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# On The Spectrum Of $(p(x), q(x))$-Laplacian In $\mathbf{R}^N$*

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**Abstract**

In this work, we study the existence of a family of eigenvalues for a nonhomogeneous problem involving variable exponents in $\mathbf{R}^N$ by using the variational approach.

## 1 Introduction

Large number of papers was devoted to study elliptic equations and variational problems with variable exponent. They were of considerable importance in the theory of partial differential equations. Some of these problems come from different areas of applied mathematics and physics such as Micro Electro-Mechanical systems, surface diffusion on solids or image processing and restoration. For more inquiries on modeling physical phenomena involving $p(x)$-growth condition, we refer to [1–10].

Motivated by the works [11, 12], we consider the following problem

$$-\Delta_{p(x)}u = \Delta_{q(x)}u + \lambda m(x)\,|u|^{r(x)-2}\,u \quad \text{in} \quad \mathbf{R}^N, \tag{1}$$

where

$$\Delta_{p(x)}u = div(|u|^{p(x)-2}u)$$

is the $p(x)$-Laplacian operator and $p, q : \mathbf{R}^N \longrightarrow \mathbf{R}$ are Lipschitz continuous functions with

$$1 < p^- := \inf_{\mathbf{R}^N} p(x) \le p(x) \le \sup_{\mathbf{R}^N} p(x) := p^+ < N, \quad N \ge 3,$$

where $m$ is a positive weight for a.e $x \in \mathbf{R}^N$ such that

$$m \in L^\infty(\mathbf{R}^N) \cap L^{\gamma(x)}(\mathbf{R}^N)$$

with

$$\gamma(x) = \frac{q^*(x)}{q^*(x) - r(x)}.$$

Suppose that

$$1 < q(x) < r^- = \inf_{\mathbf{R}^N} r(x) \le r^+ = \sup_{\mathbf{R}^N} r(x) < p(x) \tag{2}$$

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and

$$r^+ < q^*(x) \quad \text{with} \quad q^*(x) = \frac{Nq(x)}{N - q(x)}. \tag{3}$$

Define

$$\lambda_1 = \inf_{u \in W^{1,p(x)}(\mathbf{R}^N) \cap W^{1,q(x)}(\mathbf{R}^N), u \neq 0} \frac{\int_{\mathbf{R}^N} \left[\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{1}{q(x)} |\nabla u|^{q(x)}\right] dx}{\int_{\mathbf{R}^N} \frac{m(x)}{r(x)} |u|^{r(x)} dx},$$

and

$$\bar{\lambda} = \inf_{u \in W^{1,p(x)}(\mathbf{R}^N) \cap W^{1,q(x)}(\mathbf{R}^N), u \neq 0} \frac{\int_{\mathbf{R}^N} \left[|\nabla u|^{p(x)} + |\nabla u|^{q(x)}\right] dx}{\int_{\mathbf{R}^N} m(x) |u|^{r(x)} dx}.$$

We state our main result.

THEOREM 1. Assume that conditions (2) and (3) hold. Then we have:

(i) $\lambda_1 > 0$ and $\lambda \in [\lambda_1, +\infty[$ is an eigenvalue of the problem (1).

(ii) There exists $\bar{\lambda} \in ]0, \lambda_1]$ such that $\lambda \in ]0, \bar{\lambda}[$ is not an eigenvalue of problem (1).

DEFINITION 1. We say that $\lambda \in \mathbf{R}$ is an eigenvalue of the problem (1) if there exists $u \in W^{1,p(x)}(\mathbf{R}^N) \cap W^{1,q(x)}(\mathbf{R}^N) \setminus \{0\}$ such that

$$\int_{\mathbf{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla v dx + \int_{\mathbf{R}^N} |\nabla u|^{q(x)-2} \nabla u \nabla v dx = \lambda \int_{\mathbf{R}^N} m(x) |u|^{r(x)-2} uv dx,$$

for all $v \in W^{1,p(x)}(\mathbf{R}^N) \cap W^{1,q(x)}(\mathbf{R}^N) \setminus \{0\}$.

When $p(x) = q(x) = 2$ and $m(x) = 1$, the problem (1) is a normal Schrodinger equation (see [13, 14]). The case $m = 1$ and $p(x) = q(x)$ in a bounded domain has been studied by Fan, Zhang and Zhao in [15, 16]. The case of an indefinite weight $m \neq 1$ in $\mathbf{R}^N$ with $p(x) = q(x)$ was considered by [11].

This article is organized as follows. In section 2, we give the necessary notations and preliminaries. We include some useful results involving the variable exponents Lebesgue and Sobolev spaces in order to facilitate the reading of the paper. Finally, in section 3, we prove the main result.

## 2 Preliminary Notes

In order to deal with the problem (1), we need some theory of variable exponent Sobolev Space. For convenience, we only recall some basic facts which will be used later.

Define the variable exponent Lebesgue space $L^{p(x)}(\mathbf{R}^N)$ by

$$L^{p(x)}(\mathbf{R}^N) = \left\{u : \mathbf{R}^N \to \mathbf{R} \text{ measurable} : \int_{\mathbf{R}^N} |u|^{p(x)} dx < \infty\right\}.$$

Then $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\mathbf{R}^N} \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

becomes a separable and reflexive Banach space.

Define the variable exponent Sobolev space $W^{1,p(x)}(\mathbf{R}^N)$ by

$$W^{1,p(x)}(\mathbf{R}^N) = \{u \in L^{p(x)}(\mathbf{R}^N) : \nabla u \in L^{p(x)}(\mathbf{R}^N)\}$$

equipped with the norm

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_{\mathbf{R}^N} \left| \frac{\nabla u}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

which is also a separable and reflexive Banach space.

PROPOSITION 1 (cf. [17]). Let $\rho(u) = \int_{\mathbf{R}^N} |\nabla u|^{p(x)} dx$ and $u \in W^{1,p(x)}(\mathbf{R}^N)$. Then the following statements hold.

(1) If $\|u\|_{p(x)} \geq 1$, then $\|u\|_{p(x)}{}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}{}^{p^+}$.

(2) If $\|u\|_{p(x)} \leq 1$, then $\|u\|_{p(x)}{}^{p^+} \leq \rho(u) \leq \|u\|_{p(x)}{}^{p^-}$.

(3) $\lim_{n\to\infty} \|u_n\|_{p(x)} = 0$ (resp $+\infty$) if, and only if, $\lim_{n\to\infty} \rho(u_n) = 0$ (resp $+\infty$).

REMARK 1. We have similar results (1) and (2) of Propositiion 1 for $\rho_1(u) = \int_{\mathbf{R}^N} |u|^{p(x)} dx$.

PROPOSITION 2 (cf. [17]). For any $u \in L^{p(x)}(\mathbf{R}^N)$ and $v \in L^{p'(x)}(\mathbf{R}^N)$, we have

$$\left| \int_{\mathbf{R}^N} uv dx \right| \leq 2 \, |u|_{p(x)} \, |v|_{p'(x)}$$

with

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

PROPOSITION 3 (cf. [17]). Suppose that $p$ is Lipschitz continuous, $q : \mathbf{R}^N \to \mathbf{R}$ is a measurable function and $p(x) \leq q(x) \leq p^*(x)$, $\forall x \in \mathbf{R}^N$. Then there is a continuous embedding

$$W^{1,p(x)}(\mathbf{R}^N) \hookrightarrow L^{q(x)}(\mathbf{R}^N).$$

If $\Omega$ is a bounded open subset of $\mathbf{R}^N$ with cone property, then the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$$

is compact.

In what follows, for simplicity let

$$W = W^{1,p(x)}(\mathbf{R}^N) \cap W^{1,q(x)}(\mathbf{R}^N),$$

which will be endowed with the following norm

$$\|u\| = \|u\|_{p(x)} + \|v\|_{q(x)}\,,$$

which makes $W$ a Banach space separable and reflexive and denote by $\int = \int_{\mathbf{R}^N}$. The assertions of Proposition 1 remain valid with the norm $\|\cdot\|$.

PROPOSITION 4 (cf. [5]). Suppose that $p, q$ are are Lipschitz continuous functions and $r$ is a measurable function. If $q(x) \le r(x) \le p(x)$, then $L^{p(x)}(\mathbf{R}^N) \cap L^{q(x)}(\mathbf{R}^N) \hookrightarrow L^{r(x)}(\mathbf{R}^N)$ with a continuous embedding.

Analogously, we have the following interesting result (see [5]).

PROPOSITION 5. Under the assumptions of Proposition 4, we have a continuous embedding from $W$ into $L^{r(x)}(\mathbf{R}^N)$.

# 3 Proof of the Main Result

First, by the assumption (2) we can see that

$$|\nabla u|^{p(x)} + |\nabla u|^{q(x)} \ge |\nabla u|^{r^+}, \quad |\nabla u|^{p(x)} + |\nabla u|^{q(x)} \ge |\nabla u|^{r^-}$$

and

$$|u|^{r^+} + |u|^{r^-} \ge |u|^{r(x)}.$$

The continuous embedding from $W^{1,r^i}(\mathbf{R}^N)$ into $L^{r^i}(\mathbf{R}^N)$, $i = \pm$, implies that there exist two positive constants $C_1$, $C_2$ such that

$$\int |\nabla u|^{r^+}\,dx \ge C_1 \int |u|^{r^+}\,dx \text{ and } \int |\nabla u|^{r^-}\,dx \ge C_1 \int |u|^{r^-}\,dx.$$

Then there exists $C > 0$ such that

$$2\int \left[|\nabla u|^{p(x)} + |\nabla u|^{q(x)}\right] dx \ge C \int \left(|u|^{r^+} + |u|^{r^-}\right) dx \ge \frac{C}{|m|_\infty} \int m(x)\,|u|^{r(x)}\,dx$$

and thus

$$\int p^+ \left(\frac{1}{p(x)}\,|\nabla u|^{p(x)} + \frac{1}{q(x)}\,|\nabla u|^{q(x)}\right) dx \ge \frac{Cr^-}{2|m|_\infty} \int \frac{m(x)}{r(x)}\,|u|^{r(x)}\,dx.$$

So $\lambda_1 > 0$. Similarly, we get $\bar{\lambda} > 0$.

Before embarking in the proof of Theorem 1, we need some auxiliar lemmas. Let

$$I(u) = \int \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int \frac{1}{q(x)} |\nabla u|^{q(x)} \, dx$$

and

$$J(u) = \int \frac{1}{r(x)} m(x) \, |u|^{r(x)} \, dx.$$

The following lemma plays a crucial role in our different lines.

LEMMA 1.

(i) The functional $I$ is weakly lower semi-continuous, that is, $u_n \rightharpoonup u$ implies that $I(u) \leq \liminf_n I(u_n)$.

(ii) The functional $J$ is weakly-strongly continuous, that is, $u_n \rightharpoonup u$ implies that $J(u_n) \to J(u)$.

PROOF. (i) Using the convexity of the functional $I$, we see that the assertion is immediate. (ii) Assume that $u_n \rightharpoonup u$ in $W$, which is reflexive. Then $\{u_n\}$ is a bounded sequence. By Proposition 3, we have

$$W \hookrightarrow L^{q^*(x)}(\mathbf{R}^N).$$

So we obtain a boundedness of $\left\{ |u_n|_{q^*(x)} \right\}$. So, there is a positive constant $M > 0$ such that

$$\max_n \left\{ \left| |u_n|^{r(x)} \right|_{\frac{q^*(x)}{r(x)}}, \left| |u|^{r(x)} \right|_{\frac{q^*(x)}{r(x)}} \right\} \leq M.$$

Taking $\Omega_k = \{x \in \mathbf{R}^N : |x| < k\}$. Further, $m \in L^{\gamma(x)}(\mathbf{R}^N)$ implies that

$$|m|_{L^{\gamma(x)}(\mathbf{R}^N \setminus \Omega_k)} \to 0 \text{ as } k \to +\infty.$$

Giving $\varepsilon > 0$, we may find $k_1 > 0$ large enough such that

$$|m|_{L^{\gamma(x)}(\mathbf{R}^N \setminus \Omega_{k_1})} < \frac{\varepsilon}{8M}.$$

It follows from the compact embedding $W^{1,r(x)}(\Omega_{k_1}) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega_{k_1})$, that

$$\int_{\Omega_{k_1}} m(x) |u_n|^{r(x)}) dx \to \int_{\Omega_{k_1}} m(x) |u|^{r(x)} dx,$$

because $m \in L^\infty(\mathbf{R}^N)$. Hence, there exists $n_1 > 0$ such that for $n \geq n_1$,

$$\left| \int_{\Omega_{k_1}} m(x) |u_n|^{r(x)} dx - \int_{\Omega_{k_1}} m(x) |u|^{r(x)} dx \right| < \frac{\varepsilon}{2}.$$

In view of Proposition 2, we get

$$\begin{aligned}
|J(u_n)-J(u)| &\leq \left|\int_{\Omega_{k_1}}\left[m(x)|u_n|^{r(x)}-m(x)|u|^{r(x)}\right]dx\right.\\
&\quad \left.+\int_{\mathbf{R}^N\setminus\Omega_{k_1}}[m(x)|u_n|^{r(x)}-m(x)|u|^{r(x)}]dx\right|\\
&\leq \frac{\varepsilon}{2}+\int_{\mathbf{R}^N\setminus\Omega_{k_1}}m(x)\left[|u_n|^{r(x)}+|u|^{r(x)}\right]dx\\
&\leq \frac{\varepsilon}{2}+2|m|_{L^{\gamma(x)}(\mathbf{R}^N\setminus\Omega_{k_1})}\left[\left||u_n|^{r(x)}\right|_{\frac{q^*(x)}{r(x)}}+\left||u|^{r(x)}\right|_{\frac{q^*(x)}{r(x)}}\right]\\
&\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

Consequently, $J(u_n)\to J(u)$.

LEMMA 2. We have the following limits,

$$\lim_{\|u\|\to+\infty}\frac{I(u)}{J(u)}=+\infty$$

and

$$\lim_{\|u\|\to 0}\frac{I(u)}{J(u)}=+\infty. \tag{4}$$

PROOF. When $\|u\|\to 0$, we have $\|u\|_{q(x)}\to 0$. The preceding Proposition 1 together with Proposition 4 and 5 imply

$$\frac{I(u)}{J(u)}=\frac{\int\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx+\int\frac{1}{q(x)}|\nabla u|^{q(x)}\,dx}{\int m(x)\frac{1}{r(x)}|u|^{r(x)}\,dx}\geq\frac{\frac{1}{q^+}\|u\|_{q(x)}^{q^+}}{c\,|m|_\infty\,\|u\|_{q(x)}^{r^\pm}},$$

where $c>0$. Since $r^->q^+$, then the first relation hold. For the seconde assertion, let

$$K(u)=\int\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx.$$

If $\|u\|_{p(x)}\geq 1$, then we have

$$\frac{1}{p^+}\|u\|_{p(x)}^{p^-}\leq K(u)\leq\frac{1}{p^-}\|u\|_{p(x)}^{p^+}.$$

If $\|u\|_{p(x)}\leq 1$, then we have

$$\frac{1}{p^+}\|u\|_{p(x)}^{p^+}\leq K(u)\leq\frac{1}{p^-}\|u\|_{p(x)}^{p^-}.$$

In this case, we set

$$C_0 \geq \frac{1}{p^+} \|u\|_{p(x)}^{p^-} - \frac{1}{p^+} \|u\|_{p(x)}^{p^+} \geq 0.$$

Then

$$K(u) \geq \frac{1}{p^+} \|u\|_{p(x)}^{p^-} - C_0.$$

We entail that

$$K(u) \geq \frac{1}{p^+} \|u\|^{p^-} - C_0, \ \forall u \in W^{1,p(x)}(\mathbf{R}^N).$$

Thereby, there exists $C_1 > 0$ such that

$$\begin{aligned} I(u) &= \int \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int \frac{1}{q(x)} |\nabla u|^{q(x)} \, dx \\ &\geq \frac{1}{p^+} \|u\|_{p(x)}^{p^-} + \frac{1}{q^+} \|u\|_{q(x)}^{q^-} - C_1, \end{aligned}$$

whenever $u \in W$.

On the other hand, according to Proposition 4 there exists $C_3 > 0$ such that

$$J(u) = \int \frac{1}{r(x)} m(x) \, |u|^{r(x)} \, dx \leq C_3(\|u\|^{r^+} + \|u\|^{r^-}).$$

Afterwards, we have

$$\begin{aligned} \frac{I(u)}{J(u)} &= \frac{\int \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + \int \frac{1}{q(x)} |\nabla u|^{q(x)} \, dx}{\int \frac{1}{r(x)} m(x) \, |u|^{r(x)} \, dx} \\ &\geq \frac{\frac{1}{p^+} \|u\|_{p(x)}^{p^-} + \frac{1}{q^+} \|u\|_{q(x)}^{q^-} - C_1}{C_3(\|u\|^{r^+} + \|u\|^{r^-})}, \end{aligned}$$

since $r^+ < p^-$, we infer that $\frac{I(u)}{J(u)} \to +\infty$ when $\|u\| \to +\infty$.

**Proof of Theorem 1.** We show that $\lambda_1$ is an eigenvalue of (1). By the definition of $\lambda_1$, there exists $\{u_n\} \subset W \setminus \{0\}$ such that

$$\lambda_1 = \lim_n \frac{I(u_n)}{J(u_n)}.$$

Here $\{u_n\}$ is bounded in $W$, because the coercivity of $\dfrac{I(u_n)}{J(u_n)}$, and thus there exists $u \in W$ satisfying $u_n \rightharpoonup u$, so

$$I(u) \leq \liminf_n I(u_n) \ \text{ and } J(u_n) \to J(u).$$

Whether $u \neq 0$, then $\lambda_1 = \frac{I(u)}{J(u)}$ and it is done. Otherwise, assume that $u = 0$, so $I(u_n) \to 0$ as $n \to +\infty$. We have $I(u_n) = \frac{I(u_n)}{J(u_n)} J(u_n)$, passing to limit we get

$I(u_n) \to 0$ as $n \to +\infty$, along with $\|u_n\| \to 0$, hence $\frac{I(u_n)}{J(u_n)} \to +\infty$ which is paradoxical. We conclude that for any $v \in W$ we have

$$\left.\frac{\partial}{\partial \varepsilon} \frac{I(u+\varepsilon v)}{J(u+\varepsilon v)}\right|_{\varepsilon=0} = 0.$$

By a straightforward computation we have

$$\left[\int \left(|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2}\right) \nabla u \nabla v dx\right] J(u) = \left[\int m(x)|u|^{r(x)} uv dx\right] I(u), \ \forall v \in W.$$

Consequently, $\lambda_1$ is an eigenvalue of problem (1).

Now, we are to prove that any $\lambda > \lambda_1$ is an eigenvalue of (1). Set

$$L(u) = I(u) - \lambda J(u).$$

From Lemma 1, we know that I is weakly lower semi-continuous and J is weakly-strongly continuous, which yields $L$ is weakly lower semicontinuous.

If we return to relation (4), it is clear that L is of class $C^1$ and coercive, therefore, $L$ admits a global minimum $u_*$ in $W$, which is a critical point for L. We claim that $u_*$ is nontrivial. Indeed, by the characterization of $\lambda_1$, there exists $v \in W \setminus \{0\}$ such that $\lambda_1 = \frac{I(v)}{J(v)}$, since $\lambda > \lambda_1$, accordingly $L(v) < 0$, thence $u_* \neq 0$.

Hereinafter, we check that any $\lambda \in ]0, \bar{\lambda}[$ is not an eigenvalue of problem (1). First, we may observe that $\bar{\lambda} \leq \lambda_1$, because $\bar{\lambda} \leq \frac{r^+}{p^-}\lambda_1$ and $r^+ < p^-$.

Next, let $\lambda \in ]0, \bar{\lambda}[$, by contradiction we assume that $\lambda$ is an eigenvalue, then there exists $v \in W \setminus \{0\}$ which satisfies

$$\int \left(|\nabla v|^{p(x)} + |\nabla v|^{q(x)}\right) dx = \lambda \int m(x)\, |u|^{r(x)}\, dx.$$

On the other hand, according to definition of $\bar{\lambda}$ we obtain

$$\bar{\lambda} \leq \frac{\int \left(|\nabla v|^{p(x)} + |\nabla v|^{q(x)}\right) dx}{\int m(x)\, |v|^{r(x)}\, dx} = \lambda,$$

which is contradictory and then the proof is achieved.

REMARK 2. When $\lambda \leq 0$, for all $u \in W \setminus \{0\}$ we have

$$I'_\lambda(u)u = \int |\nabla u|^{p(x)}\, dx + \int |\nabla u|^{q(x)}\, dx - \lambda \int m(x)\, |u|^{r(x)}\, dx > 0,$$

and then the problem (1) has no solution.

Whether $m < 0$, we may consider that the eigenvalue $\lambda < 0$, and the similar argument works.

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