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Characterization Of Fuzzy b-Complete Set*

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Abstract

In this paper, we have dealt with the concept of fuzzy b-closed and fuzzy b-compact set, and we study the characterization of fuzzy b-compact set by using fuzzy filterbases. We consider some relations between the fuzzy b-compact space and fuzzy b-closed space. Finally, the concept of bounded and semi bounded operators between fuzzy normed space are studied and some immediate results are also proved.

1 Introduction and Preliminaries

In 1968, C. L. Chang [2], introduced and developed the concept of topological spaces based on the concept of fuzzy set introduced by L. A. Zadeh in this classical paper [17]. The theory of fuzzy topological spaces was subsequently developed by several authors [7, 8, 9]. b-Closedness occupies a very important place in fuzzy topology and so do some of its forms. In [15], the authors introduced the notion of b-closed spaces and investigated its fundamental properties and H. R. Moradi in [12] introduced these concepts in fuzzy setting. In [11], some interesting properties of fuzzy b-closed space are investigated. The notions of fuzzy vector spaces and fuzzy topological vector spaces were introduced in Katsaras and Liu [6]. These ideas were modified by Katsaras [4], and in [5] Katsaras defined the fuzzy norm on a vector space. In [8], Krishna and Sarma discussed the generation of a fuzzy vector topology from an ordinary vector topology on vector spaces. Also Krishna and Sarma [7] observed the convergence of sequence of fuzzy points. R. G. Seob et al. [16] introduced the notion of fuzzy α-Cauchy sequence of fuzzy points and fuzzy completeness. In this work, we investigate some more properties of this type of closed spaces. We also explore some expected basic properties of these concepts.

This paper includes three sections. In the first section we recall the basic concepts of fuzzy topological spaces. Some new notions of fuzzy b-closedness and fuzzy b-compactness have been introduced by using the fuzzy filterbases. We discuss their main properties in section 2. Some characterizations of bounded and semi bounded inverse theorems are provided in section 3.

Throughout this paper, X and Y mean fuzzy topological spaces (fts). The notions $Cl\,(A)$ will stand for the fuzzy closure of a fuzzy set A in a fts X. Support of a fuzzy set A in X will be denoted by $S\,(A)$. The fuzzy sets in taking on respectively the constant

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value 0 and 1 are denoted by 0_x and 1_x respectively. For the sake of completeness, we reproduce the following definitions and preliminaries result due to Kastras, which will be needed in the sequel.

DEFINITION 1 ([7]). For two fuzzy subsets μ_1 and μ_2 of X, the fuzzy subset $\mu_1 + \mu_2$ is defined by

$$(\mu_1 + \mu_2)(x) = \vee \{\mu_1(x_1) \wedge \mu_1(x_2) | x = x_1 + x_2\}.$$

And for a scalar t of K and a fuzzy subset μ of X, the fuzzy subset $t\mu$ is defined by

$$(t\mu)(x) = \begin{cases} \mu(x/t) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0 \text{ and } x \neq 0, \\ \vee \{f(y) : y \in X\} & \text{if } t = 0 \text{ and } x = 0. \end{cases}$$

DEFINITION 2 ([6]). $\mu \in I^x$ is said to be

1. convex if $t\mu + (1-t)\mu \subseteq \mu$ for each $t \in [0,1]$.
2. balanced if $t\mu \subseteq \mu$ for each $t \in K$ with $|t| \leq 1$.
3. absorbing if $\vee \{t\mu(x) | t > 0\} = 1$ for all $x \in X$.

DEFINITION 3 ([6]). Let (X, τ) be a topological space and let

$$\omega(\tau) = \{f : (X, \tau) \to [0,1] \mid f \text{ is lower semicontinuous}\}.$$

Then $\omega(\tau)$ is a fuzzy topology on X. This topology is called the fuzzy topology generated by τ on X. The fuzzy usual topology on K means the fuzzy topology generated by the usual topology of K.

DEFINITION 4 ([6]). A fuzzy linear topology on a vector space X over K is a fuzzy topology on X such that the two mappings

$$+ : X \times X \to X, \ (x, y) \to x + y$$

and

$$\cdot : K \times X \to X, \ (t, x) \to tx,$$

are continuous when K has the fuzzy usual topology and $K \times X$ and $X \times X$ have the corresponding product fuzzy topologies. A linear space with a fuzzy linear topology is called a fuzzy topological linear space or a fuzzy topological vector space.

DEFINITION 5. Let x be a point in a fuzzy topological space X. A family F of neighborhood of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0 < \theta < \mu(x)$, there exists $\mu_1 \in F$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

DEFINITION 6 ([9]). A fuzzy semi norm on X is a fuzzy set ρ in X which is convex, balanced and absorbing. If in addition $\wedge\left\{\left(t_{\rho}\right)(x)\,|t>0\right\}$ for $x \neq 0$, then ρ is called a fuzzy norm.

DEFINITION 7 ([9]). If ρ is a fuzzy semi norm on X, then the family

$$B_{\rho}=\left\{\theta \wedge\left(t_{\rho}\right)|0<\theta \leq 1, t>0\right\},$$

is a base at zero for a fuzzy linear topology τ_{ρ}. The fuzzy topology τ_{ρ} is called the fuzzy topology induced by the fuzzy semi norm ρ. And a linear space equipped with a fuzzy semi norm is called a fuzzy semi normed linear space.

DEFINITION 8. Let ρ be a fuzzy semi norm on X. $P_{\varepsilon}: X \rightarrow R_{+}$ is defined by

$$P_{\varepsilon}(x)=\wedge\{t>0 | t \rho(x)>\varepsilon\}$$

for each $\varepsilon \in(0,1)$.

THEOREM 1. The P_{ε} is a semi norm on X for each $\varepsilon \in(0,1)$. Further P_{ε} is norm on X for each $\varepsilon \in(0,1)$ if, and only if, ρ is a fuzzy norm on X.

We refer the reader to [1] for undefined notions on fuzzy normed spaces.

2 Fuzzy b-Closed Space and Fuzzy b-Compact Space

In order to derive our main results, we need the following definitions.

DEFINITION 9 ([3]). A family Ω of a fuzzy sets is called a cover of a fuzzy set A if and only if $A \leq \vee\left\{\alpha_{i}: i \in \Omega\right\}$, and it is called a fuzzy b-open cover if each member α_{i} is a fuzzy b-open set. A subcover of A is a subfamily of Ω which is also cover of A.

DEFINITION 10 ([12]). A fuzzy set A in a fts X is said to be fuzzy b-open set if and only if $A \leq(\operatorname{int}(cl(A)) \vee cl(\operatorname{int}(A)))$.

DEFINITION 11 ([12]). A fts X is said to be fuzzy b-closed if and only if for every family λ of fuzzy b-open set such that $\underset{A \in \lambda}{\vee} A=1_{x}$ there is a finite subfamily $\delta \subseteq \lambda$ such that $\left(\underset{A \in \delta}{\vee} bCl(A)\right)(x)=1_{x}$, for every $x \in X$.

DEFINITION 12. A fuzzy set U in a fts X is said to be fuzzy b-closed relative to X if and only if for every family λ of fuzzy b-open set such that $\underset{A \in \lambda}{\vee} A=1_{x}$ there is a finite subfamily $\delta \subseteq \lambda$ such that $\underset{A \in \delta}{\vee} bCl(A)(x)=U(x)$, for every $x \in S(U)$.

DEFINITION 13 ([12]). A fuzzy set U in a fts X is said to be fuzzy b-compact relative to X if and only if for every family μ of fuzzy b-open sets such that $\underset{A \in \mu}{\vee} A \geq U(x)$ there is a finite subfamily $\delta \subseteq \mu$ such that $\underset{A \in \delta}{\vee} A \geq U(x)$.

DEFINITION 14 ([12]). A fts X is said to be fuzzy b-compact if and only if for every family μ of fuzzy b-open fuzzy sets such that $\underset{A\in\delta}{\vee} A = 1_x$ there is a finite subfamily $\delta \subseteq \mu$such that $\underset{A\in\delta}{\vee} A = 1_x$, for every $x \in S(u)$.

REMARK 1. Every fuzzy b-compact space is fuzzy b-closed, but the converse is not true.

To prove the Remark 1, we prove that every fuzzy b-compact space is fuzzy b-closed and with a counterexample we will show that the contrary of above statement is not established in general.

Firstly, we need the following theorem.

THEOREM 2. Every fuzzy b-compact set in a fuzzy topological space X is fuzzy compact.

PROOF. Let A be a fuzzy b-compact set, and let $\{\alpha_i : i \in \Omega\}$ be a fuzzy cover of A. Then $A \leq \underset{i\in\Omega}{\vee} \alpha_i$. Therefore, $\{\alpha_i : i \in \Omega\}$ is a fuzzy b-open cover of A. Since A is fuzzy b-compact set, then there are finite by many indices $i_1, i_2, ..., i_n \in \Omega$ such that $A \leq \overset{n}{\underset{j=1}{\vee}} \alpha_{ij}$. Then A is fuzzy b-compact set.

THEOREM 3. Every fuzzy b-compact set is fuzzy b-closed.

PROOF. Let A be a fuzzy b-compact set in a fuzzy space X. By Theorem 2, A is fuzzy compact set. Since every fuzzy compact set of a Hausdorff topological space is fuzzy closed set (see Theorem 3-5 of [3]) then A is a fuzzy closed set. It is shown in (Remark 2.4 of [3]) every fuzzy closed sets is fuzzy b-closed set, then A is fuzzy b-closed set.

EXAMPLE 1. Let $X = \{a, b\}$ and let T be the indiscrete fuzzy space on X. Then $A : I \to X$ which is defined by $A(a) = 0.1$, $A(b) = 0.2$ is fuzzy b-compact, but it is not fuzzy b-closed.

For more information see ([14], Propositions (5.6.3), page 74).

The concept of fuzzy filterbases was introduced by A. A. Nouh in [13]. In the following theorems we obtain some known results in b-closed space by using fuzzy filterbases.

THEOREM 4. A fts X is fuzzy b-closed if and only if for every fuzzy filterbases Γin X, $\left(\underset{G\in\Gamma}{\wedge} bCl(G)\right) \neq 0_x$.

PROOF. Let μ be a fuzzy b-open set cover of X and let for every finite family of μ, $\underset{A\in\partial}{\vee} bCl(A)(x) < 1_x$ for some $x \in X$. Then $\left(\underset{A\in\partial}{\wedge} \overline{bCl(G)}\right)(x) > 0_x$ for some $x \in X$.

Thus $\left\{\left(\overline{bCl\left(A\right)}:A\in\mu\right)\right\}=\Gamma$ forms a fuzzy b-open filterbases in X. Since μ is a fuzzy b-open set cover of X, we have $\left(\underset{A\in\mu}{\wedge}A\right)=0_x$. It implies that

$$\left(\underset{A\in\mu}{\wedge}bCl\left(\overline{bCl\left(G\right)}\right)\right)(x)=0_x,$$

which is a contradiction. Then every fuzzy b-open μ of X has a finite subfamily ∂ such that $\left(\underset{A\in\partial}{\vee}bCl\left(A\right)(x)\right)=1_x$ for every $x\in X$. Hence X is fuzzy b-closed.

Conversely, suppose there exists a fuzzy b-open filterbases Γ in X, such that

$$\left(\underset{G\in\Gamma}{\wedge}bCl\left(G\right)\right)=0_x.$$

That implies $\left(\underset{G\in\Gamma}{\vee}\left(\overline{bCl\left(G\right)}\right)\right)(x)=1_x$ for $x\in X$, and hence $\mu=\left\{\overline{\left(bCl\left(G\right)\right)}:G\in\Gamma\right\}$ is a fuzzy b-open set cover of X. Since X is fuzzy b-closed, by definition μ has a finite subfamily ∂ such that $\left(\underset{G\in\partial}{\vee}bCl\left(\overline{bCl\left(G\right)}\right)\right)(x)=1_x$ for every $x\in X$, and hence $\underset{\lambda\in\partial}{\wedge}\left(\overline{bCl\left(\overline{bCl\left(G\right)}\right)}\right)=0_x$. Thus $\underset{G\in\partial}{\wedge}G=0_x$ is a contradiction. Hence $\underset{G\in\Gamma}{\wedge}bCl\left(G\right)\neq 0_x$.

THEOREM 5. Let $f:(X,\tau)\to(Y,\sigma)$ be a fuzzy b^*-continuous surjection. If X is fuzzy b-closed space, then Y is fuzzy b-closed space.

PROOF. Let $\{A_\lambda:\lambda\in\Lambda\}$ be a fuzzy b-open cover of Y. Since f is fuzzy b^*-continuous, $\left\{f^{-1}\left(A_\lambda\right):\lambda\in\Lambda\right\}$ is fuzzy b-open cover of X. By hypothesis, there exists a finite subset Δ of Γ such that $\underset{\lambda\in\Delta}{\vee}bCl\left(f^{-1}\left(A_\lambda\right)\right)=1_x$. Since f is a surjection we have,

$$1_Y=f\left(1_x\right)=f\left(\underset{\lambda\in\Delta}{\vee}bCl\left(f^{-1}\left(A_\lambda\right)\right)\right)\leq\underset{\lambda\in\Delta}{\vee}bCl\left(f\left(f^{-1}\left(A_\lambda\right)=\underset{\lambda\in\Delta}{\vee}bCl\left(A_\lambda\right)\right)\right).$$

Hence Y is a fuzzy b-closed space.

DEFINITION 14. A fuzzy set U in a fts X is said to be fuzzy b-compact relative to X if and only if for every family μ of fuzzy b-open sets such that $\underset{A\in\mu}{\vee}A\geq U(x)$ there is a finite subfamily $\delta\subseteq\mu$ such that $\underset{A\in\delta}{\vee}A\geq U(x)$ for every $x\in S(U)$.

Now, we can prove the main results in this section.

THEOREM 6. A fts X is b-compact if and only if for every collection $\{A_\lambda:\lambda\in\Lambda\}$ of fuzzy b-closed sets of X having the finite intersection property, $\underset{\lambda\in}{\wedge}A_\lambda\neq 0_x$.

PROOF. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of fuzzy b-closed sets with the finite intersection property. Suppose that $\underset{\lambda\in\Lambda}{\wedge} A_\lambda = 0_x$. Then $\underset{\lambda\in\Lambda}{\vee} \left(\overline{A_\lambda}\right) = 1_x$. Since $\left\{\overline{A_\lambda} : \lambda \in \Lambda\right\}$ is a collection of fuzzy b-compactness of X it follows that there exists a finite subset $\Delta \subseteq \Lambda$, such that $\underset{\lambda\in\Delta}{\vee} \overline{A_\lambda} = 1_x$. Then $\underset{\lambda\in\Delta}{\wedge} A_\lambda = 0_x$, which gives a contradiction. Therefore $\underset{\lambda\in\Lambda}{\wedge} A_\lambda \neq 0_x$.

Conversely, let $\{A_\lambda : \lambda \in \Lambda\}$ be a collection of fuzzy b-open sets covering X. Suppose that for every finite subset $\Delta \subseteq \Lambda$, we have $\underset{\lambda\in\Delta}{\vee} A_\lambda \neq 1_x$. Then $\underset{\lambda\in\Delta}{\wedge} \left(\overline{A_\lambda}\right) \neq 0_x$. Hence $\left\{\overline{A_\lambda} : \lambda \in \Lambda\right\}$ satisfies the finite intersection property. Then by definition we have $\underset{\lambda\in\Lambda}{\wedge} \left(\overline{A_\lambda}\right) \neq 0_x$ which implies $\underset{\lambda\in\Delta}{\vee} \left(\overline{A_\lambda}\right) \neq 1_x$, and this contradicts that $\{A_\lambda : \lambda \in \Lambda\}$ is a fuzzy b-cover of X. Thus X is fuzzy b-compact.

The following characterization on b-compactness makes use of fuzzy filterbases.

THEOREM 7. A fts X is fuzzy b-compact if and only if for every filterbases Γ in X, $\underset{G\in\Gamma}{\wedge} bClG \neq 0_x$.

PROOF. Let μ be a fuzzy b-open cover which has no finite subcover in X. Then for every finite subcollection of $\{A_1, ..., A_n\}$ of μ, there exists $x \in X$ such that $A_\lambda(x) < 1$ for every $\lambda = 1, ..., n$. Then $\overline{A_\lambda} > 0$, so that $\overset{n}{\underset{\lambda=1}{\wedge}} \overline{A_\lambda}(x) \neq 0_x$. Thus $\left\{\overline{A_\lambda}(x) : A_\lambda \in \mu\right\}$ forms a filterbase in X. Since μ is fuzzy b-open set cover of X, then $\left(\underset{A_\lambda\in\mu}{\vee} A_\lambda\right)(x) = 1_x$ for every $x \in X$ and hence $\underset{A_\lambda\in\mu}{\wedge} bCl\overline{A_\lambda}(x) = 0_x$, which is a contradiction. Then every fuzzy b-open set cover of X has a finite subcover and hence X is fuzzy b-compact.

Conversely, suppose there exists a filterbases Γ in X, $\underset{G\in\Gamma}{\wedge} bCl(G) = 0_x$ so that $\left(\underset{G\in\Gamma}{\vee} \overline{(bCl(G))}\right)(x) = 1_x$, for every and hence $\mu = \left\{\overline{bCl(G)} : G \in \Gamma\right\}$ is a fuzzy b-open cover of X. Since X is fuzzy b-compact, by definition Γ has a finite subcover such that $\left(\overset{n}{\underset{\lambda=1}{\vee}} \left(\overline{bCl(G_\lambda)}\right)(x)\right) = 1_x$ and hence $\left(\overset{n}{\underset{\lambda=1}{\vee}} \overline{(G_\lambda)}\right)(x) = 1_x$, so that $\overset{n}{\underset{\lambda=1}{\wedge}} (G_\lambda) = 0_x$, which is a contradiction. Therefore, $\underset{G\in\Gamma}{\wedge} bCl(G) \neq 0_x$ for every filterbases Γ.

THEOREM 8. A fuzzy set U in a fts X is fuzzy b-compact relative to X if and only if for every filterbase Γ such that every finite members of Γ is quasi coincident with U,

$$\left(\underset{G\in\Gamma}{\wedge} bCl(G)\right) \wedge U \neq 0_x.$$

PROOF. Suppose U is not fuzzy b-compact relative to X. Then there exists a fuzzy b-open set μ covering of U with no finite subcover υ. Then $\left(\underset{A_\lambda\in\upsilon}{\vee} A_\lambda(x)\right) < U(x)$, for some $x \in S(U)$, so that $\left(\underset{A_\lambda\in\upsilon}{\wedge} \overline{A_\lambda}\right)(x) > \overline{U}(x) \geq 0_x$ and hence $\Gamma = \left\{\overline{A_\lambda}(x) : A_\lambda \in \mu\right\}$

forms a filterbases and $\underset{A_\lambda \in \upsilon}{\wedge} \overline{A_\lambda}$ q U. By hypothesis $\left(\underset{A_\lambda \in \upsilon}{\wedge} bCl\overline{A_\lambda}\right) \wedge U \neq 0_x$ and hence $\left(\underset{A_\lambda \in \upsilon}{\wedge} \overline{A_\lambda}\right) \wedge U \neq 0_x$. Then for some $x \in S(U)$, $\left(\underset{A_\lambda \in \mu}{\wedge} \overline{A_\lambda}\right)(x) > 0_x$ that is $\left(\underset{A_\lambda \in \mu}{\vee} A_\lambda\right)(x) < 1_x$ which is a contradiction. Hence U is fuzzy b-compact relative to X.

Conversely, suppose that there exists a filterbases Γ such that every finite members of Γ is quasi coincident with U and $\left(\underset{G \in \Gamma}{\wedge} bCl(G)\right) \wedge U \neq 0_x$. Then for every $x \in S(U)$, $\left(\underset{G \in \Gamma}{\wedge} bCl(G)\right)(x) = 0_x$ and hence $\underset{G \in \Gamma}{\vee} \overline{bCl(G)}(x) = 1_x$ for every $x \in S(U)$. Thus $\mu = \left\{\overline{bCl(G)} : G \in \Gamma\right\}$ is a fuzzy b-open set cover of U. Since U is fuzzy b-compact relative to X, there exists a finite subcover, $\left\{\left(\overline{bCl(G_\lambda)}\right) : \lambda = 1, ...n\right\}$ such that $\left(\overset{n}{\underset{\lambda=1}{\wedge}} \overline{bCl(G_\lambda)}\right)(x) \geq U(x)$ for every $x \in S(U)$. Hence $\left(\overset{n}{\underset{\lambda=1}{\wedge}} bCl(G_\lambda)\right)(x) \leq \overline{U}(x)$ for every $x \in S(U)$. So that $\overset{n}{\underset{\lambda=1}{\wedge}} bCl(G_\lambda)\ \overline{q}\ U$, which is a contradiction. Therefore, for every filterbases Γ, every finite member of Γ is quasi coincident with U. So $\left(\underset{G \in \Gamma}{\wedge} bCl(G)\right) \wedge u \neq 0_x$.

The following theorem proves the hereditary property for fuzzy b-compact spaces.

THEOREM 9. Every fuzzy b-closed subset of a fuzzy b-compact space is fuzzy b-compact relative to X.

PROOF. Suppose Γ be a fuzzy filterbases, Δ be its finite subcollection in X and for a fuzzy b-closed set U, let U q $\wedge\{G \in \Gamma\}$. Let $\Gamma^* = \{U\} \cup \Gamma$. For any finite subcollection Δ^* of Γ^*, if $U \notin \Delta^*$ then $\wedge\Delta^* \neq 0_x$. If $U \in \Delta^*$ and U q $\wedge\{G : G \in \Delta^* - U\}$, then $\wedge\Delta^* \neq 0_x$. Hence Δ^* is a fuzzy filterbases in X. Since X is a fuzzy b-compact space, we have $\left(\underset{G \in \Gamma}{\wedge} bCl(G)\right) \neq 0_x$, such that

$$\left(\underset{G \in \Gamma}{\wedge} bCl(G)\right) \underset{G \in \Gamma}{\wedge} U = \left(\underset{G \in \Gamma}{\wedge} bCl(G)\right) \wedge bCl(U) \neq 0_x.$$

Hence by Theorem 4, is fuzzy b-compact relative to X.

In the following theorem it shows that the image of a fuzzy b-compact space under a fuzzy b^*-continuous mapping is fuzzy b-compact.

THEOREM 10. If a function $f : (X, \tau) \to (Y, \sigma)$ is fuzzy b^*-continuous and U is fuzzy b-compact relative to X, then $f(U)$ is fuzzy b-compact.

PROOF. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a fuzzy b-open cover of $S(f(u))$ in Y. For $x \in S(U)$, $f(x) \in f(S(U))$. Since f is fuzzy b^*-continuous, $\left\{f^{-1}(A_\lambda) : \lambda \in \Lambda\right\}$ is fuzzy b-open cover of $S(U)$ in X. Since U is fuzzy b-compact relative to X, there is a finite subfamily $\{f^{-1}(A_\lambda) : \lambda = 1, ..., n\}$ such that

$$S(U) \leq \bigvee_{\lambda=1}^{n} f^{-1}(A_\lambda) = f^{-1}\left(\bigvee_{\lambda=1}^{n} A_\lambda\right).$$

Hence,

$$S(f(U)) = f(S(U)) \leq f\left(f^{-1}\left(\bigvee_{\lambda=1}^{n} A_\lambda\right)\right) \leq \bigvee_{\lambda=1}^{n} A_\lambda.$$

Therefore $f(U)$ is λ fuzzy b-compact relative to Y.

The pre image of a fuzzy b-compact space under fuzzy b^*-open bijective mapping is fuzzy b-compact.

THEOREM 11. If a function $f : (X, \tau) \to (Y, \sigma)$ is a fuzzy b^*-open bijective mapping and Y fuzzy b-compact, then X is fuzzy b-compact.

PROOF. Let $\{A_\lambda : \lambda \in \Lambda\}$ be a family of fuzzy b-open covering of X. Then

$$\{f(A_\lambda) : \lambda \in \Lambda\}$$

is a family of fuzzy b-open sets that covering Y. Since Y is fuzzy b-compact, by definition there exists a finite family $\Delta \subseteq \Lambda$ such that $\{f(A_\lambda) : \lambda \in \Delta\}$ covers Y. Also, since f is bijective we have

$$1_x = f^{-1}(1_Y) = f^{-1}f\left(\bigvee_{\lambda\in\Delta} A_\lambda\right) = \bigvee_{\lambda\in\Delta} A_\lambda.$$

Thus X is fuzzy b-compact.

3 Bounded and Semi-Bounded Inverse Theorem

The main result of this section is a characterization of bounded and semi bounded inverse theorems.

THEOREM 12. Let $(X_i, \|.\|, L_i, R_r)$ be complete fuzzy normed space satisfying $\lim_{a\to 0^+} R_i(a, a) = 0$, $(i = 1, 2)$ and T a bounded linear operator from X_1 into X_2 satisfying $\delta(T) = X_2$ and $\chi(T) = \{\theta\}$. Then T^{-1} is bounded.

PROOF. For each $a \in (0, 1]$, there is $\mu = \mu(a) \in (0, a]$ such that for each $u \in N(\mu, \mu)$ there exists $\{x_n\} \subset X_1$ such that

$$\lim_{n\to\infty} x_n = x \in N(a, a) \text{ and } u = \lim_{n\to\infty} Tx_n.$$

Since T is bounded, T is continuous. Thus, $u = Tx$, i.e, $N(\mu, \mu) \subset TN(a, a)$. By $\delta(T) = X_2$ and $\chi(T) = \{\theta\}$, T^{-1} exists. Hence $T^{-1}N(\mu, \mu) \subset N(a, a)$. Suppose $\nu \in X_2$. Then

$${}^{\mu\nu}/_{(\|\nu\|_\mu^+ + \varepsilon)} \in N(\mu, \mu), \quad \left\|T^{-1}\left[{}^{\mu\nu}/_{(\|\nu\|_\mu^+ + \varepsilon)}\right]\right\|_a^+ < a,$$

where ε is arbitrary. Namely, $\left\|T^{-1}\nu\right\|_a^+ < \frac{a}{\mu}\left(\|\nu\|_\mu^+ + \varepsilon\right)$ letting $\varepsilon \to 0^+$, we have $\left\|T^{-1}\nu\right\| \leq \frac{a}{\mu}\|\nu\|_\mu^+$, showing that T^{-1} is bounded.

THEOREM 13. Let $(X_i, \|.\|, L_i, R_r)$ be complete fuzzy normed spaces $(i = 1, 2)$ and T a proper ρ-bounded linear operator from X_1 into X_2 satisfying $\delta(T) = X_2$ and $\chi(T) = \{\theta\}$. Suppose that

1. $\lim_{a\to 0^+} (a, a) = 0$ and $a \leq R_1(a, a) < 1$ for each $a \in (0, 1)$;
2. $R_1(a, a) \leq \max(a, b)$;
3. $\|.\|$ maps X_2 into ζ_*^+, $L_2(1, 1)$ and $\lim_{a\to 0^+} R_2(a, b) \leq b$ for each $b \in [0, 1]$;
4. $\lim_{a\to 0^+} R_2(a, a) = 0$ and $a \leq R_2(a, a) < 1$ for each $a \in (0, 1)$;
5. $R_2(a, a) \leq \max(a, b)$;

If $(X_1, \|.\|, L_1, R_1)$ satisfies (1) or (2) and $(X_2, \|.\|, L_2, R_2)$ satisfies (3) or (4) or (5), then T^{-1} is λ-bounded, where $\lambda \in (0, 1)$.

PROOF. By $\delta(T) = X_2$ and $\chi(T) = \{\theta\}$, T^{-1} exists. Since $(X_2, \|.\|, L_2, R_2)$ satisfies (3) or (4) or (5), we have T is ω-closed, where $\omega \in [\rho, 1)$. Since T is proper, we get that T^{-1} is σ-closed, where $\sigma \in (0, 1)$. Since $(X_1, \|.\|, L_1, R_1)$ satisfies (1) or (2), we obtain T^{-1} is λ-bounded, where $\lambda \in (0, 1)$.

EXAMPLE 2. Let $(X, \|.\|, L_i, R_i)$ be complete fuzzy random variables satisfying $\lim_{a\to 0^+} R_i(a, a) = 0$, $(i = 1, 2)$ and $\{x_n\}_{n=1}^\infty \subset X$. If $\lim_{n\to\infty} x_n = \theta$ in $(X, \|.\|_1, L_1, R_1)$ implies $\lim_{n\to\infty} x_n = \theta$ in $(X, \|.\|_2, L_2, R_2)$, then we have convergence in $(X, \|.\|_1, L_1, R_1)$ and in $(X, \|.\|_2, L_2, R_2)$ are equivalent.

In fact, let I be the identity operator from $(X, \|.\|_1, L_1, R_1)$ onto $(X, \|.\|_2, L_2, R_2)$. Since $\lim_{n\to\infty} x_n = \theta$ implies $\lim_{n\to\infty} Ix_n = \theta$, i.e. I is continuous, applying Theorem 12, we obtain that I^{-1} is countinuous, i.e., $\lim_{n\to\infty} x_n = \theta$ in $(X, \|.\|_2, L_2, R_2)$ implies $\lim_{n\to\infty} x_n = \theta$ in $(X, \|.\|_1, L_1, R_1)$. Hence convergence in $(X, \|.\|_1, L_1, R_1)$ and in $(X, \|.\|_2, L_2, R_2)$ are equivalent.

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