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Energy Decay Result In A Quasilinear Parabolic System With Viscoelastic Term*

Mohamed Ferhat†, Ali Hakem‡

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Abstract

In this paper we consider a quasilinear parabolic system of the form:

$$A(t)|u_t|^{m-2}u_t - Lu + \int_0^t g(t-s)Lu(s)ds = 0$$

in a bounded domain Ω, $A(t)$ is a bounded and positive definite matrix, $\Omega \subset R^n (n \geq 1)$, initial data (u_0, u_1) are given functions belonging to suitable spaces and g a continuously differentiable decaying function. We use the lemma of Martinez to establish a general decay result. This improves the result obtained by Messaoudi and Tellab [3].

1 Introduction

In this paper we consider

$$A(t)|u_t|^{m-2}u_t - Lu + \int_0^t g(t-s)Lu(s)ds = 0, \qquad m > 2, \tag{1}$$

subjected to the following boundary conditions

$$u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0, \tag{2}$$

and initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{3}$$

where Ω is a bounded open subset of $R^n (n \geq 1)$, $A(t)$ is a bounded and positive definite matrix,

$$Lu = -div(M\nabla u) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_i}\right).$$

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†Department of Mathematics, USTO University, 31000 Oran, Algeria
‡Laboratory ACEDP, Djillali Liabes University, 22000 Sidi Bel Abbes, Algeria

The matrix $M = (a_{i,j}(x))$, where $a_{i,j} \in C^1(\overline{\Omega})$, is symmetric and there exists a constant $a_0 > 0$ such that for all $x \in \overline{\Omega}$ and $\zeta = (\zeta_1, \zeta_2, ..., \zeta_N) \in R^N$, we have $\sum_{i,j=1}^{N} a_{i,j}(x)\zeta_j\zeta_i \geq a_0|\zeta|^2$. Also,

$$a(u(t), \upsilon(t)) = \sum_{i,j=1}^{N} a_{i,j}(x)\frac{\partial u(t)}{\partial x_j}\frac{\partial \upsilon(t)}{\partial x_i}dx = \int_{\Omega} M\nabla u(t).\nabla \upsilon(t)dx,$$

and

$$a_1 = \max\left(\sum_{i=1}^{N} \|a_{i,j}\|_{\infty}^2\right).$$

The values of u are taken in R^n and $A \in C(R^+)$ is a bounded square matrix satisfying

$$c_0|\upsilon|^2 \leq (A(t)\upsilon, \upsilon) \leq c_1|\upsilon|^2, \ \forall t \in R^+, \ \upsilon \in R^n. \tag{4}$$

The initial data (u_0, u_1) are given functions belonging to suitable spaces and g a continuously differentiable decaying function. To motivate our work, let us recall some results regarding quasilinear parabolic system. This type of equation arises from a variety of mathematical models in engineering and physical sciences. For example, in the study of a heat conduction in materials with memory, the classical Fourier's law is replaced by the following form (cf. [7]):

$$q = -d\nabla u - \int_{-\infty}^{t} k(x,t)u(x,s)ds,$$

where u is the temperature, d is the diffusion coefficient and the integral term represents the memory effect in the material. The study of this type of equations has drawn a considerable attention and many results have been obtained, see ([2, 3, 6, 9]). From a mathematical point of view one would expect that the integral term should be dominated by the leading term in the equation. Messaoudi and Tellab [3] studied the following system

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0,$$

with the same conditions in (2)–(3) and obtained an energy decay result although the memory term makes more complex situation. Berrimi and Messaoudi [6] showed that if A satisfies

$$(A(t)\upsilon, \upsilon) \geq c_0|\upsilon|^2, \quad \forall t \in R^+, \upsilon \in R^n$$

then the solutions with small initial energy decay exponentially for $m = 2$ and polynomially if $m > 2$. Very recently for a framework of blow-up in finite time Liu and Chen [2] studied the following system

$$A(t)|u_t|^{m-2}u_t - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = |u|^{p-2}u,$$

with the same conditions in (2)–(3) and proved a blow-up result for both positive and negative initial energy under suitable conditions on g and p. Motivated by the previous works, in the present paper we investigate problem (1) in which we generalize the results obtained in [3], supposing new conditions with which the stability is assured, by using the lemma of Martinez.

Our work is organized as follows. In section 2, we present some preliminaries and some lemmas. In section 3, the decay property is derived. Our result improves the one in Messaoudi and Tellab [3].

2 Preliminary Results

In this section, we present some material needed for the proof of our main result. For the relaxation function g we assume

(A_0) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)ds = l < 1,$$

and there exists a nonincreasing differentiable function $\xi : R^+ \to R^+$ satisfying

$$g'(t) \leq -\xi(t)g(t),\ t \geq 0; \quad \int_0^{+\infty} \xi(s)ds = +\infty.$$

(A_1) We also assume that

$$2 \leq m \leq \frac{2n}{n-2} \quad \text{if} \quad n \geq 3; \quad m \geq 2, \quad \text{if} \quad n = 1, 2.$$

REMARK 1. The same as in [3], there are many functions satisfying (A_0). Examples of such functions are

$$g_1(t) = e^{-(t+1)^\alpha}, \ \ 0 < \alpha \leq 1 \quad \text{and} \quad g_2(t) = (1+t)^\epsilon, \ \ \epsilon < -1.$$

We will also be using the embedding

$$H_0^1(\Omega) \hookrightarrow L^p(\Omega), \quad H_0^1(\Omega) \hookrightarrow L^m(\Omega).$$

LEMMA 1 ([2]). Let $E : R_+ \to R_+$ be a nonincreasing function and $\psi : R_+ \to R_+$ be a C^2 increasing function with $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = +\infty$. Assume that there exists $c > 0$ for which

$$\int_S^T E(t)\psi'(t)dt \leq cE(S),\ \forall S \geq 0.$$

Then

$$E(t) \leq \alpha E(0) e^{-\left(\int_0^t \xi(s)ds\right)},\ \forall t \geq 0,$$

where α and ω are positive constants.

LEMMA 2 (Sobolev-Poincaré's inequality). Let $2 \leq m \leq \frac{2n}{n-2}$. The inequality

$$\|u\|_m \leq c_s \|\nabla u\|_2 \quad \text{for} \quad u \in H_0^1(\Omega)$$

holds with some positive constant c_s.

LEMMA 3 ([3]). For $u(.,t) \in H_0^1(\Omega)$, we have

$$\int_\Omega \left(\int_0^t g(t-s)(u(x,t) - u(x,s))ds \right)^2 dx \leq (1-l)\frac{{c_s}^2}{a_0}(g \circ u)(t),$$

where c_s is the Poincaré constant and l is given in (A_1), and

$$(g \circ u)(t) = \int_0^t g(t-s) \int_\Omega a(u(x,t) - u(x,s), u(x,t) - u(x,s))dxds.$$

3 Asymptotic Behavior

In this section, we consider the energy decay of solutions associated to the system (1)–(3). Similarly as in [7] we give a definition of a weak solution of the system (1)–(3).

DEFINITION 1. A weak solution of (1)–(3) is a function $u(x,t)$ such that

$$u(x,t) \in C\left([0,T); [H_0^1(\Omega)]^n\right) \cap C^1\left((0,T); [L^m(\Omega)]^n\right),$$

which satisfies

$$\int_0^t \int_\Omega A(s)|u_s(s)|^{m-2} u_s(x,s)\phi(x,s)dsdx + \int_0^t \int_\Omega Lu(x,s)\phi(x,s)dsdx$$
$$+ \int_0^s \int_0^t \int_\Omega g(t-\nu)\phi(x,s)Lu(x,\nu)d\nu dxds = 0,$$

for all $t \in [0,T]$ and $\phi \in C\left([0,T); [H_0^1(\Omega)]^n\right)$.

REMARK 2. Similar to ([3, 9]), we assume the existence of a solution. For the linear case ($m = 2$), one can easily establish the existence of a weak solution by the Galerkin method. In the one-dimensional case ($n = 1$), the existence is established, in a more general setting, by Yin [9].

Now we define the "modified" energy equation related with problem (1)–(3) by

$$E(t) = \frac{1}{2}(g \circ u)(t) + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right) a(u(t), u(t)).$$

LEMMA 4. Let $u(x,t)$ be the solution of (1)–(3). Then the energy equation satisfies

$$E'(t) \leq \frac{1}{2}(g' \circ u)(t) - \frac{1}{2}g(t)a(u(t),u(t)) - \int_\Omega A(t)|u_t(t)|^m dx.$$

PROOF. By multiplying (1) by $u_t(t)$, and integrating over Ω we get

$$\int_\Omega A(t)|u_t(t)|^m dx - \frac{1}{2}\frac{d}{dt}a(u(t),u(t)) + \int_\Omega \int_0^t g(t-s)M\nabla u(s)\nabla u_t(t)dsdx = 0. \quad (5)$$

Note that

$$a(u(t),u_t(t)) = \frac{1}{2}\frac{d}{dt}a(u(t),u(t)),$$

following the ideas of [10], we can obtain

$$\begin{aligned}
& \int_0^t g(t-s)\int_\Omega M\nabla u(s)\nabla u_t(t)dxds \\
= \;& \sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\frac{\partial u(s)}{\partial x_j}\frac{\partial u_t(t)}{\partial x_i}dxds \\
= \;& \sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\frac{\partial u(t)}{\partial x_i}\frac{\partial u_t(t)}{\partial x_i}dxds \\
& -\sum_{i,j=1}^N \int_0^t \int_\Omega g(t-s)a_{i,j}(x)\left(\frac{\partial u(t)}{\partial x_i} - \frac{\partial u_t(s)}{\partial x_j}\right)\frac{\partial u_t(t)}{\partial x_i}dxds \\
= \;& \frac{1}{2}\int_0^t g(t-s)\left(\frac{d}{dt}a(u(t),u(t))\right)ds \\
& -\frac{1}{2}\int_0^t g(t-s)\left(\frac{d}{dt}a(u(t)-u(s),u(t)-u(s))\right)ds \\
= \;& \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s)a(u(t),u(t)\right)ds \\
& -\frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t-s)a(u(t)-u(s),u(t)-u(s))\right)ds \\
& -\frac{1}{2}g(t)a(u(t),u(t)) + \frac{1}{2}\int_0^t g'(t-s)a(u(t)-u(s),u(t)-u(s))ds \\
= \;& -\frac{1}{2}\frac{d}{dt}(g\circ u)(t) + \frac{1}{2}(g'\circ u)(t) + \frac{1}{2}\frac{d}{dt}\left[a(u(t),u(t))\int_0^t g(s)ds\right] \\
& -\frac{1}{2}g(t)a(u(t),u(t)), \qquad (6)
\end{aligned}$$

where

$$(g\circ u)(t) = \int_0^t g(t-s)a(u(x,t)-u(x,s),u(x,t)-u(x,s))ds.$$

By (5) and (6), we obtain

$$E'(t) \leq -\frac{1}{2}(g' \circ u)(t) - \frac{1}{2}g(t)a(u(t),u(t)) - \int_\Omega A(t)|u_t(t)|^m dx \leq 0.$$

THEOREM 1. Let $(u_0, u_1) \in (H_0^1(\Omega))^2$ be given. Suppose that (A_0)–(A_1) and (4) hold. Then there exist two positive constants w and K, depending on the initial data and c_0 for which the solution of (1)–(3) satisfies

$$E(t) \leq Ke^{-w\int_0^t \xi(s)ds}.$$

PROOF. From now on, we denote by c_i various positive constants which may be different at different occurrences. We multiply the equation (1) by $\xi(t)u$, integrate over $\Omega \times (S,T)$, and use the boundary conditions to get

$$\int_S^T \int_\Omega \xi(t)A(t)|u_t(t)|^{m-2}u_t(t)u(t)dxdt - \int_S^T \xi(t)a(u(t),u(t))dtdx + \int_S^T \xi(t)\int_\Omega M\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx = 0. \tag{7}$$

We then estimate

$$\begin{aligned} & -\int_\Omega \xi(t).M\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx \\ = & \int_\Omega \xi(t).M\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))\nabla u(t)dsdx \\ & -\int_0^t g(s)ds.\xi(t)a(u(t),u(t)), \end{aligned} \tag{8}$$

by substituting (8) in (7) and adding the following term in (7)

$$\frac{1}{2}\int_S^T \xi(t)(g \circ u)(t) - \frac{1}{2}\int_S^T \xi(t)(g \circ u)(t). \tag{9}$$

So (7) becomes

$$\begin{aligned} \int_S^T \xi(t)E(t)dt = & -\int_S^T \int_\Omega \xi(t)A(t)|u_t(t)|^{m-2}u_t(t)u(t)dxdt \\ & -\int_S^T \int_\Omega \xi(t)M.\int_0^t g(t-s)(\nabla u(t)-\nabla u(s)).\nabla u(t)dsdxdt \\ & +\int_S^T \xi(t)(g \circ u)(t)dt. \end{aligned} \tag{10}$$

By Lemma 4, equation (4), the boundedness of $A(t)$ and condition (A_1), we see that, $\forall\beta>0$,

$$\begin{aligned}\int_\Omega A(t)|u_t(t)|^{m-2}u_t(t)u(t)dx &\leq \beta\int_\Omega |u(t)|^m dx + c_\beta\int_\Omega |u_t(t)|^m dx\\ &\leq \beta c_s^m\|\nabla u(t)\|^m + c_\beta\int_\Omega |u_t(t)|^m dx\\ &\leq \beta c_s^m\left(\frac{2E(0)}{l}\right)^{\frac{m-2}{2}}E(t)-\left(\frac{c_\beta}{c_0}\right)E'(t).\end{aligned}$$

Then

$$\begin{aligned}\int_S^T\int_\Omega A(t)|u_t(t)|^{m-2}u_t(t)u(t)dxdt &\leq \left(\beta c_s^m\left(\frac{2E(0)}{l}\right)^{\frac{m-2}{2}}\right)\int_S^T E(t)\xi(t)dt\\ &\quad -\left(\frac{c_\beta}{c_0}\right)\int_S^T E'(t)\xi(t)dt,\quad \forall\beta>0. \quad (11)\end{aligned}$$

We also have

$$\begin{aligned}&\int_\Omega\int_0^t Mg(t-s)(\nabla u(t)-\nabla u(s))\nabla u(t)dsdx\\ =\ & \sum_{i,j=1}^N\int_0^t g(t-s)\int_\Omega a_{ij}(x)\frac{\partial u(t)}{\partial x_j}\left(\frac{\partial u(t)}{\partial x_i}-\frac{\partial u(s)}{\partial x_i}\right)dxds\\ \leq\ & \mu\sum_{i,j=1}^N\int_\Omega\left(\int_0^t a_{ij}(x)\frac{\partial u(t)}{\partial x_j}ds\right)^2dx\\ &+\frac{1}{\mu}\sum_{i,j=1}^N\int_\Omega\left(\int_0^t g(t-s)\left(\frac{\partial u(s)}{\partial x_i}-\frac{\partial u(t)}{\partial x_i}\right)ds\right)^2dx\\ \leq\ & \frac{\mu}{a_0}\left(\max\sum_{i,j=1}^N\|a_{ij}\|_\infty^2\right)a(u(t),u(t))+\frac{N}{4a_0\mu}(1-l)(g\circ u)(t)\\ \leq\ & \frac{\mu}{a_0}\left(\sum_{i,j=1}^N\|a_{ij}\|_\infty^2\right)E(t)+\frac{N}{4a_0\mu}(1-l)(g\circ u)(t), \quad (12)\end{aligned}$$

and using the fact that

$$|\xi(t)(g\circ u)(t)| = \xi(t)(g\circ u)(t)\leq\beta(g\circ u)(t)\leq\beta(-E'(t)),$$

we obtain

$$\int_S^T\xi(t)(g\circ u)(t)dt\leq\int_S^T\beta(-E'(t))dt\leq CE(S). \quad (13)$$

By combining (10)–(13), we easily deduce the following

$$\begin{aligned}
&\left\{1-\beta c_s^m\left(\frac{2E(0)}{l}\right)^{\frac{m-2}{2}}-\left[(\alpha+1)\left(\max_{i\le i\le N}\sum_{i,j=1}^{N}\|a_{ij}\|_\infty^2\right)+\frac{N}{4a_0\mu}(1-l)\right]\right\}\\
&\times\int_S^T\xi(t)E(t)dt\\
\le\ &\left(\frac{c_\beta}{c_0}\xi(0)+e^\beta\right)E(S).
\end{aligned}$$

Finally, we get

$$\int_S^T\xi(t)E(t)dt\le CE(S),\quad \forall S\ge 0,$$

by choosing β, δ_2, ϵ small enough and by the hypothesis $l<1$. By letting T go to infinity, one can easily see that (A_0) is satisfied with $\psi(t)=\int_0^t\xi(s)ds$. This completes the proof.

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