Applied Mathematics E-Notes, 16(2016), 45-55 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

A Note On "Periodic Solutions Of A Three-Species Food Chain Model" [Applied Math E-Notes, 9 (2009), 47-54] *

Rana Durga Parshad and Aladeen Al Basheer †

Received 21 April 2015

Abstract

The work of Zhang et al. [18] investigates the existence of periodic solutions in a ODE model, of a three species food chain, based on a modified Leslie-Gower scheme. They consider time dependent periodic coefficients to model periodicity of the natural environment. Their main result is that under certain restrictions on these coefficients, there exists at least one periodic solution to the three species model. In the current manuscript we *prove* that this result is not true. We then derive certain global existence conditions, which when enforced in conjunction with the earlier conditions of [18], yield at least one periodic solution to the model. We support all of our results via numerical simulations.

1 Introduction

Interactions of predator and prey species form the cornerstone of modern ecology. Therein a predator or a "hunting" organism, hunts down and attempts to kill its prey, in order to feed. The situation becomes even more interesting, if one considers three or more interacting species, instead of two. Such is the cases where there is both a specialist predator and a generalist predator, or perhaps two competing predators, for a single prey [3, 7, 12, 15]. In the context of ODE models, moving from two species to three species can bring about rich dynamic behavior such as chaos [3]. However, there is much discrepancy between chaotic dynamics seen in mathematical three species models, and actual observations in nature [15]. In [15] Upadhyay and Rai proposed a model to understand in particular, the reasons why chaos is rarely observed in natural populations of three interacting species. They model the top predator as a generalist, so it can change its food source, in the absence of its favorite food. The model and its variants have been intensely studied [1, 2, 5, 8, 9, 10, 11, 13, 14, 16]. All of these works consider constant coefficient models. Note, there is a fair amount of evidence, that mating rates, death rates and environmental protection rates in natural populations, vary seasonally [7]. Thus in [18] Zhang et al. considered a variation of the model in [1, 15], with time dependent, periodic coefficients, to mimic periodicity of the natural

*Mathematics Subject Classifications: 34C11, 34C25, 92D25, 92D40.

†Department of Mathematics, Clarkson University, Potsdam, New York 13699, USA

environment. The model they considered is described by the following system of equations, where x, y, z represent the numbers at any instant of time of the prey, middle predator and top predator respectively,

$$\frac{dx(t)}{dt} = a_1(t)x(t) - b_1(t)x(t)^2 - w_0(t)\left(\frac{x(t)y(t)}{x(t)+d_0}\right), \tag{1}$$

$$\frac{dy(t)}{dt} = -a_2(t)y(t) + w_1(t)\left(\frac{x(t)y(t)}{x(t)+d_1}\right) - w_2(t)\left(\frac{y(t)z(t)}{y(t)+d_2}\right), \tag{2}$$

$$\frac{dz(t)}{dt} = c_0(t)z(t)^2 - w_3(t)\frac{z(t)^2}{y(t)+d_3}. \tag{3}$$

The interaction between the middle predator y and prey x is modeled via a Holling type II functional response [7], and the interaction between the middle predator y and top predator z is modeled via a modified Leslie Gower scheme [6]. The various parameters in the model are: $a_1(t),\ a_2(t),\ b_2(t), w_0(t),\ w_1(t),\ w_2(t), w_3(t),$ and $c_0(t)$, which are time dependent, periodic functions, that are bounded above and below by positive constants. The other parameters are d_0, d_1, d_2, and d_3, which are all positive constants. They are defined as follows: $a_1(t)$ is the growth rate of prey x; $a_2(t)$ measures the rate at which y dies out when there is no x to prey on and no z; $w_i(t), 0 \leq i \leq 3$, is the maximum value that the per-capita rate can attain; d_0 and d_1 measure the level of protection provided by the environment to the prey; $b_1(t)$ is a measure of the competition among prey, x; d_2 is the half saturation value of y; d_3 represents the loss in z due to the lack of its favorite food, y; $c_0(t)$ describes the growth rate of z via sexual reproduction. We also assume suitable positive initial conditions (x_0, y_0, z_0).

Since the coefficients are time dependent periodic functions, the analysis is tricky. Zhang et al. in [18], follow the work of [4], apply Fredholm operator theory to investigate the existence of periodic solutions in (1)–(3). The main result of [18], is that under certain restrictions on the coefficients, (1)–(3) always has at least one periodic solution. The main theorem from [18] is recalled:

THEOREM 1. Suppose that

$$\bar{w}_1 > \bar{a}_2,\ \bar{w}_2 > \bar{w}_1,\ \text{and } d_2 > d_3. \tag{4}$$

Then system (1)–(3) has at least one ω-periodic positive solution.

Here in general we define $\overline{g} = \frac{1}{\omega}\int_0^{\omega} g(t)dt$. In the event that all coefficients in (1)–(3) are taken to be pure constants, the model reduces to the one considered in [1, 15]. Therein the first global existence result for (1)–(3) was established in [1]. However, recent work on the original model considered in [1, 15] shows that solutions can blow-up in finite time, for large initial data [11, 14]. Our contributions in the current manuscript are the following:

(1) We show that Theorem 1 from [18] is incorrect. That is enforcing the conditions of Theorem 1 via (4), is **not sufficient** to guarantee the existence of a periodic solution. This is demonstrated via Theorem 1 in the current manuscript.

(2) We show that depending on the coefficients, finite time blow-up can occur in (1)–(3). This is demonstrated via Theorems 2 & 3.

(3) We show where the error is, in the proof of Theorem 1, provided in [18].

(4) We derive new global existence conditions, under which there can be small data global periodic solutions. This is demonstrated via Theorem 4.

(5) We derive certain additional restrictions on the coefficients, under which we state a new theorem for the existence of periodic solutions. This is demonstrated via Theorem 5.

(6) The numerical example provided in [18] is incorrect. We support all of our results via numerical simulations, and new examples.

2 Non-existence of Periodic Solution

We first show that Theorem 1 from [18] is incorrect. We state the following theorem

THEOREM 2. Suppose that

$$\bar{w}_1 > \bar{a}_2, \bar{w}_2 > \bar{w}_1, \text{ and } d_2 > d_3. \tag{5}$$

Then system (1)–(3) has no ω-periodic positive solutions for any initial condition (x_0, y_0, r_0).

PROOF. The right hand side of (1)–(3) is quasi-positive [17], hence we have positivity of solutions (x, y, z) here. This entails

$$\left(c_0(t) - \frac{w_3}{y(t)+d_3}\right) > \left(c_0(t) - \frac{w_3(s)}{d_3}\right).$$

Let us consider

$$\frac{dz}{dt} = \left(c_0(t) - \frac{w_3(s)}{d_3}\right) z^2. \tag{6}$$

We integrate (6) to yield

$$\frac{1}{z} = \frac{1}{z_0} - \int_0^t \left(c_0(s) - \frac{w_3(s)}{d_3}\right) ds. \tag{7}$$

Thus

$$z = \frac{1}{\frac{1}{z_0} - \int_0^t \left(c_0(s) - \frac{w_3(s)}{d_3}\right) ds}. \tag{8}$$

Now we maintain (5), but choose $d_3 \int_0^t c_0(s)ds > \int_0^t w_3(s)ds$. We notice that z solving (6) blows up at a finite time $t = T^*$, *no matter what initial condition one chooses.* Here T^* is given by the first time such that, $\frac{1}{z_0} = \int_0^{T^*} \left(c_0(s) - \frac{w_3(s)}{d_3}\right) ds$. This follows trivially by the intermediate value theorem applied to the continuous function $g(t) = \int_0^t \left(c_0(s) - \frac{w_3(s)}{d_3}\right) ds$. Thus by comparison the z solving (3) blows-up, before time $t = T^*$ [17]. So there is no periodic solution to (1)–(3), even if (5) holds. This proves the theorem.

Note that choosing $d_3 \int_0^t c_0(s)ds > \int_0^t w_3(s)ds$, does not invalidate any of the conditions in (4) of Theorem 1. Thus Theorem 1 is incorrect, in that (4) is **not sufficient** to yield the existence of a ω-periodic solution. We next analyze situations where $d_3 \int_0^t c_0(s)ds < \int_0^t w_3(s)ds$. Here large data blow-up is still possible. This is easily seen via Theorem 3, which is a modification of our result from [14]. We first state and prove the following lemma, which will show that due to continuity of the solutions of (1)–(3), y can remain large for a "sufficient" period of time, if y_0 is chosen large enough. We will then use this property to prove Theorem 3.

LEMMA 1. Consider the model (1)–(3). Suppose that

$$\bar{w}_1 > \bar{a}_2, \bar{w}_2 > \bar{w}_1 \text{ and } d_2 > d_3. \tag{9}$$

Then $y(t)$ solving (2) satisfies the following lower estimate

$$y(t) > y_0 e^{-\int_0^t \left(a_2(s) + \frac{w_2(s)C}{d_2}\right)ds}. \tag{10}$$

Furthermore, given a $\delta > 0$, one can always choose initial data y_0 large enough such that

$$y_0 e^{-\int_0^t \left(a_2(s) + \frac{w_2(s)C}{d_2}\right)ds} + d_3 - C_1 > 0, \ \forall t \in [0, \delta), \tag{11}$$

where $C_1 = \frac{2 \sup w_3(t)}{\inf c_0(t)}$, and C is an upper bound on $z(t)$ on its maximal interval of existence.

PROOF. First note C_1 exists, as by assumption, all the coefficients in (1)–(3), are bounded above and below by positive constants [18]. Now, from equation (2) one easily obtains

$$\frac{dy(t)}{dt} \geq -a_2(t)y(t) - w_2(t)\left(\frac{y(t)z(t)}{d_2}\right). \tag{12}$$

Now note, solutions to model (1)–(3) are classical, thus they are uniformly bounded on any interval $[0, T]$, $T < T_{max}$. Here T_{max} is the maximal interval of existence of the solutions. Thus we obtain $z(t) < C$ on $[0, T]$, and applying this in (12) we obtain

$$\frac{dy(t)}{dt} \geq -a_2(t)y(t) - w_2(t)\left(\frac{y(t)C}{d_2}\right), \ t \in [0, T]. \tag{13}$$

Dividing both sides of (13) by $y(t)$, followed by integration in time yields the lower bound on $y(t)$ given in (10). Now note as mentioned earlier the solutions of (1)–(3) are classical in time (locally at least). Thus

$$\phi(t) = y_0 e^{-\int_0^t \left(a_2(s) + \frac{w_2(s)C}{d_2}\right)ds} + d_3 - C_1,$$

where $C_1 = \frac{2\sup w_3(s)}{\inf c_0(s)}$, is easily seen to be continuous. By the continuity of ϕ, given a $\delta > 0$, one can always choose initial data y_0 large enough such that

$$y_0 e^{-\int_0^t \left(a_2(s) + \frac{w_2(s)C}{d_2}\right)ds} + d_3 - C_1 > 0,\ \forall t \in [0, \delta).$$

This proves the second part of the lemma.

THEOREM 3. Consider the model (1)–(3). Suppose that

$$\bar{w}_1 > \bar{a}_2,\ \bar{w}_2 > \bar{w}_1, \text{ and } d_2 > d_3.$$

Then the system (1)–(3) can blow-up in finite time, that is

$$\lim_{t \to T^* < \infty} |z(t)| \to \infty, \tag{14}$$

even if $d_3 \int_0^t c_0(s)ds < K \int_0^t w_3(s)ds$, for any constant $K << 1$, if (y_0, z_0) is chosen large enough.

PROOF. The proof is a simple modification of methods in [14]. Consider (1)–(3), with positive initial conditions (x_0, y_0, z_0). By integrating (3), we obtain

$$z = \frac{1}{\frac{1}{z_0} - \int_0^t \left(c_0(s) - \frac{w_3(s)}{y(s)+d_3}\right) ds}.$$

Thus our aim is to show the continuous function:

$$\psi(t) = \frac{1}{z_0} - \int_0^t \left(c_0(s) - \frac{w_3(s)}{y(s) + d_3}\right) ds,$$

vanishes at some time $T > 0$. Now we know from Lemma 1 that for $\delta > 0$, we can choose y_0 sufficiently large such that

$$y + d_3 > y_0 e^{-\int_0^t \left(a_2(s) + \frac{w_2(s)C}{d_2}\right)ds} + d_3 > C_1 > 2\frac{w_3(t)}{c_0(t)},\ \forall t \in [0, \delta).$$

Thus $y + d_3 > 2\frac{w_3(t)}{c_0(t)}$, $\forall t \in [0, \delta)$, and so

$$\frac{w_3(t)}{y(t) + d_3} < \frac{c_0(t)}{2},\ \forall t \in [0, \delta).$$

This implies

$$\frac{1}{t}\int_0^t \left(\frac{w_3(s)}{y(s) + d_3}\right) ds < \frac{1}{t}\int_0^t \left(\frac{c_0(s)}{2}\right) ds < \frac{K}{2}\int_0^t \left(\frac{w_3(s)}{d_3}\right) ds,\ \forall t \in [0, \delta).$$

Thus

$$\frac{1}{z_0} - \left[\frac{1}{t}\int_0^t \left(c_0(s) - \frac{w_3(s)}{y(s) + d_3}\right) ds\right] t < \frac{1}{z_0} - \int_0^t \left(\frac{c_0(s)}{2}\right) ds,\ \forall t \in [0, \delta).$$

If z_0 is chosen sufficiently large, then we can find $T^* \in (0, \delta)$ such that

$$\frac{1}{z_0} - \int_0^{T^*} \left(\frac{c_0(s)}{2} \right) ds = 0.$$

This entails

$$\psi\left(T^*\right) = \frac{1}{z_0} - \int_0^{T^*} \left(c_0(s) - \frac{w_3(s)}{y(s) + d_3} \right) ds < \frac{1}{z_0} - \int_0^{T^*} \left(\frac{c_0(s)}{2} \right) ds = 0.$$

Thus one has $\psi\left(T^*\right) < 0$, but $\psi\left(0\right) > 0$, and by application of the mean value theorem, we obtain the existence of some $T \in (0, \delta)$, $T < T^*$, such that $\psi\left(T\right) = 0$. This implies the solution z of (3) blows up in finite time, at $t = T^*$, by a standard comparison argument [17].

Note that now since we have the desired blow-up, it must be that $T_{max} < T^*$. Here T_{max} was the maximal interval of existence assumed on $z(t)$, so that we could make the formal estimates. Whereas T^* is the *actual* eventual blow-up time.

Thus we see that even if $d_3 \int_0^t c_0(s)ds < K \int_0^t w_3(s)ds$, $K << 1$, (1)–(3) does not have any ω-periodic solution, for large initial data. Our next goal is to investigate restrictions on the initial data (under the condition $d_3 \int_0^t c_0(s)ds < \int_0^t w_3(s)ds$), that yield firstly a global solution, so that the search for a periodic solution can ensue. This is stated via the following theorem.

THEOREM 4. Consider the model (1)–(3). Suppose that

$$\bar{w}_1 > \bar{a}_2, \bar{w}_2 > \bar{w}_1, \text{ and } d_2 > d_3. \tag{15}$$

Assume that there exists an initial data (x_0, y_0) for which x, y are ω-periodic. Then z will be ω-periodic if $\left(c_0(t) - \frac{w_3}{y(t)+d_3}\right)$ is ω-periodic and switches sign between $[0, \frac{n\omega}{2}]$ and $[\frac{n\omega}{2}, n\omega]$, for all $n > N + k$, where $k \in \mathbb{Z}^+$, where transient behavior is possible for $t \in [0, N\omega]$, for some integer N. Also we require z_0 to be such that

$$\frac{\omega}{2} < \frac{1}{\delta_1 |z_0|},$$

where $\delta_1 = ||c_0(t)||_\infty + \frac{||w_3(t)||_\infty}{d_3}$.

PROOF. First note, The periodicity of a solution in the z variable depends on the coefficient $\left(c_0(t) - \frac{w_3(s)}{y(t)+d_3}\right)$. That is for z to be periodic with period ω, we need $\left(c_0(t) - \frac{w_3(s)}{y(t)+d_3}\right)$ to be periodic with period ω, where $\left(c_0(t) - \frac{w_3(s)}{y(t)+d_3}\right)$, must switch sign between $[0, n\frac{\omega}{2}]$ and $[\frac{n\omega}{2}, n\omega]$, for all $n > N + k$ (where $k \in \mathbb{Z}^+$, and transient behavior is possible for $t \in [0, N\omega]$, for some integer N). Else, if $\left(c_0(t) - \frac{w_3}{y(t)+d_3}\right) > 0$, z will blow up in finite time, for large enough data, in comparison with

$$\frac{dz}{dt} = \delta_{min} z^2,$$

where $\delta_{min} = \min\left(c_0(t) - \frac{w_3(t)}{d_3}\right)$. Hence z cannot be periodic. If $\left(c_0(t) - \frac{w_3(t)}{y(t)+d_3}\right) < 0$, then z will decay to zero, in comparison with

$$\frac{dz}{dt} = -\delta_{max} z^2,$$

where $\delta_{max} = \max\left(c_0(t) - \frac{w_3(t)}{d_3}\right)$. Hence z cannot be periodic. Now we know that the solution z to (3) will only blow-up (if it does) after the solution to

$$\frac{dz}{dt} = \delta_1 z^2, \tag{16}$$

where

$$\left(c_0(t) - \frac{w_3(t)}{y(t) + d_3}\right) < ||c_0(t)||_\infty + \frac{1}{d_3}||w_3(t)||_\infty = \delta_1.$$

This follows via a simple comparison argument [17]. Thus, if we enforce

$$\frac{\omega}{2} < \frac{1}{\delta_1 |z_0|},$$

then $\left(c_0(t) - \frac{w_3}{y(t)+d_3}\right)$ will switch sign before the z solving (16) can blow up, so the z solving (3), could certainly not have blown up by this time, by comparison. Once the sign of $\left(c_0(t) - \frac{w_3}{y(t)+d_3}\right)$ switches, z solving (3) decays till the sign becomes positive again, and this repeats in the periodic intervals.

It is important to address where exactly the flaws in the main result, Theorem 1 in [18] occur. The authors therein proceed by following the techniques of [4] to define $J_1(t) = \ln(x), J_2(t) = \ln(y), J_3(t) = \ln(z)$, and then bound from above and below, each of these quantities, under the restriction (4) in Theorem 1. (see (23)-(24) in [18] for the derivation of the upper bound on J_3). However, this is incorrect, as we have seen via Theorem 2 that z can blow-up in finite time for any initial condition, even under the restriction imposed via (4). Hence $J_3(t) = \ln(z)$, will also blow-up, and is **not bounded** from above. Thus the preceding analysis in [18] is incorrect. We now state the following result

THEOREM 5. Consider the model (1)–(3). Suppose that

$$\bar{w}_1 > \bar{a}_2, \bar{w}_2 > \bar{w}_1, d_2 > d_3 \text{ and } \bar{w}_3 > d_3\bar{c}_0. \tag{17}$$

Then there exists ω_1 such that, if we restrict the size of the initial data via

$$|y_0| < \frac{\bar{w}_3}{\bar{c_0}} - d_3, \text{ and } |z_0| < \frac{2}{\omega_1 \delta_1}, \tag{18}$$

then the system (1)–(3) has at least one ω-periodic solution, where $\omega < \omega_1$. However, the system (1)–(3) can blow-up in finite time for large initial data.

PROOF. The proof to show the existence of a periodic solution follows by the methods of Theorem 4, taken in conjunction with the proof of Theorem 1 in [18]. That is enforcing (17), for small data such as via (18), we can now bound $J_3(t) = \ln(z)$ from above, and use the Fredholm theory as in [18], to give the existence of a ω-periodic solution. For large data however, we can follow the methods of Lemma 1 and Theorem 3, to show that finite time blow-up occurs.

3 Numerical Simulations

We first point out that the numerical example provided in [18] is incorrect. The coefficients are not bounded below by positive constants. We consider the following counter example instead:

$$\begin{aligned} \frac{dx}{dt} &= \left[\left(9.9+\sin t\right)/4\right]x(t) - \left[\left(3+\sin t\right)/55\right]x(t)^2 - \frac{x(t)y(t)}{1+x(t)}, \\ \frac{dt}{dt} &= -\left(1.01+\cos t\right)y(t) + \frac{\left(1.5+\cos t\right)x(t)y(t)}{1/2+x(t)} - \frac{3z(t)y(t)}{(30+y(t))}, \\ \frac{dz}{dt} &= \left(0.65+0.02\sin(t)\right)z^2(t) - \frac{\left(1.4+\sin t\right)z^2(t)}{y(t)+21}. \end{aligned} \tag{19}$$

Note

$$\bar{w}_1 = 1.5 > 1.1 = \bar{a}_2,\ \bar{w}_2 = 3 > 1.5 = \bar{w}_1,\ \text{and } d_2 = 2.5 > 2 = d_3.$$

Also the coefficients are positive continuous ω-periodic functions that satisfy the conditions of Theorem 1. However,

$$\begin{aligned} d_3\int_0^t c_0(s)ds &= 21\int_0^t \left[0.65+0.02\sin(x)\right]ds \\ &> \int_0^t \left[1.4+\sin(s)\right]ds = \int_0^t w_3(s)ds. \end{aligned}$$

Thus the system will blow-up in finite time for any initial condition (x_0, y_0, z_0). Hence there is no periodic solution. We demonstrate the blow-up with a specific initial condition in figure 1.

Next we consider the following system:

$$\begin{aligned} \frac{dx}{dt} &= \left[\left(9.9+\sin t\right)/4\right]x(t) - \left[\left(3+\sin t\right)/55\right]x(t)^2 - \frac{x(t)y(t)}{1+x(t)}, \\ \frac{dt}{dt} &= -\left(1.01+\cos t\right)y(t) + \frac{\left(1.5+\cos t\right)x(t)y(t)}{1/2+x(t)} - \frac{3z(t)y(t)}{(30+y(t))}, \\ \frac{dz}{dt} &= \left(0.035+0.002\sin(t)\right)z^2(t) - \frac{\left(1.4+\sin t\right)z^2(t)}{y(t)+21}. \end{aligned} \tag{20}$$

Applying Theorem 4 we see that,

$$\bar{w}_1 = 1.5 > 1.01 = \bar{a}_2,\ \bar{w}_2 = 3 > 1.5 = \bar{w}_1,\ \text{and } d_2 = 30 > 21 = d_3.$$

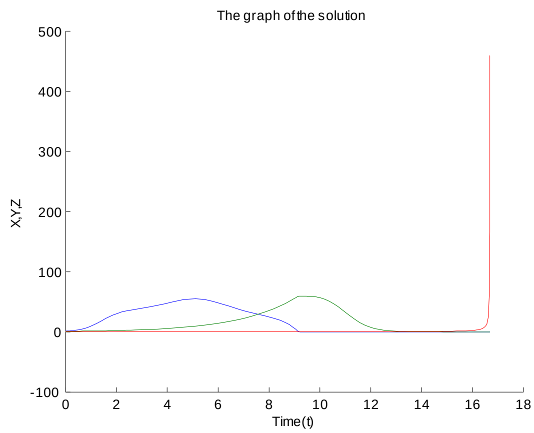

Figure 1: We demonstrate finite time blow-up with the initial condition $(1.25, 1.25, 0.1)$ in system (1)– (3).

Also

$$\bar{w}_3 = 1.4 > 21(0.037) = 0.798 = d_3\bar{c}_0.$$

Furthermore we see that the condition on the initial data $(1.25, 1.25, 0.1)$ via Theorem 5 are satisfied. That is,

$$1.25 = y_0 < \frac{\bar{w}_3}{\bar{c}_0} - d_3 = 16.838,$$

and there exists ω_1 such that

$$70 = \omega < \omega_1 < \frac{2}{|z_0|\delta_1} = \frac{2}{(0.1)(0.066)} = 303.03.$$

Here $\omega = 70$, is the period of the solution. We graphically show the existence of a periodic solution in Figure 2.

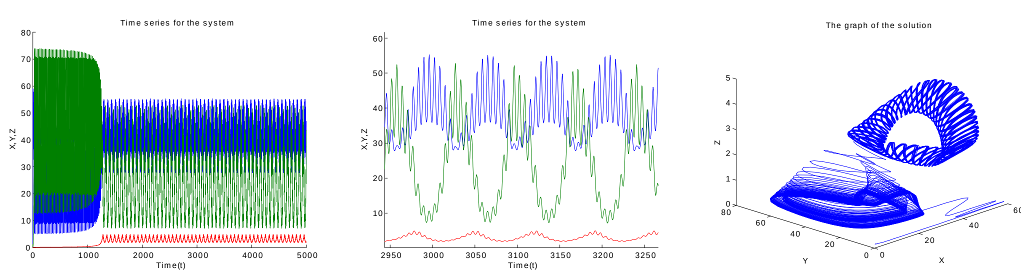

Figure 2: Periodic solution in system (1)–(3). A zoom in and phase plot are also shown.

4 Conclusion

We see in Figure 2 that transient behavior takes place till $t = 1000$, after which the system settles into a periodic orbit. We would next like to discuss possible future directions. It is an interesting question to consider the case of (1)–(3), with a time delay. This is interesting even in the case where the coefficients are pure constants. It is claimed in [4], that the periodic solutions remain bounded, in case of a constant time delay. However, in [4], one can ensure a global bound for solutions to the model considered, for any initial condition. This is certainly not the case in (1)–(3). Thus investigating the effect of a constant delay τ in all species, or perhaps different constant delay's τ_1 in y, and τ_2 in z, is in our opinion an interesting future direction. One should perhaps consider the global existence question first, and then the question of periodic solutions. The effect of time delay on known Turing instability in the constant coefficient diffusion model [13] might also be an interesting question. Lastly, since the environment is inherently stochastic, it would also be interesting to consider the effect of noise on system (1)–(3).

Acknowledgment. We would like to acknowledge the valuable comments of the anonymous referee that helped us improve the quality of our manuscript.

References

[1] M. A. Aziz-Alaoui, Study of a Leslie-Gower type tri-trophic population model, Chaos. Soliton. Fract., 14(2002), 1275–1293.

[2] S. Gakkhar and B. Singh, Complex dynamic behavior in a food web consisting of two preys and a predator, Chaos. Soliton. Fract., 24(2005), 789–801.

[3] A. Hastings and T. Powell, Chaos in a three species food chain model, Ecology, 72(1991), 896–903.

[4] H. F. Huo, Periodic solutions for a semi-ratio-dependent predator-prey system with functional responses, Appl. Math. Lett., 18(2005), 313–320.

[5] C. Letellier and M. A. Aziz-Alaoui, Analysis of the dynamics of a realistic ecological model, Chaos. Soliton. Fract., 13(2002), 95–107.

[6] P.H. Leslie, Some further notes on the use of matricies in population mathematics, Biometrika, 35(1948), 213–245.

[7] J. D. Murray, Mathamatical Biology, Springer-Verlag, Berlin, 1993.

[8] N. Kumari, Pattern Formation in Spatially Extended Tritrophic Food Chain Model Systems: Generalist versus Specialist Top Predator, ISRN Biomatheatics, 2013(2013), Article ID 198185.

[9] R. D. Parshad and R. K. Upadhyay, Investigation of long time dynamics of a diffusive three species aquatic model, Dynam. Part. Differ. Eq., 7(2010), 217–244.

[10] R. D. Parshad, H. A. Abderrahmanne, R. K. Upadhyay, and N. Kumari, Finite time blowup in a realistic food chain model, ISRN Biomathematics, 2013(2013), Article ID 424062.

[11] R. D. Parshad, E. Quansah, K. Black, and M. Beauregard, Biological control via "ecological" damping: An approach that attenuates non-target effects. Math. Biosci., 273(2016), 23–44.

[12] R. D. Parshad, E. Quansah, K. Black, R. K. Upadhyay, S. K. Tiwari and N, Kumari, Long time dynamics of a three-species food chain model with Allee effect in the top predator, Comput. Math. Appl., 71(2016), 503–528.

[13] R. D. Parshad, N. Kumari, A. R. Kasimov and H. A. Abderrahmane, Turing Patterns and long time behavior in a three-species model, Math. Biosci., 254(2014), 83–102.

[14] R. D. Parshad, N. Kumari and S. Kouachi, A remark on "Study of a Leslie-Gower-type tritrophic population model" [Chaos. Soliton. Fract. 14 (2002) 1275-1293], Chaos. Soliton. Fract., 71(2015), 22–28.

[15] R. K. Upadhyay and V. Rai, Why chaos is rarely observed in natural populations?, Chaos. Soliton. Fract., 8(1997), 1933–1939.

[16] R. K. Upadhyay, S. R. K. Iyengar and V. Rai, Stability and complexity in ecological systems, Chaos. Soliton. Fract., 11(2000), 533–542.

[17] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York, 1983.

[18] L. Zhang and H. Zhao, Periodic solutions of a three-species food chain model, Appl. Math. E-Notes, 9(2009), 47–54.