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A Note On Strongly Quotient Graphs And Strongly Sum Difference Quotient Graphs*

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Abstract

A graph with n vertices is called a *strongly quotient graph* if its vertices can be labeled with $1, 2, \dots, n$, such that the quotient function $f_q = \min\{\frac{f(u)}{f(v)}, \frac{f(v)}{f(u)}\}$ is injective, where $f(v)$ is the label of vertex v. A graph with n vertices is called a *strongly sum difference quotient graph* if its vertices can be labeled with $1, 2, \dots, n$, such that the sum difference quotient function $f_{sdq} = \frac{|f(u)+f(v)|}{|f(u)-f(v)|}$ is injective, where $f(v)$ is the label of vertex v. In this paper, we show that a graph is a strongly quotient graph if and only if it is a strongly sum difference quotient graph, i.e., these two graph labelings are essentially the same.

1 Introduction

During the past fifty years, an enormous amount of research has been done on graph labeling. In paper [8], Gallian introduced the history of graph labeling and almost all concepts of graph labeling. Graphs with labeled edges are commonly used to model networks, with restrictions on the network represented as restrictions on the labels of edges. For instance, when modeling transportation networks, such labels can be used to represent a variety of factors, from cost to level of traffic flow. More generally, Ahuja, Magnati and Orlin [3] point out various applications in statistical physics, particle physics, computer science, biology, economics, operations research, and sociology. Interesting applications of graph labeling also can be found in Bloom and Golomb [6, 7]. In paper [5], Beineke and Hegde introduced the concept of strongly multiplicative graph. Motivated by this, Adiga and Zaferani [2] introduced the concept of strongly quotient graph. Later Adiga and Swamy [1] introduced the concept of strongly sum difference quotient graph.

The graphs considered in this paper are finite, undirected, connected and simple (no loops or multiple edges). The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively.

A *labeling* f of a graph G with n vertices means an injective mapping

$$f : V(G) \to \{1, 2, \dots, n\}.$$

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We define the quotient function

$$f_q : E(G) \to Q$$

by

$$f_q(e) = \min\left\{\frac{f(u)}{f(v)}, \frac{f(v)}{f(u)}\right\}$$

if e joints u and v. Note that $0 < f_q(e) < 1$ for any $e \in E(G)$. A graph with n vertices is called a *strongly quotient (SQ) graph* if its vertices can be labeled with $1, 2, \ldots, n$, such that the quotient function f_q is injective, i.e., the values $f_q(e)$ on the edges are all distinct.

Similarly we define the sum difference quotient function

$$f_{sdq} : E(G) \to Q$$

by

$$f_{sdq}(e) = \frac{|f(u) + f(v)|}{|f(u) - f(v)|}$$

if e joints u and v. Note that $f_{sdq}(e) > 1$ for any $e \in E(G)$. A graph with n vertices is called a *strongly sum difference quotient (SSDQ) graph* if its vertices can be labeled with $1, 2, \ldots, n$, such that the quotient function f_{sdq} is injective, i.e., the values $f_{sdq}(e)$ on the edges are all distinct.

In [10], Zaferani showed that some families of graphs such as ladder, triangular ladder, star, double star and fan are strongly quotient graphs. In paper [9], Shivakumar Swamy, Shrikanth and Sriraj showed that some families of graphs such as Mycielskian of the path and the cycle, the Cartesian product of the path and the cycle, double triangular snake graphs and total graph of the cycle are strongly sum difference quotient graphs. In [4], Akwu showed that one-point union of graphs which have SSDQ labelings are still strongly sum difference quotient graphs. Additionally, Akwu gave SSDQ labeling of the corona graph $C_n \odot mK_1$.

Up to now, we can only find SQ labeling or SSDQ labeling for special graph families. It seems that it is not easy to find SQ labeling or SSDQ labeling for complex graphs.

In this paper, we show that SQ labeling is equivalent to SSDQ labeling. Since SSDQ labeling is more complicated than SQ labeling, we may only focus on SQ labeling according to this result.

2 Main Results

THEOREM 1. A graph is a strongly quotient graph if and only if it is a strongly sum difference quotient graph.

PROOF. For sufficiency, we may assume that a graph G with n vertices is a strongly sum difference quotient graph. So there exists a SSDQ labeling for G. Then we may define two injective mappings as follows:

$$f : V(G) \to \{1, 2, \ldots, n\} \quad \text{and} \quad f_{sdq} : E(G) \to Q$$

by

$$f_{sdq}(e) = \frac{|f(u) + f(v)|}{|f(u) - f(v)|}$$

if e joints u and v.

Since f_{sdq} is an injective mapping, $f_{sdq}(e_1) \neq f_{sdq}(e_2)$ for any $e_1, e_2 \in E(G)$ and $e_1 \neq e_2$.

Based on the vertex labeling, we define the quotient function

$$f_q : E(G) \to Q$$

by

$$f_q(e) = \min\{\frac{f(u)}{f(v)}, \frac{f(v)}{f(u)}\}$$

if e joints u and v. We will prove f_q is injective as follows.

Now we consider the adjacency matrix $A(G)$ of G, in which the vertices of rows and columns are arrayed in labeling sequence. Since G is a simple graph, $A(G)$ is a $(0,1)$-matrix with the principal diagonal elements all zeros. Note that $A(G)$ is a symmetric matrix. As a result, we only need to consider the elements above the principal diagonal, in which each nonzero (i,j)-element $(i<j)$ represents an edge e_{ij} which joints labeled vertex i and j.

$$A(G) = \begin{array}{cc} & \begin{matrix} 1 & 2 & \cdots & \cdots & j_1 & \cdots & j_2 & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ i_1 \\ \vdots \\ i_2 \\ \vdots \\ n-1 \\ n \end{matrix} & \begin{pmatrix} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & (i_1, j_1) & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & (i_2, j_2) & & \\ & & & & & & \ddots & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{pmatrix} \end{array}.$$

We arbitrarily select two different nonzero elements (i_1, j_1) and (i_2, j_2) in $A(G)$ $(i_k < j_k, k = 1, 2)$. Then the corresponding edge $e_{i_1 j_1} \neq e_{i_2 j_2}$ and $f_{sdq}(e_{i_1 j_1}) \neq f_{sdq}(e_{i_2 j_2})$, where

$$f_{sdq}(e_{i_1 j_1}) = \frac{i_1 + j_1}{j_1 - i_1} \quad \text{and} \quad f_{sdq}(e_{i_2 j_2}) = \frac{i_2 + j_2}{j_2 - i_2}.$$

Since

$$\frac{i_1 + j_1}{j_1 - i_1} \neq \frac{i_2 + j_2}{j_2 - i_2},$$

we have

$$i_1 j_2 + j_1 j_2 - i_1 i_2 - i_2 j_1 \neq i_2 j_1 + j_1 j_2 - i_1 i_2 - i_1 j_2,$$

$$2 i_1 j_2 \neq 2 i_2 j_1, \quad i_1 j_2 \neq i_2 j_1, \quad \frac{i_1}{j_1} \neq \frac{i_2}{j_2}, \quad \text{and} \quad f_q(e_{i_1 j_1}) \neq f_q(e_{i_2 j_2}).$$

Therefore, f_q is injective. Thus the sufficiency holds.

Since each step of the above is reversible, the necessity holds too.

This completes the proof.

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