1}{h})^\gamma}\right), \quad \text{as} \quad h \to 0,$$

then

$$\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O\left(\frac{r^{-qm}\psi(r^{-q})}{(\log r)^{q\gamma}}\right), \quad \text{as} \quad r \to \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, ...$

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