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Explicit Approximation Of The Sums Over The Imaginary Part Of The Non-Trivial Zeros Of The Riemann Zeta Function*

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Abstract

Based on the recent improved upper bound for the argument of the Riemann zeta-function $\zeta(s)$ on the critical line, we obtain explicit sharp bounds for the sum $\sum_{0<\gamma\leqslant T}\gamma^{-1}$, where γ denote the imaginary part of the non-trivial zeros $\rho=\beta+i\gamma$ of $\zeta(s)$.

1 Introduction

The Riemann zeta-function is defined by $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ for $\Re(s)>1$, and extended by analytic continuation to the complex plan with a simple pole at $s=1$. We let

$$\gamma_1=\min\{\gamma>0:\zeta(\beta+i\gamma)=0\}\approxeq 14.1347251417,$$

which is the imaginary part of the first nontrivial zero of $\zeta(s)$. It is known [2] that

$$N(T):=\sum_{\substack{0<\gamma\leqslant T\\ \zeta(\beta+i\gamma)=0}}1=\frac{T}{2\pi}\log\frac{T}{2\pi\mathrm{e}}+O(\log T), \tag{1}$$

from which by integration we infer that

$$A(T):=\sum_{\substack{0<\gamma\leqslant T\\ \zeta(\beta+i\gamma)=0}}\frac{1}{\gamma}=K(T)+O(1),$$

where for the whole text we let

$$K(T)=\frac{1}{4\pi}\log^2 T-\frac{\log(2\pi)}{2\pi}\log T.$$

The aim of present note is to determine an interval for which the term $O(1)$ in the above approximation belongs in. To this end, we show the following.

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THEOREM 1. For each $T \geqslant \gamma_1$, we have $0.015 < A(T) - K(T) < 0.482$.

We note that in our previous paper [1] we have done similar computation, based on the approximation of $N(T)$ due to Rosser [3]. Unfortunately, we have misquoted in Rosser's result, taking 0.433 instead of 0.443 which occurs in the approximation of $N(T)$ given by him, and ends in the double side approximation $0.06 < A(T) - K(T) < 0.436$, so our previous calculation needs some corrections. The present paper is indeed such correction.

We give the proof of Theorem 1 in the next two sections, and then we give some computational remarks concerning the difference $A(T) - K(T)$ and we propose finding its limit value as $T \to \infty$.

2 Preliminaries

To get an explicit result as the above, we need an explicit form of the approximation (1). We note that the term $O(\log T)$ in (1) comes from the approximation of the function $S(T)$, which is defined traditionally by $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$, where the argument is determined via continuous variation along the line segments connecting 2, $2 + iT$ and $\frac{1}{2} + iT$, with taking the argument of $\zeta(s)$ at $s = 2$ to be zero. If T is an ordinate of a zero of $\zeta(s)$, then we set $S(T) = \frac{1}{2}(S(T^+) + S(T^-))$. Indeed, the approximation of the function $N(T)$ related strongly to the approximation of $S(T)$, by considering the known (see [4]) inequality

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leqslant |S(T)| + \mathcal{E}(T), \tag{2}$$

which holds for each $T \geqslant 1$ with

$$\mathcal{E}(T) = \frac{1}{4\pi} \arctan \frac{1}{2T} + \frac{T}{4\pi} \log \left(1 + \frac{1}{4T^2} \right) + \frac{1}{3\pi T}.$$

On the other hand, the best known unconditional approximations of $S(T)$ is due to Trudgian [4], where he shows for each $T \geqslant \mathrm{e}$ that

$$|S(T)| \leqslant 0.112 \log T + 0.278 \log \log T + 2.510. \tag{3}$$

By using the above bound, we imply the following approximation.

LEMMA 1. For each $T \geqslant \gamma_1$, we have $|N(T) - \mathcal{F}(T)| \leqslant \mathcal{R}(T)$ with

$$\mathcal{F}(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8}, \quad \text{and} \quad \mathcal{R}(T) = \frac{14}{125} \log T + \frac{139}{500} \log \log T + \frac{1261}{500}.$$

This approximation allows us to get bounds for the sum $\sum_{0 < \gamma \leqslant T} g(\gamma)$ where g is a suitable function, with the conditions as in the following result.

PROPOSITION 1. For the functions f and g and for the real numbers a and b, we define

$$\Im(a,b;g,f) := \int_a^b g(t)f'(t)dt + g(a)f(a).$$

If we assume that $g(t)$ is a positive, differentiable and decreasing function defined for $t \geqslant \gamma_1$, then for each $T \geqslant \gamma_1$, we have

$$\left| \sum_{0<\gamma\leqslant T} g(\gamma) - \Im(\gamma_1, T; g, \mathcal{F}) \right| \leqslant \Im(\gamma_1, T; g, \mathcal{R}),$$

where $\mathcal{F}$ and $\mathcal{R}$ are as in Lemma 1.

If we take $g(t) = \frac{1}{t}$, then by following some computations, we will obtain bounds as in Theorem 1. To perform computations we note that

$$\int \frac{\mathcal{F}'(t)}{t} dt = \frac{1}{4\pi} \log^2\left(\frac{t}{2\pi}\right) := \hat{F}(t). \tag{4}$$

Also we have

$$\int \frac{\mathcal{R}'(t)}{t} dt = -\frac{14}{125t} - \frac{139}{500} J(t) := \hat{R}(t), \tag{5}$$

where the function J is defined for each real $t > 1$ by

$$J(t) = \int_1^\infty \frac{dt}{st^s},$$

which is strictly decreasing and $J(t) \sim \frac{1}{t\log t}$ as $t \to \infty$. More precisely, we show the following.

LEMMA 2. For each real $t > 1$, we have

$$\frac{1}{t\log t} - \frac{1}{t\log^2 t} < J(t) < \frac{1}{t\log t} - \frac{1}{t\log^2 t} + \frac{2}{t\log^3 t}. \tag{6}$$

Moreover, for each $t > 12.7$, we have

$$\frac{1}{t\log t} - \frac{1}{t\log^2 t} + \frac{1}{t\log^3 t} < J(t) < \frac{1}{t\log t} - \frac{1}{t\log^2 t} + \frac{2}{t\log^3 t}. \tag{7}$$

As an immediate corollary, by considering the relation (5) together with the bounds (7), we obtain the following required bounds.

COROLLARY 1. For each $t > 12.7$, we have $\hat{R}_\ell(t) < \hat{R}(t) < \hat{R}_u(t)$, where

$$\hat{R}_\ell(t) = -\frac{14}{125t} - \frac{139}{500}\left(\frac{1}{t\log t} - \frac{1}{t\log^2 t} + \frac{2}{t\log^3 t}\right),$$

and

$$\hat{R}_u(t) = -\frac{14}{125t} - \frac{139}{500}\left(\frac{1}{t\log t} - \frac{1}{t\log^2 t} + \frac{1}{t\log^3 t}\right).$$

By applying the above preliminary results, deduction of Theorem 1 is based on a simple computation. We follow the computational details in the next section.

REMARK 1. Assume that the bounds in (3) were to improve dramatically, say to $|S(T)| \leqslant \delta_1 \log T$ for some fixed $\delta_1 > 0$ and for each $T \geqslant \gamma_1$. Proposition 1 implies that

$$\left|A(T) - \left(\hat{F}(T) - \hat{F}(\gamma_1) + \frac{F(\gamma_1)}{\gamma_1}\right)\right| \leqslant w(T) - w(\gamma_1) + \frac{\delta_1 \log \gamma_1 + \mathcal{E}(\gamma_1)}{\gamma_1},$$

where

$$w(t) := \int \frac{\frac{d}{dt}(\delta_1 \log t + \mathcal{E}(t))}{t} dt \sim \frac{\log 2}{\pi} - \frac{\delta_1}{t} + \frac{1}{\pi}\sum_{j=1}^{\infty} \frac{(-1)^{j-1} c_j}{t^{2j}} \qquad (\text{as } t \to \infty),$$

and c_js are positive absolute (more precisely independent from δ_1) rational constants, satisfying $c_j = o(1)$ as $j \to \infty$. Thus, by considering the relation $\hat{F}(t) = K(t) + \frac{1}{4\pi}\log^2(2\pi)$, we get

$$C_\ell(\delta_1, \gamma_1) - w(T) \leqslant A(T) - K(T) \leqslant C_u(\delta_1, \gamma_1) + w(T),$$

where $C_\ell(\delta_1, \gamma_1)$ and $C_u(\delta_1, \gamma_1)$ are constants depending on δ_1 and γ_1.

3 Proofs

In this section, we prove Lemmas 1, 2, Proposition 1, and Theorem 1.

PROOF of Lemma 1. We apply (2), and also (3) with the known [4] parameters $a = 0.112$, $b = 0.278$, $c = 2.510$, and $T_0 = \mathrm{e}$. Since the function $\mathcal{E}(T)$ is strictly decreasing for $T > 0$, we imply that $\mathcal{E}(T) \leqslant \mathcal{E}(\gamma_1) < 0.012$ for each $T \geqslant \gamma_1$, and hence $|S(T)| + \mathcal{E}(T) < \mathcal{R}(T)$.

PROOF of Proposition 1. For each smooth function g, we have

$$\sum_{0<\gamma\leqslant T} g(\gamma) = \int_{\gamma_1^-}^{T} g(t)dN(t) = g(T)N(T) + \int_{\gamma_1}^{T} N(t)\left(-g'(t)\right) dt. \tag{8}$$

If we assume that g is decreasing, then $g'(t) \geqslant 0$. For each smooth function f integration by parts implies that

$$\int_{\gamma_1}^{T} f(t)\left(-g'(t)\right) dt = -f(T)g(T) + f(\gamma_1)g(\gamma_1) + \int_{\gamma_1}^{T} g(t)f'(t)dt. \tag{9}$$

Hence, by applying the bound $N(T) \leqslant \mathcal{F}(T) + \mathcal{R}(T)$ in (8), and also by utilizing (9) with $f(t) = \mathcal{F}(t) + \mathcal{R}(t)$ we get validity of

$$\sum_{0<\gamma\leqslant T} g(\gamma) \leqslant \int_{\gamma_1}^{T} g(t)\left(\mathcal{F}'(t) + \mathcal{R}'(t)\right) dt + g(\gamma_1)\left(\mathcal{F}(\gamma_1) + \mathcal{R}(\gamma_1)\right), \qquad (10)$$

for each $T \geqslant \gamma_1$, and similarly, by applying the bound $N(T) \geqslant \mathcal{F}(T) - \mathcal{R}(T)$ in (8), and also by utilizing (9) with $f(t) = \mathcal{F}(t) - \mathcal{R}(t)$, for each $T \geqslant \gamma_1$ we obtain

$$\sum_{0<\gamma\leqslant T} g(\gamma) \geqslant \int_{\gamma_1}^{T} g(t)\left(\mathcal{F}'(t) - \mathcal{R}'(t)\right) dt + g(\gamma_1)\left(\mathcal{F}(\gamma_1) - \mathcal{R}(\gamma_1)\right). \qquad (11)$$

This completes the proof.

PROOF of Lemma 2. We have $\frac{d}{dt}J(t) = -\frac{1}{t^2 \log t}$, hence J is strictly decreasing. Also $\lim_{t\to\infty}(t\log t)J(t) = 1$. We set $J_0(t) = J(t)$, and for each $n \geqslant 1$ we let $J_n(t) = J_{n-1}(t) + \frac{(-1)^n(n-1)!}{t\log^n t}$. By summing over the difference $J_k(t) - J_{k-1}(t)$, we imply that

$$J_n(t) - J_0(t) = \sum_{k=1}^{n} \frac{(-1)^k(k-1)!}{t\log^k t},$$

and hence

$$J_n(t) = J(t) - \sum_{k=0}^{n-1} \frac{(-1)^k k!}{t\log^{k+1} t}.$$

The function $h(t) = (t\log^3 t)J_2(t)$ is strictly increasing for $t \in (1,\infty)$, and it admits limit values $\lim_{t\to 1^+} h(t) = 0$ and $\lim_{t\to\infty} h(t) = 2$. Hence we obtain validity of (6). Moreover, by considering the value $h(12.7) \approxeq 1.00017 > 1$ we get (7).

PROOF of Theorem 1. We apply the approximation (10) with $g(t) = \frac{1}{t}$, and then the upper bound in Corollary 1 to obtain

$$\begin{aligned} A(T) &\leqslant \hat{F}(T) - \hat{F}(\gamma_1) + \hat{R}(T) - \hat{R}(\gamma_1) + \frac{\mathcal{F}(\gamma_1)}{\gamma_1} + \frac{\mathcal{R}(\gamma_1)}{\gamma_1} \\ &\leqslant \hat{F}(T) - \hat{F}(\gamma_1) + \hat{R}_u(T) - \hat{R}_\ell(\gamma_1) + \frac{\mathcal{F}(\gamma_1)}{\gamma_1} + \frac{\mathcal{R}(\gamma_1)}{\gamma_1} := K(T) + U(T), \end{aligned}$$

say, for each $T \geqslant \gamma_1$. The function $U(T)$ is strictly increasing for $T \geqslant \gamma_1$ and also we have $\lim_{T\to\infty} U(T) < \frac{482}{1000}$. Hence $A(T) < K(T) + \frac{482}{1000}$ is valid for each $T \geqslant \gamma_1$.

By following similar argument as the above, we apply the approximation (11) with $g(t) = \frac{1}{t}$, and then the lower bound in Corollary 1 to obtain

$$\begin{aligned} A(T) &\geqslant \hat{F}(T) - \hat{F}(\gamma_1) - \hat{R}(T) + \hat{R}(\gamma_1) + \frac{\mathcal{F}(\gamma_1)}{\gamma_1} - \frac{\mathcal{R}(\gamma_1)}{\gamma_1} \\ &\geqslant \hat{F}(T) - \hat{F}(\gamma_1) - \hat{R}_u(T) + \hat{R}_\ell(\gamma_1) + \frac{\mathcal{F}(\gamma_1)}{\gamma_1} - \frac{\mathcal{R}(\gamma_1)}{\gamma_1} := K(T) + L(T), \end{aligned}$$

for each $T \geqslant \gamma_1$. The function $L(T)$ is strictly decreasing for $T \geqslant \gamma_1$ and also we have $\lim_{T\to\infty} L(T) > \frac{15}{1000}$. Hence $A(T) > K(T) + \frac{15}{1000}$ is valid for each $T \geqslant \gamma_1$. This completes the proof.

4 Computational Remarks

Regarding to the truth of Theorem 1, naturally we ask about the limit value

$$\lim_{T\to\infty} A(T) - K(T).$$

Does this limit exists? If yes, what is its value? Our computations suggests the above limit exists. To perform such computations, we define the sequence Δ_N with general term

$$\Delta_N = A(\gamma_N) - K(\gamma_N) = \sum_{n=1}^{N} \frac{1}{\gamma_n} - \left(\frac{1}{4\pi} \log^2 \gamma_N - \frac{\log(2\pi)}{2\pi} \log \gamma_N \right).$$

Figure 1 pictures the points (N, Δ_N) for $1 \leqslant N \leqslant 20000$, and Figure 2 shows the points (N, Δ_N) $9.9 \times 10^5 \leqslant N \leqslant 10^6$ and $1.99 \times 10^6 \leqslant N \leqslant 2 \times 10^6$. As these figures show, the values of Δ_N seems to tend toward a limit with approximate value 0.25163.

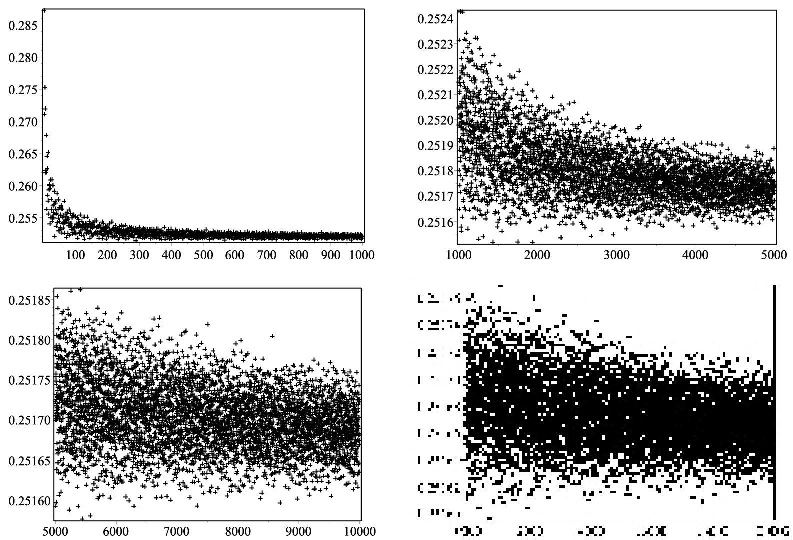

Figure 1: Graphs of the points (N, Δ_N) in several intervals from 1 to 20000, with end-points 1000, 5000, 10000, 20000.

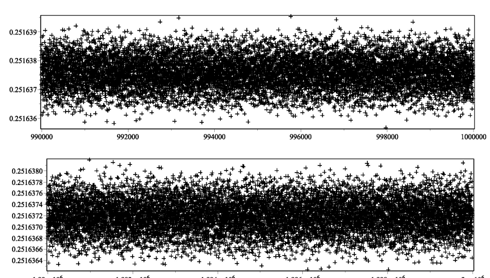

Figure 2: Graphs of the points (N, Δ_N) for $9.9 \times 10^5 \leqslant N \leqslant 10^6$ and $1.99 \times 10^6 \leqslant N \leqslant 2 \times 10^6$.

To avoid oscillation behavior of the values of the sequence Δ_N we define the modified sequence M_N by

$$M_N = \frac{1}{2000} \sum_{n=1+2000(N-1)}^{2000N} \Delta_n.$$

The values of M_N are indeed a clustering in averaging of the values of Δ_N in short intervals. Figure 3 shows the points (N, M_N) for $500 \leqslant N \leqslant 1000$. Also, Figure 4 shows the values of the difference $M_N - M_{N-1}$ for $500 \leqslant N \leqslant 1000$. As this figure shows, the sequence M_N is not decreasing.

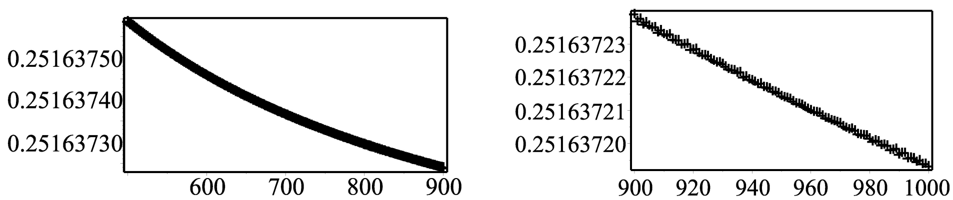

Figure 3: Graphs of the points (N, M_N) for $500 \leqslant N \leqslant 1000$.

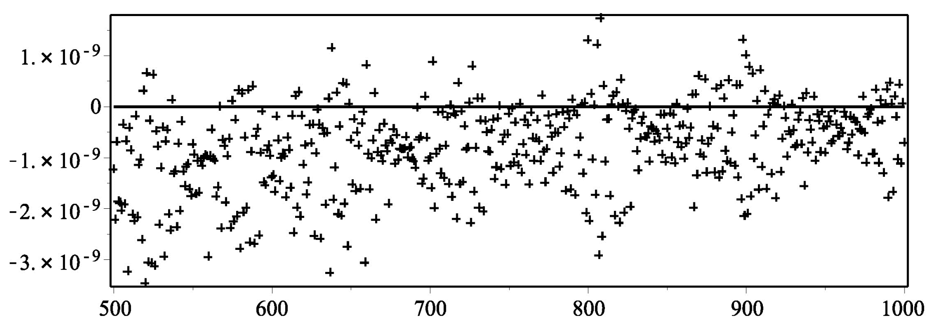

Figure 4: Graphs of the points $(N, M_N - M_{N-1})$ for $500 \leqslant N \leqslant 1000$.

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