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Existence Of Nontrivial Solutions For A p-Laplacian Problems With Impulsive Effects*

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Abstract

In this paper, we study a class of p-Laplacian problems with impulsive conditions depending on a real parameter λ. Using variational methods and Bonnano's critical points theorem [4], we give some appropriate conditions on the nonlinear term and the impulsive functions to find a range of the control parameter for which the impulsive problem admits at least one nontrivial solution.

1 Introduction

In this work, we study the existence of nontrivial solutions for the following p-Laplacian problem with the impulsive conditions

$$\begin{cases} -(\rho(x)\phi_p(u'))' + s(x)\phi_p(u) = \lambda f(x,u(x)), \quad \text{a.e. } x \in (a,b), \\ \alpha_1 u'(a^+) - \alpha_2 u(a) = 0, \quad \beta_1 u'(b^-) + \beta_2 u(b) = 0, \\ \Delta(\rho(x_j)\phi_p(u'(x_j))) = \lambda I_j(u(x_j)), \quad j = 1,2,\ldots,n, \end{cases} \tag{1}$$

where $\phi_p(t) = |t|^{p-2}t$ and $a, b \in \mathbb{R}$ with $a < b$, $p > 1$, α_1, α_2, β_1, β_2 and λ are positive constants, $\rho, s \in L^\infty([a,b])$ with $\rho_0 := \operatorname{ess\,inf}_{x\in[a,b]} \rho(x) > 0$, $s_0 := \operatorname{ess\,inf}_{x\in[a,b]} s(x) > 0$, $\rho(a^+) = \rho(a) > 0$, $\rho(b^-) = \rho(b) > 0$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for $j = 1,\ldots,n$, and $x_0 = a < x_1 < x_2 < \cdots < x_n < x_{n+1} = b$. We note that

$$\Delta(\rho(x_j)\phi_p(u'(x_j))) = \rho(x_j^+)\phi_p(u'(x_j^+)) - \rho(x_j^-)\phi_p(u'(x_j^-)),$$

where $z(y^+)$ and $z(y^-)$ denote the right and left limits of $z(y)$ at y respectively.

The theory of impulsive differential equations has become an important area of investigation in the past two decades because of their applications to various problems arising in communications, control technology, electrical engineering, population dynamics, biotechnology processes, chemistry and biology (see [2, 3, 8, 12, 14]).

There have been many papers to study impulsive problems by variational method and critical point theory, we refer the reader to [13, 15, 20, 23] and references cited

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therein. In [19] Tian and Ge obtained sufficient conditions that guarantee the existence of at least two positive solutions of a p-Laplacian boundary value problem with impulsive effects

$$\begin{cases} -(\rho(t)\phi_p(u'(t)))' + s(t)\phi_p(u(t)) = f(t,u(t)), \quad \text{a.e. } t \in (a,b), \\ \alpha u'(a^+) - \beta u(a) = A, \quad \gamma u'(b^-) + \sigma u(b) = B, \\ \Delta(\rho(t_i)\phi_p(u'(t_j))) = I_i(u(t_i)), \quad i = 1,2,\dots,l, \end{cases}$$

where $a, b \in \mathbb{R}$ with $a < b$, $p > 1$, $\phi_p(t) = |t|^{p-2}t$, $\rho, s \in L^\infty([a,b])$ with $\operatorname{ess\,inf}_{t\in[a,b]} \rho(t) > 0$, $\operatorname{ess\,inf}_{t\in[a,b]} s(t) > 0$, $0 < \rho(a), \rho(b) < +\infty$, $A \le 0$, $B \ge 0$, α, β, γ, σ are positive constants, $I_i \in C([0,+\infty),\ [0,+\infty))$ for $i = 1,\dots,l$, $f \in C([a,b] \times [0,+\infty),[0,+\infty))$, $f(t,0) \neq 0$ for $t \in [a,b]$, $t_0 = a < t_1 < t_2 \dots < t_l < t_{l+1} = b$.

In [1], by virtue of Ricceri's three critical points theorem [18], Bai and Dai studied the existence of at least three solutions for the following p-Laplacian impulsive problem

$$\begin{cases} -(\rho(t)\phi_p(u'(t)))' + s(t)\phi_p(u(t)) = \lambda f(t,u(t)), \quad \text{a.e. } t \in (a,b), \\ \alpha_1 u'(a^+) - \alpha_2 u(a) = 0, \quad \beta_1 u'(b^-) + \beta_2 u(b) = 0, \\ \Delta(\rho(t_i)\phi_p(u'(t_j))) = I_i(u(t_i)), \quad i = 1,2,\dots,l. \end{cases}$$

In [5], Bonnano et al. considered the second-order impulsive differential equations with Dirichlet boundary conditions, depending on two real parameters

$$\begin{cases} -u''(t) + a(t)u'(t) + b(t)u(t) = \lambda g(t,u(t)), \quad t \in [0,T], \quad t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \mu I_j(u(t_j)), \quad j = 1,2,...,n, \end{cases}$$

where λ, $\mu > 0$, $g : [0,T]\times\mathbb{R} \to \mathbb{R}$, $a, b \in L^\infty([0,T])$ satisfy the conditions $\operatorname{ess\,inf}_{t\in[0,T]} a(t) \ge 0$, $\operatorname{ess\,inf}_{t\in[0,T]} b(t) \ge 0$, $0 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$, and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for every $j = 1,2,\dots,n$. Under an appropriate growth condition of the nonlinear function, and a small perturbations of impulsive terms, they established the existence of at least three solutions by choosing μ in a suitable way and for every λ lying in a precise interval.

Recently, the authors in [10] studied the following nonlinear perturbed problem

$$\begin{cases} -(\rho(t)\phi_p(u'(t)))' + s(t)\phi_p(u(t)) = \lambda f(t,u(t)) + \mu g(t,u(t)), \quad \text{a.e. } t \in (a,b), \\ \alpha_1 u'(a^+) - \alpha_2 u(a) = 0, \quad \beta_1 u'(b^-) + \beta_2 u(b) = 0, \\ \Delta(\rho(t_i)\phi_p(u'(t_j))) = I_i(u(t_i)), \quad i = 1,2,\dots,l. \end{cases}$$

They utilized Bonnano's theorem [6], to establish precise values of λ and μ for which the above problem admits at least three weak solutions.

Motivated by the above mentioned works, our goal in this paper is to obtain some sufficient conditions to guarantee that problem (1) admits at least one nontrivial solution when the parameter λ lies in different intervals. Our analysis is mainly based on the critical point theorems obtained by Bonanno [4]. This theorem has been used in several works to obtain existence results for different kinds of problems. For review on the subject, we refer the reader to [7, 9, 11].

2 Preliminaries

Our main tools are two consequences of a local minimum theorem ([4], Theorem 3.1) which is a more general version of the Ricceri variational principle (see [17]). Given a set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we put

$$\beta(r_1, r_2) := \inf_{u \in \Phi^{-1}(]r_1, r_2[)} \frac{\left(\sup_{v \in \Phi^{-1}(]r_1, r_2[)} \Psi(v)\right) - \Psi(u)}{r_2 - \Phi(u)}, \tag{2}$$

$$\rho_1(r_1, r_2) := \sup_{u \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(u) - \left(\sup_{v \in \Phi^{-1}(]-\infty, r_1[)} \Psi(v)\right)}{\Phi(u) - r_1}, \tag{3}$$

and

$$\rho(r) := \sup_{u \in \Phi^{-1}(]r, +\infty[)} \frac{\Psi(u) - \left(\sup_{v \in \Phi^{-1}(]-\infty, r[)} \Psi(v)\right)}{\Phi(u) - r} \tag{4}$$

for all $r, r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$.

THEOREM 1 ([4], Theorem 5.1). Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gateaux differentiable function whose Gateaux derivative admits a continuous inverse on X^*, and $\Psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative is compact. Put $I_\lambda = \Phi - \lambda\Psi$ and assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that

$$\beta(r_1, r_2) < \rho_1(r_1, r_2), \tag{5}$$

where β, ρ_1 are given by (2) and (3). Then, for each $\lambda \in \left]\frac{1}{\rho_1(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$, there is a $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

THEOREM 2 ([4], Theorem 5.3). Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative admits a continuous inverse on X^*.$\Psi : X \to \mathbb{R}$ be a continuously Gateaux differentiable function whose Gateaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that

$$\rho(r) > 0, \tag{6}$$

where ρ is given by (4), and for each $\lambda > \frac{1}{\rho(r)}$ the function $I_\lambda = \Phi - \lambda\Psi$ is coercive. Then, for $\lambda > \frac{1}{\rho(r)}$, there is a $u_{0,\lambda} \in \Phi^{-1}(]r, +\infty[)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]r, +\infty[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Let X be the Sobolev space $W^{1,p}([a, b])$ equipped with the norm

$$\|u\| := \left(\int_a^b \rho(x) \left|u'(x)\right|^p dx + \int_a^b s(x)|u(x)|^p dx\right)^{1/p},$$

which is equivalent to the usual one. We define the norm in $C^0([a,b])$ as

$$\|u\|_\infty = \max_{x\in[a,b]} |u(x)|.$$

Since $p > 1$, X is compactly embedded in $C^0([a,b])$.

LEMMA 1 ([19], lemma 2.6). For $u \in X$, we have $\|u\|_\infty \leq M\|u\|$, where

$$M = 2^{1/q} \max\left\{ \frac{(b-a)^{-1/p}}{s_0^{1/p}}, \frac{(b-a)^{1/q}}{\rho_0^{1/p}} \right\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Throughout the sequel, we assume that the functions f and I_j satisfy the following assumptions:

(F) $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function namely: $t \to f(t,x)$ is measurable for every $x \in \mathbb{R}$, $x \to f(t,x)$ is continuous for almost every $t \in [a,b]$, and for every $\varrho > 0$ there exists a function $l_\varrho \in L^1([a,b])$ such that

$$\sup_{|\xi|\leq\varrho} |f(x,\xi) \leq l_\varrho(x) \text{ for a.e. } x \in [a,b].$$

(H) The impulsive functions I_j have sublinear growth, i.e., there exist constants a_j, $b_j > 0$ and $\sigma_j \in [0, p-1)$ such that

$$|I_j(x)| \leq a_j + b_j|x|^{\sigma_j} \quad \text{for all } x \in \mathbb{R},\ j = 1,2,\ldots,n.$$

DEFINITION 1. We say that $u \in X$ is a weak solution of problem (1) if, for $v \in X$,

$$\int_a^b \rho(x)\phi_p(u'(x))v'(x)dx + \int_a^b s(x)\phi_p(u(x))v(x)dx + \rho(a)\phi_p\left(\frac{\alpha_2 u(a)}{\alpha_1}\right)v(a)$$
$$+\rho(b)\phi_p\left(\frac{\beta_2 u(b)}{\beta_1}\right)v(b) - \lambda\left(\int_a^b f(x,u(x))v(x)dx - \sum_{j=1}^n I_j(u(x_j))v(x_j)\right) = 0.$$

Now, Put

$$F(x,\xi) = \int_0^\xi f(x,t)dt \quad \text{for all } (x,\xi) \in [a,b] \times \mathbb{R}.$$

We introduce the functional $I_\lambda : X \to \mathbb{R}$ defined, for each $u \in X$, by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u),$$

where

$$\begin{cases} \Phi(u) = \frac{1}{p}\|u\|^p + \frac{\rho(a)\alpha_2^{p-1}}{p\alpha_1^{p-1}}|u(a)|^p + \frac{\rho(b)\beta_2^{p-1}}{p\beta_1^{p-1}}|u(b)|^p, \\ \Psi(u) = \int_a^b F(x,u)dx - \sum_{j=1}^n \int_0^{u(x_j)} I_j(t)dt. \end{cases} \tag{7}$$

By the property of f and the continuity of I_j $(j = 1, 2, ..., n)$, we have that Φ and Ψ are well defined and Gâteaux differentiable functionals, whose Gâteaux derivatives at $u \in X$ are given by

$$\Phi'(u)v = \int_a^b \rho(x)\phi_p(u'(x))v'(x)dx + \int_a^b s(x)\phi_p(u(x))v(x)dx + \rho(a)\phi_p\left(\frac{\alpha_2 u(a)}{\alpha_1}\right)v(a) + \rho(b)\phi_p\left(\frac{\beta_2 u(b)}{\beta_1}\right)v(b)$$

and

$$\Psi'(u)v = \int_a^b f(x,u)vdx - \sum_{j=1}^n I_j(u(x_j))v(x_j)$$

for all $v \in X$.

We need the following Proposition in the proofs of our main results.

PROPOSITION 1 ([10], Proposition 2.4). Let $T : X \to X^*$ be defined by

$$T(u)h = \int_a^b \rho(x)\phi_p(u'(x))h'(x)dx + \int_a^b s(x)\phi_p(u(x))h(x)dx + \rho(a)\phi_p\left(\frac{\alpha_2 u(a)}{\alpha_1}\right)h(a) + \rho(b)\phi_p\left(\frac{\beta_2 u(b)}{\beta_1}\right)h(b),$$

for every $u, h \in X$. Then the operator T admits a continuous inverse on X^*.

3 Main Results

For the sake of convenience, we put

$$k := \frac{2(p+1)\rho_0}{2^p(p+1)||\rho||_\infty + (p+2)(b-a)^p||s||_\infty} \tag{8}$$

and

$$\Gamma_c := \sum_{j=1}^n \left[\frac{a_j}{c} + \left(\frac{b_j}{\sigma_j+1}\right)c^{\sigma_j-1}\right] \quad \text{and} \quad \mu(d) = \left(\frac{\rho_0 d^p M^p}{k(b-a)^{p-1}}\right)^{2/p},$$

where a_j, b_j, σ_j are given by (H), M is given in Lemma 1 and c, d are two positive constants. Moreover, given a nonnegative constant η and a positive constant θ such that

$$\frac{\eta^p}{M^p} \neq \frac{\rho_0(1+C_1)\theta^p}{k(b-a)^{p-1}}, \text{ where } C_1 = M^p\left(\frac{\rho(a)\alpha_2^{p-1}}{\alpha_1^{p-1}} + \frac{\rho(b)\beta_2^{p-1}}{\beta_1^{p-1}}\right).$$

We set

$$\mathcal{A}_\theta(\eta) := \frac{\int_a^b \max_{|t|\le\eta} F(x,t)dx + \eta^2\Gamma_\eta + \mu(\theta)\Gamma_{\mu(\theta)} - \int_{\frac{a+b}{2}}^b F(t,\theta)\,dt}{\frac{\eta^p}{M^p} - \frac{\rho_0(1+C_1)\theta^p}{k(b-a)^{p-1}}}.$$

THEOREM 3. Assume that there exists a nonnegative constant η_1 and two positive constants η_2 and θ with

$$\eta_1^p < \frac{\rho_0 M^p}{(b-a)^{p-1}}\theta^p < \frac{k}{(1+C_1)}\eta_2^p, \tag{9}$$

such that

(*A*1) $\mathcal{A}_\theta(\eta_2) < \mathcal{A}_\theta(\eta_1)$,

(*A*2) $F(x,t) \geq 0$ for every $(x,t) \in [a, \frac{a+b}{2}] \times [0,\theta]$.

Then, for each $\lambda \in \frac{1}{p}\left]\frac{1}{\mathcal{A}_\theta(\eta_1)}, \frac{1}{\mathcal{A}_\theta(\eta_2)}\right[$, problem (1) admits at least one nontrivial weak solution $\bar{u} \in X$ such that

$$\frac{\eta_1^p}{pM^p} < \Phi(\bar{u}) < \frac{\eta_2^p}{pM^p}.$$

PROOF. Let Φ and Ψ be the functionals defined in (7). It is well known that Φ is coercive and sequentially weakly lower semicontinuous. From Proposition 1, of course, Φ' admits a continuous inverse on X^*. Moreover, Ψ has a compact derivative, it results sequentially weakly continuous. Hence Φ and Ψ satisfy all regularity assumptions requested in Theorem 1. So, our aim is to verify condition (5). To this end, let

$$r_1 = \frac{\eta_1^p}{pM^p}, \quad r_2 = \frac{\eta_2^p}{pM^p}, \quad \text{and} \quad u_0(x) = \begin{cases} \frac{2\theta}{b-a}(x-a), & x \in [a, \frac{a+b}{2}[, \\ \theta, & x \in [\frac{a+b}{2}, b]. \end{cases} \tag{10}$$

Clearly $u_0 \in X$. Moreover, one has

$$\|u_0\|^p = \frac{2^p\theta^p}{(b-a)^p}\int_a^{\frac{a+b}{2}} \rho(x)dx + \frac{2^p\theta^p}{(b-a)^p}\int_a^{\frac{a+b}{2}} (x-a)^p s(x)dx + \theta^p \int_{\frac{a+b}{2}}^b s(x)dx.$$

Using (8), we observe that

$$\frac{\rho_0\theta^p}{(b-a)^{p-1}} \leq \|u_0\|^p \leq \frac{\rho_0\theta^p}{k(b-a)^{p-1}}. \tag{11}$$

From the definition of Φ, we have

$$\frac{1}{p}\|u\|^p \leq \Phi(u) \leq \frac{1}{p}\left(1+C_1\right)\|u\|^p.$$

In particular, we infer

$$\frac{\rho_0\theta^p}{p(b-a)^{p-1}} \leq \Phi(u_0) \leq (1+C_1)\,\frac{\rho_0\theta^p}{pk(b-a)^{p-1}}. \tag{12}$$

Hence, it follows from (9) that

$$r_1 < \Phi(u_0) < r_2.$$

Now, let $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r_2[)$. By Lemma 1, we obtain

$$|u(x)| \leq \eta_2 \quad \text{for each } x \in [a,b]. \tag{13}$$

Moreover, thanks to (H), we get

$$\left| \sum_{j=1}^{n} \int_0^{u(t_j)} I_j(x)\,dx \right| \leq \sum_{j=1}^{n} \left(a_j \|u\|_\infty + \frac{b_j}{\sigma_j+1} \|u\|_\infty^{\sigma_j+1} \right), \tag{14}$$

which combined with (13) yields that

$$\begin{aligned} \sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) &= \sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \left(\int_a^b F(x,u)dx - \sum_{j=1}^{n} \int_0^{u(x_j)} I_j(t)dt \right) \\ &\leq \int_a^b \max_{|t| \leq \eta_2} F(x,t)dx + \sum_{j=1}^{n} \left(a_j \|u\|_\infty + \frac{b_j}{\sigma_j+1} \|u\|_\infty^{\sigma_j+1} \right) \\ &\leq \int_a^b \max_{|t| \leq \eta_2} F(x,t)dx + \eta_2^2 \Gamma_{\eta_2}. \end{aligned} \tag{15}$$

Arguing as before, we obtain

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1[)} \Psi(u) \leq \int_a^b \max_{|t| \leq \eta_1} F(x,t)dx + \eta_1^2 \Gamma_{\eta_1}. \tag{16}$$

On the other hand, due to Lemma 1, (H), $(A2)$ and (11), we have

$$\begin{aligned} \Psi(u_0) &\geq \int_{\frac{a+b}{2}}^b F(t,\theta)\,dt - \sum_{j=1}^{n} \int_0^{u_0(x_j)} I_j(t)dt \\ &\geq \int_{\frac{a+b}{2}}^b F(t,\theta)\,dt - \sum_{j=1}^{n} \left(a_j \|u_0\|_\infty + \frac{b_j}{\sigma_j+1} \|u_0\|_\infty^{\sigma_j+1} \right) \\ &\geq \int_{\frac{a+b}{2}}^b F(t,\theta)\,dt - \mu(\theta)\Gamma_{\mu(\theta)}. \end{aligned} \tag{17}$$

Therefore, from (12) and (15)–(17), we get

$$\begin{aligned} \beta(r_1, r_2) &\leq \frac{\left(\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) \right) - \Psi(u_0)}{r_2 - \Phi(u_0)} \\ &\leq \frac{\int_a^b \max_{|t| \leq \eta_2} F(x,t)dx + \eta_2^2 \Gamma_{\eta_2} + \mu(\theta)\Gamma_{\mu(\theta)} - \int_{\frac{a+b}{2}}^b F(t,\theta)\,dt}{\frac{\eta_2^p}{pM^p} - \frac{\rho_0(1+C_1)\theta^p}{pk(b-a)^{p-1}}} \\ &= p\mathcal{A}_\theta(\eta_2). \end{aligned}$$

We also obtain

$$\begin{aligned}
\rho_1(r_1,r_2) &\geq \frac{\Psi(u_0)-\left(\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)\right)}{\Phi(u_0)-r_1}\\
&\geq \frac{\int_{\frac{a+b}{2}}^{b}F(t,\theta)\,dt-\mu(\theta)\Gamma_{\mu(\theta)}-\eta_1^2\Gamma_{\eta_1}-\int_a^b\max_{|t|\leq\eta_1}F(x,t)dx}{\frac{\rho_0(1+C_1)\theta^p}{pk(b-a)^{p-1}}-\frac{\eta_1^p}{pM^p}}\\
&= p\mathcal{A}_\theta(\eta_1).
\end{aligned}$$

So, by our assumption it follows that $\beta(r_1,r_2)<\rho_1(r_1,r_2)$. Hence, from Theorem 1 for each $\lambda\in\frac{1}{p}\left]\frac{1}{\mathcal{A}_\theta(\eta_1)},\frac{1}{\mathcal{A}_\theta(\eta_2)}\right[$, the functional I_λ admits at least one critical point $\overline{u}$ such that

$$\frac{\eta_1^p}{pM^p}<\Phi(\overline{u})<\frac{\eta_2^p}{pM^p},$$

and the proof of Theorem 3 is achieved.

Now, we point out the following consequence of Theorem 3.

THEOREM 4. Assume that there exist two constants η and θ with

$$\frac{\rho_0M^p}{(b-a)^{p-1}}\theta^p<\frac{k}{(1+C_1)}\eta^p,$$

such that assumption $(A2)$ in Theorem 3 holds. Furthermore, suppose that

$$\frac{\int_a^b\max_{|t|\leq\eta}F(x,t)dx+\eta^2\Gamma_\eta}{\eta^p}<\frac{k(b-a)^{p-1}}{\rho_0M^p(1+C_1)}\frac{\int_{\frac{a+b}{2}}^{b}F(t,\theta)\,dt-\mu(\theta)\Gamma_{\mu(\theta)}}{\theta^p}. \tag{18}$$

Then, for each

$$\lambda\in\frac{1}{p}\left]\frac{\rho_0\theta^pM^p(1+C_1)}{\int_{\frac{a+b}{2}}^{b}F(t,\theta)\,dt-\mu(\theta)\Gamma_{\mu(\theta)}},\frac{\eta^pk(b-a)^{p-1}}{\int_a^b\max_{|t|\leq\eta}F(x,t)dx+\eta^2\Gamma_\eta}\right[,$$

problem (1) admits at least one nontrivial weak solution $\overline{u}$ such that $|\overline{u}(x)|<\eta$ for all $x\in[a,b]$.

PROOF. Our aim is to apply Theorem 3. To this end we pick $\eta_1=0$ and $\eta_2=\eta$.

From (18), one has

$$\begin{aligned}
\mathcal{A}_\theta(\eta) &= \frac{\int_a^b \max_{|t|\le\eta} F(x,t)dx + \eta^2\Gamma_\eta + \mu(\theta)\Gamma_{\mu(\theta)} - \int_{\frac{a+b}{2}}^b F(t,\theta)\,dt}{\left(\dfrac{\eta}{M}\right)^p - \dfrac{\rho_0(1+C_1)\theta^p}{k(b-a)^{p-1}}} \\
&< \frac{\left[1 - \frac{\rho_0\theta^p M^p(1+C_1)}{\eta^p k(b-a)^{p-1}}\right]\left[\int_a^b \max_{|t|\le\eta} F(x,t)dx + \eta^2\Gamma_\eta\right]}{\left(\dfrac{\eta}{M}\right)^p - \dfrac{\rho_0(1+C_1)\theta^p}{k(b-a)^{p-1}}} \\
&= \frac{\int_a^b \max_{|t|\le\eta} F(x,t)dx + \eta^2\Gamma_\eta}{\left(\dfrac{\eta}{M}\right)^p} < \frac{\int_{\frac{a+b}{2}}^b F(t,\theta)\,dt - \mu(\theta)\Gamma_{\mu(\theta)}}{\dfrac{\rho_0(1+C_1)\theta^p}{k(b-a)^{p-1}}} \\
&= \mathcal{A}_\theta(0).
\end{aligned}$$

Hence, Theorem 3 ensures the existence of nontrivial weak solution $\overline{u}$ of problem (1) such that

$$\frac{1}{p}\|\overline{u}\|^p \le \Phi(\overline{u}) < \frac{\eta^p}{pM^p},$$

and clearly by Lemma 1, $|\overline{u}(x)| < \eta$ for all $x \in [a,b]$.

Finally, we also give an application of Theorem 2.

THEOREM 5. Assume that there exist two constants $\bar{\eta}$ and $\bar{\theta}$ with

$$\bar{\eta}^p < \frac{\rho_0 M^p}{(b-a)^{p-1}}\bar{\theta}^p,$$

such that

$$\int_a^b \max_{|t|\le\bar{\eta}} F(x,t)dx + \bar{\eta}^2\Gamma_{\bar{\eta}} < \int_{\frac{a+b}{2}}^b F(x,\bar{\theta})dx - \mu(\bar{\theta})\Gamma_{\mu(\bar{\theta})}, \tag{19}$$

and

$$\limsup_{|\xi|\to+\infty} \frac{F(x,\xi)}{|\xi|^p} \le 0 \text{ uniformly in } x. \tag{20}$$

Then, for each $\lambda > \overline{\lambda}$, where

$$\overline{\lambda} = \frac{\frac{\bar{\eta}^p}{M^p} - \dfrac{\rho_0(1+C_1)\bar{\theta}^p}{k(b-a)^{p-1}}}{p\left[\int_a^b \max_{|t|\le\bar{\eta}} F(x,t)dx + \bar{\eta}^2\Gamma_{\bar{\eta}} + \mu(\bar{\theta})\Gamma_{\mu(\bar{\theta})} - \int_{\frac{a+b}{2}}^b F(x,\bar{\theta})dx\right]},$$

problem (1) admits at least one nontrivial weak solution $\overline{u}$ such that $||\overline{u}|| > \frac{\bar{\eta}}{M(1+C_1)^{1/p}}$.

PROOF. The functionals Φ and Ψ given by (7) satisfy all regularity assumptions requested in Theorem 2. Moreover, by standard computations, condition (20) implies that I_λ, $\lambda > 0$, is coercive. To apply Theorem 2, it suffices to verify condition (6). Indeed, put $u_0(x)$ as in (10) and $r = \frac{\bar{\eta}^p}{pM^p}$. Arguing as in the proof of Theorem 3 we obtain

$$\begin{aligned} \rho(r) &\geq \frac{\Psi(u) - \left(\sup_{v\in\Phi^{-1}(]-\infty,r[)} \Psi(v)\right)}{\Phi(u) - r} \\ &\geq \frac{\int_{\frac{a+b}{2}}^{b} F(t,\bar{\theta})\,dt - \mu(\bar{\theta})\Gamma_{\mu(\bar{\theta})} - \bar{\eta}^2\Gamma_{\bar{\eta}} - \int_a^b \max_{|t|\leq\bar{\eta}} F(x,t)dx}{\dfrac{\rho_0(1+C_1)\bar{\theta}^p}{pk(b-a)^{p-1}} - \frac{\bar{\eta}^p}{pM^p}}. \end{aligned}$$

So, from our assumption it follows that $\rho(r) > 0$. Hence, in view of Theorem 2 for each $\lambda > \overline{\lambda}$, I_λ admits at least one local minimum $\overline{u}$ such that

$$\frac{\bar{\eta}^p}{pM^p} < \Phi(\overline{u}) \leq \frac{1}{p}(1+C_1)||\overline{u}||^p,$$

and our conclusion is achieved.

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References

[1] L. Bai and B. Dai, Three solutions for a p-Laplacian boundary value problem with impulsive effects, Appl. Math. Comput., 217(2011), 9895–9904.

[2] D. Bainov and P. Simeonov, Systems with Impulse Effect, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, 1989.

[3] M. Benchohra, J. Henderson and S. Ntouyas, Theory of Impulsive Differential Equations, Contemporary Mathematics and Its Applications, 2. Hindawi Publishing Corporation, New York, (2006).

[4] G. Bonanno, A critical point theorem via the Ekeland variational principle, Nonlinear Anal., 75 (2012) 2992–3007.

[5] G. Bonanno, B. Di Bella and J. Henderson, Existence of solutions to second-order boundary-value problems with small perturbations of impulses, Electron. J. Differential Equations, 2013(2013), 1–14.

[6] G. Bonanno and S. A. Marano, On the structure of the critical set of non-differentiable functions with a weak compactness condition, Appl. Anal., 89(2010), 1–10.

[7] G. Bonanno and A. Sciammetta, Existence and multiplicity results to Neumann problems for elliptic equations involving the p-Laplacian, J. Math. Anal. Appl., 390(2012), 59–67.

[8] T. E. Carter, Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion, Dyn. Control, 10(2000), 219–227.

[9] M. Ferrara, S. Khademloo and S. Heidarkhani, Multiplicity results for perturbed fourth-order Kirchhoff type elliptic problems, Appl. Math. Comput., 234(2014), 316–325.

[10] M. Ferrara and S. Heidarkhani, Multiple solutions for perturbed p-Laplacian boundary-value problems with impulsive effects, Electron. J. Differential Equations, 2014(2014), 1–14.

[11] El. M. Hssini, M. Massar and N. Tsouli, Existence and multiplicity of solutions for a $p(x)$-Kirchhoff type problems, Bol. Soc. Paran. Mat., 33(2015), 201–215.

[12] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics, vol.6, World Scientific, Teaneck, NJ, 1989.

[13] X. N. Lin and D. Q. Jiang, Multiple positive solutions of Dirichlet boundary value problems for second order impulsive differential equations, J. Math. Anal. Appl., 321(2006), 501–514.

[14] X. Liu and A. R. Willms, Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft, Math. Probl. Eng., 2(1996), 277–299.

[15] J. J. Nieto and D. O'Regan, Variational approach to impulsive differential equations, Nonlinear Anal. Real World Appl., 10(2009), 680–690.

[16] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., Vol. 65, Amer. Math. Soc., Providence, RI, 1986.

[17] B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math., 133(2000), 401–410.

[18] B. Ricceri, On a three critical points theorem, Arch. Math., 75(2000), 220–226.

[19] Y. Tian and W. Ge, Applications of variational methods to boundary value problem for impulsive differential equations, Proc. Edinburgh Math. Soc., 51(2008), 509–527.

[20] Y. Tian and W. Ge, Variational methods to Sturm-Liouville boundary value problem for impulsive differential equations, Nonlinear Anal., 72(2010), 277–287.

[21] M. Xiang, B. Zhang and M. Ferrara, Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian, J. Math. Anal. Appl., 424(2015), 1021–1041.

[22] J. Xu, Z. Wei and Y. Ding, Existence of weak solutions for p-Laplacian problem with impulsive effects, Taiwanese J. Math., 17(2013), 501–515.

[23] D. Zhang and B. Dai, Existence of solutions for nonlinear impulsive differential equations with Dirichlet boundary conditions, Math. Comput. Modelling, 53(2011), 1154–1161.