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Value Distribution And Uniqueness Of Certain Type Of Difference Polynomials*

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Abstract

In the paper we investigate the distribution of zeros as well as the uniqueness problems of certain type of difference polynomials sharing a small function with finite weight. The research findings also include IM analogues of the theorem in which the small function is allowed to be shared ignoring multiplicities. The results of the paper improve and generalize the recent results due to Bhoosnurmath and Kabbur [International Journal of Analysis and Applications, 2(2013), 124-136].

1 Introduction, Definitions and Results

In this paper, a meromorphic function means meromorphic in the whole complex plane. We shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions as explained in [9, 11, 18]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f - a$ and $g - a$ coincide in locations and multiplicity, we say that f and g share the value a CM (counting multiplicities). On the other hand, if the zeros of $f - a$ and $g - a$ coincide only in their locations, then we say that f and g share the value a IM (ignoring multiplicities). We say f and g share a function h CM or IM if $f - h$ and $g - h$ share 0 CM or IM respectively. For a positive integer p, we denote by $N_p(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. A meromorphic function $\alpha(\not\equiv 0,\infty)$ is called a small function with respect to f, if $T(r,\alpha) = S(r,f)$. We define difference operators $\triangle_c f(z) = f(z+c) - f(z)$ and $\triangle_c^n f(z) = \triangle_c^{n-1}(\triangle_c f(z))$ where c is a nonzero complex number and $n \geq 2$ is a positive integer. In particular, if $c = 1$, we use the usual difference notation $\triangle_c f(z) = \triangle f(z)$.

A lot of research works on entire and meromorphic functions whose differential polynomials share certain value or fixed points have been done by many mathematicians

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in the world (see [5, 6, 15, 16, 17]). Recently, value distribution in difference analogue has become a subject of great interest among the researchers. In 2006, R. G. Halburd and R. J. Korhonen [7] established a version of Nevanlinna theory based on difference operators. The difference logarithmic derivative lemma, given by R. G. Halburd and R. J. Korhonen [8] in 2006, Y. M. Chiang and S. J. Feng [4] in 2008 plays an important role in considering the difference analogues of Nevanlinna theory. With the development of difference analogue of Nevanlinna theory, many mathematicians paid their attention on the distribution of zeros of difference polynomials. In 2007, I. Laine and C. C. Yang [12] proved the following result for difference polynomials.

THEOREM A. Let f be a transcendental entire function of finite order and c be a nonzero complex constant. Then for $n \geq 2$, $f^n(z)f(z+c)$ assumes every nonzero value $a \in \mathbb{C}$ infinitely often.

In 2010, X. G. Qi, L. Z. Yang and K. Liu [14] proved the following uniqueness result which corresponded to Theorem A.

THEOREM B. Let f and g be two transcendental entire functions of finite order, and c be a nonzero complex constant, and let $n \geq 6$ be an integer. If $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share the value 1 CM, then either $fg = t_1$ or $f = t_2 g$ for some constants t_1 and t_2 satisfying $t_1^{n+1} = t_2^{n+1} = 1$.

In the same year, J. L. Zhang [19] considered the zeros of one certain type of difference polynomial and obtained the following result.

THEOREM C. Let f be a transcendental entire function of finite order, $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and c be a nonzero complex constant. If $n \geq 2$ is an integer, then $f^n(z)(f(z)-1)f(z+c) - \alpha(z)$ has infinitely many zeros.

In the same paper the author also proved the following uniqueness result which corresponds to Theorem C.

THEOREM D. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that c is a nonzero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = g$.

In 2013, S. S. Bhoosnurmath and S. R. Kabbur [2] considered the zeros of difference polynomial of the form $f^n(z)(f^m(z)-1)f(z+c)$, where n, m are positive integers and c is a nonzero complex constant and obtained the following theorem.

THEOREM E. Let f be an entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a nonzero complex constant and n, m are positive integers. If $n \geq 2$, then $f^n(z)(f^m(z)-1)f(z+c) - \alpha(z)$ has infinitely many zeros.

The following two theorems are the uniqueness results corresponding to Theorem E proved by S. S. Bhoosnurmath and S. R. Kabbur [2].

THEOREM F. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant and n, m are positive integers such that $n \geq m+6$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = tg$ where $t^m = 1$.

THEOREM G. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant and n, m are positive integers satisfying $n \geq 4m+12$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ IM, then $f = tg$ where $t^m = 1$.

An increment to uniqueness theory has been to considering weighted sharing instead of sharing IM or CM, this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing has been introduced by I. Lahiri around 2001, which measure how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

DEFINITION 1 ([10]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f, g share the value a with weight k.

The definition implies that if f, g share the value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write *f, g share* (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share the value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If α is a small function of f and g, then f, g share α with weight k means that $f - \alpha$, $g - \alpha$ share the value 0 with weight k.

Regarding the results of Bhoosnurmath and Kabbur [2] it is natural to ask the following questions which are the motivation of the paper.

QUESTION 1. What can be said if we consider the difference polynomials of the form $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ and $(f^n(z)(f(z)-1)^m f(z+c))^{(k)}$ where $k(\geq 0)$ is an integer ?

QUESTION 2. Is it possible to relax in any way the nature of sharing the small function in Theorem F keeping the lower bound of n fixed ?

In the paper, our main concern is to find the possible answer of the above questions. The following are the main results of the paper.

THEOREM 1. Let f be a transcendental entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers. If $n \geq k+2$, then $(f^n(z)(f^m(z)-1)f(z+c))^{(k)} - \alpha(z)$ has infinitely many zeros.

THEOREM 2. Let f be a transcendental entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers. If $n \geq k+2$ when $m \leq k+1$ and $n \geq 2k-m+3$ when $m > k+1$, then $(f^n(z)(f(z)-1)^m f(z+c))^{(k)} - \alpha(z)$ has infinitely many zeros.

THEOREM 3. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$. If $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ and $(g^n(z)(g^m(z)-1)g(z+c))^{(k)}$ share $(\alpha, 2)$, then $f = tg$ where $t^m = 1$.

REMARK 1. Theorem 3 improves and generalizes Theorem F.

THEOREM 4. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$ when $m \leq k+1$ and $n \geq 4k-m+10$ when $m > k+1$. If $(f^n(z)(f(z)-1)^m f(z+c))^{(k)}$ and $(g^n(z)(g(z)-1)^m g(z+c))^{(k)}$ share $(\alpha, 2)$, then either $f = g$ or f and g satisfy the algebraic equation $R(f,g) = 0$ where $R(f,g)$ is given by

$$R(w_1, w_2) = w_1^n(w_1-1)^m w_1(z+c) - w_2^n(w_2-1)^m w_2(z+c).$$

THEOREM 5. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 5k+4m+12$. If $(f^n(z)(f^m(z)-1)f(z+c))^{(k)}$ and $(g^n(z)(g^m(z)-1)g(z+c))^{(k)}$ share $\alpha(z)$ IM, then $f = tg$ where $t^m = 1$.

REMARK 2. Theorem 5 improves and generalizes Theorem G.

THEOREM 6. Let f and g be two transcendental entire functions of finite order, and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a nonzero complex constant, $n(\geq 1)$, $m(\geq 1)$ and $k(\geq 0)$ are integers satisfying $n \geq 5k+4m+12$ when $m \leq k+1$ and $n \geq 10k-m+19$ when $m > k+1$. If $(f^n(z)(f(z)-1)^m f(z+c))^{(k)}$ and $(g^n(z)(g(z)-1)^m g(z+c))^{(k)}$ share $\alpha(z)$ IM, then the conclusions of Theorem 4 hold.

2 Lemmas

Let F and G be two nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$. We denote by H the following function:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

LEMMA 1 ([13]). Let f be a meromorphic function of finite order ρ and let $c(\neq 0)$ be a fixed nonzero complex constant. Then

$$\bar{N}(r,\infty;f(z+c)) \leq \bar{N}(r,\infty;f) + S(r,f),$$

outside a possible exceptional set of finite logarithmic measure.

LEMMA 2 ([3]). Let f be an entire function of finite order and $F = f^n(z)(f^m(z) - 1)f(z+c)$. Then

$$T(r,F) = (n+m+1)T(r,f) + S(r,f).$$

Arguing in a similar manner as in Lemma 2.6 ([3]) we obtain the following lemma.

LEMMA 3. Let f be an entire function of finite order and $F = f^n(z)(f(z) - 1)^m f(z+c)$. Then

$$T(r,F) = (n+m+1)T(r,f) + S(r,f).$$

LEMMA 4 ([20]). Let f be a nonconstant meromorphic function, and p, k be two positive integers. Then

$$N_p\left(r,0;f^{(k)}\right) \leq T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f) \tag{1}$$

and

$$N_p\left(r,0;f^{(k)}\right) \leq k\bar{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{2}$$

LEMMA 5 ([10]). Let f and g be two nonconstant meromorphic functions sharing $(1,2)$. Then one of the following cases holds:

(i) $T(r) \leq N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g) + S(r)$,

(ii) $f = g$,

(iii) $fg = 1$,

where $T(r) = \max\{T(r,f), T(r,g)\}$ and $S(r) = o\{T(r)\}$.

LEMMA 6 ([1]). Let F and G be two nonconstant meromorphic functions sharing the value 1 IM and $H \not\equiv 0$. Then

$$\begin{aligned} T(r,F) \leq\ & N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) \\ & +2\bar{N}(r,0;F) + \bar{N}(r,0;G) + 2\bar{N}(r,\infty;F) + \bar{N}(r,\infty;G) \\ & +S(r,F) + S(r,G), \end{aligned}$$

and the same inequality holds for $T(r,G)$.

LEMMA 7. Let f and g be two entire functions, and $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$ be integers, and let

$$F = (f^n(z)(f^m(z)-1)f(z+c))^{(k)},\ G = (g^n(z)(g^m(z)-1)g(z+c))^{(k)}.$$

If there exists nonzero constants c_1 and c_2 such that $\bar{N}(r,c_1;F) = \bar{N}(r,0;G)$ and $\bar{N}(r,c_2;G) = \bar{N}(r,0;F)$, then $n \leq 2k+m+3$.

PROOF. We put $F_1 = f^n(z)(f^m(z)-1)f(z+c)$ and $G_1 = g^n(z)(g^m(z)-1)g(z+c)$. By the second fundamental theorem of Nevanlinna we have

$$\begin{aligned} T(r,F) &\leq \bar{N}(r,0;F) + \bar{N}(r,c_1;F) + S(r,F) \\ &\leq \bar{N}(r,0;F) + \bar{N}(r,0;G) + S(r,F). \end{aligned} \tag{3}$$

Using (3), Lemmas 2 and 4, we obtain

$$\begin{aligned} (n+m+1)T(r,f) &\leq T(r,F) - \bar{N}(r,0;F) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq \bar{N}(r,0;G) + N_{k+1}(r,0;F_1) + S(r,f) \\ &\leq N_{k+1}(r,0;F_1) + N_{k+1}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq (k+m+2)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \end{aligned} \tag{4}$$

Similarly,

$$(n+m+1)T(r,g) \leq (k+m+2)(T(r,f)+T(r,g)) + S(r,f) + S(r,g). \tag{5}$$

Combining (4) and (5) we obtain

$$(n-2k-m-3)(T(r,f)+T(r,g)) \leq S(r,f) + S(r,g),$$

which gives $n \leq 2k+m+3$. This proves the lemma.

LEMMA 8. Let f and g be two entire functions, $n(\geq 1)$, $m(\geq 1)$, $k(\geq 0)$ be integers, and let

$$F = (f^n(z)(f(z)-1)^m f(z+c))^{(k)},\ G = (g^n(z)(g(z)-1)^m g(z+c))^{(k)}.$$

If there exists nonzero constants c_1 and c_2 such that $\bar{N}(r, c_1; F) = \bar{N}(r, 0; G)$ and $\bar{N}(r, c_2; G) = \bar{N}(r, 0; F)$, then $n \leq 2k + m + 3$ for $m \leq k + 1$ and $n \leq 4k - m + 5$ for $m > k + 1$.

PROOF. By the same reasoning as in the proof of Lemma 7 we can easily deduce the result. Here we omit the details.

LEMMA 9 ([2]). Suppose that f and g are two transcendental entire functions of finite order, $c(\neq 0)$ is a fixed nonzero complex constant, and n, m are positive integers. If $n \geq m + 5$ and

$$f^n(z)(f^m(z) - 1)f(z + c) = g^n(z)(g^m(z) - 1)g(z + c),$$

then $f = tg$, where $t^m = 1$.

NOTE 1. Though the authors [2] claimed that the conclusion of Lemma 9 holds for $n \geq m + 6$, from the proof of it one can easily checked that it is true for $n \geq m + 5$.

3 Proof of the Theorem

PROOF OF THEOREM 1. Let $F_1 = f^n(z)(f^m(z) - 1)f(z + c)$. Then F_1 is a transcendental entire function. If possible, we assume that $F_1^{(k)} - \alpha(z)$ has only finitely many zeros. Then we have

$$N(r, \alpha; F_1^{(k)}) = O\{\log r\} = S(r, f). \tag{6}$$

Using (1), (6) and Nevanlinna's three small function theorem we obtain

$$\begin{aligned} T(r, F_1^{(k)}) &\leq \bar{N}(r, 0; F_1^{(k)}) + \bar{N}(r, \alpha; F_1^{(k)}) + S(r, f) \\ &\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+1}(r, 0; F_1) + S(r, f). \end{aligned} \tag{7}$$

Applying Lemma 2 we obtain from (7)

$$\begin{aligned} (n + m + 1)T(r, f) &\leq N_{k+1}(r, 0; F_1) + S(r, f) \\ &\leq (k + m + 2)T(r, f) + S(r, f). \end{aligned}$$

This gives

$$(n - k - 1)T(r, f) \leq S(r, f),$$

a contradiction with the assumption that $n \geq k + 2$. This proves the theorem.

PROOF OF THEOREM 2. Let $F_2 = f^n(z)(f(z) - 1)^m f(z + c)$. Then F_2 is also a transcendental entire function. If possible, suppose that $F_2^{(k)} - \alpha(z)$ has only finitely many zeros. Then we have

$$N(r, \alpha; F_2^{(k)}) = O\{\log r\} = S(r, f). \tag{8}$$

Now using (1), (8) and Nevanlinna's three small function theorem we obtain

$$\begin{aligned} T(r,F_2^{(k)}) &\leq \bar{N}(r,0;F_2^{(k)}) + \bar{N}(r,\alpha;F_2^{(k)}) + S(r,f) \\ &\leq T(r,F_2^{(k)}) - T(r,F_2) + N_{k+1}(r,0;F_2) + S(r,f). \end{aligned} \tag{9}$$

Applying Lemma 3, we obtain from (9)

$$(n+m+1)T(r,f) \leq N_{k+1}(r,0;F_2) + S(r,f). \tag{10}$$

If $m \leq k+1$, we deduce from (10) that

$$(n-k-1)T(r,f) \leq S(r,f),$$

a contradiction with the assumption that $n \geq k+2$.

If $m > k+1$, by (10) we obtain

$$(n+m-2k-2)T(r,f) \leq S(r,f),$$

a contradiction with the assumption that $n \geq 2k-m+3$. This proves Theorem 2.

PROOF OF THEOREM 3. Let $F_1 = f^n(z)(f^m(z)-1)f(z+c)$, $G_1 = g^n(z)(g^m(z)-1)g(z+c)$, $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share $(1,2)$ except the zeros and poles of $\alpha(z)$. Using (1) and Lemma 2 we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;(F_1)^{(k)}) + S(r,f) \\ &\leq T(r,(F_1)^{(k)}) - (n+m+1)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f) \\ &\leq T(r,F) - (n+m+1)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f). \end{aligned}$$

From this we get

$$(n+m+1)T(r,f) \leq T(r,F) + N_{k+2}(r,0;F_1) - N_2(r,0;F) + S(r,f). \tag{11}$$

Again by (2) we have

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\leq N_{k+2}(r,0;F_1) + S(r,f). \end{aligned} \tag{12}$$

Suppose, if possible, that (i) of Lemma 5 holds. Then using (12) we obtain from (11)

$$\begin{aligned} (n+m+1)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + N_{k+2}(r,0;F_1) \\ &\quad +S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq (k+m+3)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \end{aligned} \tag{13}$$

In a similar manner we obtain

$$(n+m+1)T(r,g) \leq (k+m+3)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{14}$$

(13) and (14) together gives

$$(n-2k-m-5)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

contradicting with the fact that $n \geq 2k+m+6$. Therefore, by Lemma 5 we have either $FG=1$ or $F=G$. Let $FG=1$. Then

$$\begin{gathered}(f^n(z)(f^m(z)-1)f(z+c))^{(k)}(g^n(z)(g^m(z)-1)g(z+c))^{(k)} = \alpha^2 \\ i.e.\ (f^n(z)(f(z)-1)(f^{m-1}(z)+f^{m-2}(z)+...+1)f(z+c))^{(k)} \\ (g^n(z)(g(z)-1)(g^{m-1}(z)+g^{m-2}(z)+...+1)g(z+c))^{(k)} = \alpha^2.\end{gathered}$$

It can be easily viewed from above that $N(r,0;f)=S(r,f)$ and $N(r,1;f)=S(r,f)$. Thus we obtain

$$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3,$$

which is not possible. Therefore, we must have $F=G$, and then

$$(f^n(z)(f^m(z)-1)f(z+c))^{(k)} = (g^n(z)(g^m(z)-1)g(z+c))^{(k)}.$$

Integrating above we obtain

$$(f^n(z)(f^m(z)-1)f(z+c))^{(k-1)} = (g^n(z)(g^m(z)-1)g(z+c))^{(k-1)} + c_{k-1},$$

where c_{k-1} is a constant. If $c_{k-1} \neq 0$, using Lemma 7 it follows that $n \leq 2k+m+1$, a contradiction. Hence $c_{k-1}=0$. Repeating the process k-times, we deduce that

$$f^n(z)(f^m(z)-1)f(z+c) = g^n(z)(g^m(z)-1)g(z+c),$$

which by Lemma 9 gives $f=tg$, where t is a constant satisfying $t^m=1$. This proves Theorem 3.

PROOF OF THEOREM 4. Let $F_1 = f^n(z)(f(z)-1)^m f(z+c)$, $G_1 = g^n(z)(g(z)-1)^m g(z+c)$, $F = \frac{F_1^{(k)}}{\alpha(z)}$ and $G = \frac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share the value 1 with weight two except possibly the zeros and poles of $\alpha(z)$. Arguing in a manner similar to the proof of Theorem 3 we obtain either $FG=1$ or $F=G$. If $F=G$, then applying the same technique as in the proof of Theorem 3 and using Lemma 8 we obtain

$$f^n(z)(f(z)-1)^m f(z+c) = g^n(z)(g(z)-1)^m g(z+c). \tag{15}$$

Set $h = \frac{f}{g}$. If h is a constant, then substituting $f=gh$ in (15), we deduce that

$$g(z+c)[g^m(h^{n+m+1}-1) - {}^mC_1 g^{m-1}(h^{n+m}-1) + ... + (-1)^m(h^{n+1}-1)] = 0.$$

Since g is a transcendental entire function, we have $g(z+c) \neq 0$. So from above we obtain

$$g^m(h^{n+m+1}-1) - {}^mC_1 g^{m-1}(h^{n+m}-1) + ... + (-1)^m(h^{n+1}-1) = 0,$$

which implies $h = 1$ and hence $f = g$. If h is not a constant, then it follows from (15) that f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by

$$R(w_1, w_2) = w_1^n(w_1 - 1)^m w_1(z + c) - w_2^n(w_2 - 1)^m w_2(z + c).$$

If $FG = 1$, proceeding in a like manner as in the proof of Theorem 3 we arrive at a contradiction. This completes the proof of Theorem 4.

PROOF OF THEOREM 5. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 3. Then F and G are transcendental meromorphic functions that share the value 1 IM except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H \not\equiv 0$. Using Lemma 6 and (12) we obtain from (11)

$$\begin{aligned}(n + m + 1)T(r, f) \quad &\leq \quad N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\bar{N}(r, 0; F) + \bar{N}(r, 0; G)\\ &\quad\; + N_{k+2}(r, 0; F_1) + 2\bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + S(r, f) + S(r, g)\\ &\leq \quad N_{k+2}(r, 0; F_1) + N_{k+2}(r, 0; G_1) + 2N_{k+1}(r, 0; F_1)\\ &\quad\; + N_{k+1}(r, 0; G_1) + S(r, f) + S(r, g)\\ &\leq \quad (3k + 3m + 7)T(r, f) + (2k + 2m + 5)T(r, g)\\ &\quad\; + S(r, f) + S(r, g)\\ &\leq \quad (5k + 5m + 12)T(r) + S(r). \end{aligned} \tag{16}$$

Similarly,

$$(n + m + 1)T(r, g) \leq (5k + 5m + 12)T(r) + S(r). \tag{17}$$

(16) and (17) together yields

$$(n - 5k - 4m - 11)T(r) \leq S(r),$$

which is a contradiction with the assumption that $n \geq 5k + 4m + 12$. We now assume that $H \equiv 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right) = 0.$$

Integrating both sides of the above equality twice we get

$$\frac{1}{F - 1} = \frac{A}{G - 1} + B, \tag{18}$$

where $A(\neq 0)$ and B are constants. From (18) it is obvious that F, G share the value 1 CM and hence they share $(1, 2)$. Therefore $n \geq 2k + m + 6$. We now discuss the following three cases separately.

Case 1. Suppose that $B \neq 0$ and $A = B$. Then from (18) we obtain

$$\frac{1}{F - 1} = \frac{BG}{G - 1}. \tag{19}$$

If $B = -1$, then from (19) we obtain $FG = 1$, which is a contradiction as in the proof of Theorem 3.

If $B \neq -1$, from (19), we have $\frac{1}{F} = \frac{BG}{(1+B)G-1}$ and so $\bar{N}(r, \frac{1}{1+B}; G) = \bar{N}(r, 0; F)$. Using (1), (2) and the second fundamental theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r,G) &\leq \bar{N}(r,0;G) + \bar{N}\left(r, \frac{1}{1+B}; G\right) + \bar{N}(r,\infty;G) + S(r,G) \\ &\leq \bar{N}(r,0;F) + \bar{N}(r,0;G) + \bar{N}(r,\infty;G) + S(r,G) \\ &\leq N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1) \\ &\quad -(n+m+1)T(r,g) + S(r,g). \end{aligned}$$

This gives

$$(n+m+1)T(r,g) \leq (k+m+2)\{T(r,f) + T(r,g)\} + S(r,g).$$

Thus we obtain

$$(n-2k-m-3)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction as $n \geq 2k+m+6$.

Case 2. Let $B \neq 0$ and $A \neq B$. Then from (18) we get $F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)}$ and so $\bar{N}(r, \frac{B-A+1}{B+1}; G) = \bar{N}(r, 0; F)$. Proceeding in a manner similar to case 1 we can arrive at a contradiction.

Case 3. Let $B = 0$ and $A \neq 0$. Then from (18) we get $F = \frac{G+A-1}{A}$ and $G = AF - (A-1)$. If $A \neq 1$, it follows that $\bar{N}(r, \frac{A-1}{A}; F) = \bar{N}(r, 0; G)$ and $\bar{N}(r, 1-A; G) = \bar{N}(r, 0; F)$. Now applying Lemma 7 it can be shown that $n \leq 2k+m+3$, which is a contradiction. Thus $A = 1$ and then $F = G$. Now the result follows from the proof of Theorem 3. This completes the proof of Theorem 5.

PROOF OF THEOREM 6. Arguing in a like manner as in the proof of Theorem 5, the conclusion of Theorem 6 follows. Here we omit the details.

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