Applied Mathematics E-Notes, 15(2015), 304-316 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Positive Periodic Solutions For A Kind Of Prescribed Mean Curvature Duffing-Type Equation*

Fan Chao Kong†, Shi Ping Lu‡

Received 15 May 2015

Abstract

In this paper, we study the existence of periodic solutions to the following prescribed mean curvature Duffing-type equation with a singularity and a deviating argument:

$$\left(\frac{u'(t)}{\sqrt{1+(u')^2}}\right)' + cu'(t) + g(t, u(t-\delta)) = p(t),$$

where g has a strong singularity at $x = 0$ and satisfies a small force condition at $x = \infty$, which are different from the known literatures.

1 Introduction

In recent years, the problems of periodic solution have been studied widely for some types of differential equations with a singularity, see [3, 6–8, 13–16] and references therein. For example, Wang [15] studied periodic solutions for the Liénard equation with a singularity and a deviating argument of the form

$$x''(t) + f(x(t))x'(t) + g(t, x(t-\sigma)) = 0,$$

where $0 \leq \sigma < T$ is a constant, $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is an L^2-Carathéodory function, $g(t, x)$ is a T-periodic function in the first argument and can be singular at $x = 0$, *i.e.*, $g(t, x)$ can be unbounded as $x \to 0^+$.

Nowadays, the prescribed mean curvature equation

$$\left(\frac{u'(t)}{\sqrt{1+(u')^2}}\right)' = f(u(t)),$$

and its modified forms, which arises from some problems associated to differential geometry and combustible gas dynamics, were studied extensively, see [1, 2, 11, 12] and the references therein. Moreover, we note that the existence of periodic solutions for the prescribed curvature mean equations has attracted much attention from researchers.

---

*Mathematics Subject Classifications: 34B16, 34K13.

†College of Mathematics and Computer Science, Anhui Normal University, Wuhu 241000, P. R. China

‡College of Mathematics and Statistics, NUIST, Nanjing 210044, P. R. China

However, it is not easy to study the periodic solutions for the prescribed curvature mean equations. The main difficulty lies in the nonlinear term $(\frac{u'(t)}{\sqrt{1+(u')^2}})'$, the existence of which obstructs the usual method of finding a priori bounds for the Liénard or the Rayleigh equations from working. Until, in [4] , Feng considered a kind of prescribed mean curvature Liénard equation

$$\left(\frac{u'(t)}{\sqrt{1+(u')^2}}\right)' + f(u(t))u'(t) + g(t, u(t-\tau(t))) = e(t), \tag{1}$$

where $\tau, e \in C(\mathbb{R}, \mathbb{R})$ are T-periodic, and $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is T-periodic in the first argument, $T > 0$ is a constant. Through the transformation, Feng asserts that Eq.(1) is equivalent to the following system

$$\begin{cases} u'(t) = \varphi(v(t)) = \frac{v(t)}{\sqrt{1-v^2(t)}}, \\ v'(t) = -f(t, \varphi(v(t))) - g(t, u(t-\tau(t))) + e(t). \end{cases}$$

Then by applying Mawhin's continuation theorem under some sufficient conditions, the author show that Eq.(1) has at least one periodic solution.

On the basis of Feng's work, various types of prescribed curvature mean equations have been studied, see [9, 10, 17] and the references therein.

However, to the best of our knowledge, the study of positive periodic solutions for the prescribed mean curvature equation with a singularity is relatively infrequent. This is due to the fact that the mechanism on which how the solution is influenced by the singularity and the nonlinear term $(\frac{u'(t)}{\sqrt{1+(u')^2}})'$ associated to prescribed mean curvature equation is far away from clear.

Inspired by the above facts, in this paper, we consider the following prescribed mean curvature Duffing-type equation with a singularity and a deviating argument

$$\left(\frac{u'(t)}{\sqrt{1+(u')^2}}\right)' + cu'(t) + g(t, u(t-\delta)) = p(t), \tag{2}$$

where c is a constant, $0 \leq \delta < T$, $g : [0, T] \times (0, +\infty) \to \mathbb{R}$ is a continuous function. g can be singular at $u = 0$, $p(t)$ is continuous and T-periodic with $\int_0^T p(t)dt = 0$. By applying Mawhin's continuation theorem, we prove that Eq.(2) has at least one positive T-periodic solution.

The structure of the rest of this paper is as follows. In Section 2, we state some necessary definitions and lemmas. In Section 3, we prove the main result. Finally, we give an example of an application in Section 4.

2 Preliminary

In order to use Mawhin's continuation theorem, we first recall it.

Let X and Y be two Banach spaces, a linear operator $L : D(L) \subset X \to Y$ is said to be a Fredholm operator of index zero provided that

(a) $\operatorname{Im} L$ is a closed subset of Y,

(b) $\dim \ker L = co \dim \operatorname{Im} L < \infty$.

Let X and Y be two Banach spaces, $\Omega \subset X$ be an open and bounded set, and $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero. A continuous operator $N : \Omega \subset X \to Y$ is said to be L-compact in $\bar{\Omega}$ provided that

(c) $K_p(I - Q)N(\bar{\Omega})$ is a relative compact set of X,

(d) $QN(\bar{\Omega})$ is a bounded set of Y,

where we define $X_1 = \ker L$, $Y_2 = \operatorname{Im} L$, and

$$X = X_1 \bigoplus X_2 \text{ and } Y = Y_1 \bigoplus Y_2.$$

Let $P : X \to X_1$, $Q : Y \to Y_1$ be continuous linear projectors (meaning $P^2 = P$ and $Q^2 = Q$), and $K_p = L\mid_{\ker P \cap D(L)}^{-1}$.

LEMMA 1 ([5]). Let X and Y be two real Banach spaces, and Ω be an open and bounded set of X, and $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero. The operator $N : \bar{\Omega} \subset X \to Y$ is said to be L-compact in $\bar{\Omega}$. In addition, if the following conditions hold:

(1) $Lx \neq \lambda N x$, $\forall (x, \lambda) \in \partial\Omega \times (0, 1)$;

(2) $QNx \neq 0$, $\forall x \in \ker L \cap \partial\Omega$;

(3) $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$ where $J : \operatorname{Im} Q \to \ker L$ is a homeomorphism.

Then $Lx = Nx$ has at least one solution in $D(L) \cap \bar{\Omega}$.

In order to use Lemma 1, let us consider the problem

$$\begin{cases} u'(t) = \phi(v(t)) = \frac{v(t)}{\sqrt{1-v^2(t)}}, \\ v'(t) = -c\phi(v(t)) - g(t, u(t-\delta)) + p(t). \end{cases} \tag{3}$$

Obviously, if $(u(t), v(t))^\top$ is a solution of (3), then $u(t)$ is a solution of (2). Let

$$X = Y = \{x : x(t) = (u(t), v(t))^\top \in C^1(\mathbb{R}, \mathbb{R}^2),\ x(t) = x(t+T)\},$$

where the normal

$$\|x\| = \max\{||u||_0, ||v||_0\},\ ||u||_0 = \max_{t\in[0,T]} |u|,\ \text{ and }\ ||v||_0 = \max_{t\in[0,T]} |v|.$$

It is obvious that X and Y are Banach spaces.

Now we define the operator

$$L : D(L) \subset X \to Y, \quad Lx = x' = (u'(t), v')^\top,$$

where

$$D(L) = \left\{x : x = (u(t), v(t))^\top \in C^1(\mathbb{R}, \mathbb{R}^2) \text{ and } x(t) = x(t+T)\right\}.$$

Let

$$X_0 = \left\{x = (u(t), v(t))^\top \in C^1(\mathbb{R}, \mathbb{R} \times (-1, 1)) : x(t) = x(t+T)\right\}.$$

Define a nonlinear operator $N : \overline{\Omega} \subset (X \cap X_0) \subset X \to Y$ as follows:

$$Nx = \left(\frac{v(t)}{\sqrt{1 - v^2(t)}}, -\frac{cv(t)}{\sqrt{1 - v^2(t)}} - g(t, u(t - \delta)) + p(t)\right)^\top,$$

where $\overline{\Omega} \subset X_0 \subset X$ and Ω is an open and bounded set. Then problem (3) can be written as $Lx = Nx$ in $\overline{\Omega}$. We know

$$\ker L = \left\{x : x \in X,\ x' = (u'(t),\ v'(t))^\top = (0, 0)^\top\right\}.$$

Then we have $u'(t) = 0$, $v'(t) = 0$ for $t \in \mathbb{R}$. Obviously $u \in \mathbb{R}$, $v \in \mathbb{R}$, thus $\ker L = \mathbb{R}^2$, and it is also easy to prove that

$$\operatorname{Im} L = \left\{y \in Y : \int_0^T y(s) ds = 0\right\}.$$

Therefore, L is a Fredholm operator of index zero. Let

$$P : X \to \ker L, \quad Px = \frac{1}{T}\int_0^T x(s) ds,$$

$$Q : Y \to \operatorname{Im} Q, \quad Qy = \frac{1}{T}\int_0^T y(s) ds.$$

Let $K_p = L|^{-1}_{\ker L \cap D(L)}$. Then it is easy to see that

$$(K_p y)(t) = \int_0^T G_k(t, s) y(s) ds,$$

where

$$G_k(t) = \begin{cases} \frac{s-T}{T} & \text{for } 0 \le t \le s, \\ \frac{s}{T} & \text{for } s \le t \le T. \end{cases}$$

For all $\overline{\Omega}$ with $\overline{\Omega} \subset (X \cap X_0) \subset X$, we see that $K_p(I - Q)N(\overline{\Omega})$ is a relative compact set of X and $QN(\overline{\Omega})$ is a bounded set of Y. So the operator N is L-compact in $\overline{\Omega}$.

For the sake of convenience, we list the following assumptions

$[H_1]$ There exist positive constants A_1 and A_2 with $A_1 < A_2$ such that

(1) For each positive continuous T-periodic function $x(t)$ satisfying

$$\int_0^T g(t,x(t))dt = 0,$$

there exists a positive point $\tau \in [0,T]$ such that

$$A_1 \leq x(\tau) \leq A_2.$$

(2) $\overline{g}(x) < 0$ for all $x \in (0, A_1)$ and $\overline{g}(x) > 0$ for all $x > A_2$ where

$$\overline{g}(x) = \frac{1}{T}\int_0^T g(t,x)dt, x > 0.$$

$[H_2]$ $g(t,x) = g_1(t,x) + g_0(x)$ where $g_1 : [0,T] \times (0,+\infty) \to \mathbb{R}$ is a continuous function and

(1) There exist positive constants a and b such that

$$g(t,x) \leq ax + b \text{ for all } (t,x) \in [0,T] \times (0,+\infty).$$

(2) $\int_0^1 g_0(x)dx = -\infty$.

Throughout this paper, define

$$B := \left(\int_0^T |p(t)|^2 dt)^{\frac{1}{2}} + \sup_{t\in[0,T]} |p(t)|\right) < +\infty.$$

3 Main Results

THEOREM 1. Suppose the conditions $[H_1]$–$[H_2]$ hold, $|c| > aT$ and

$$\frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B) < 1.$$

Then Eq.(2) has at least one positive T-periodic solution.

PROOF. Let

$$\Omega_1 = \{z \in \overline{\Omega} : Lz = \lambda Nz \text{ and } \lambda \in (0,1)\}.$$

If $z \in \Omega_1$, we have

$$\begin{cases} u'(t) = \lambda\phi(v(t)) = \lambda\frac{v(t)}{\sqrt{1-v^2(t)}}, \\ v'(t) = -\lambda c\phi(v(t)) - \lambda g(t,u(t-\delta)) + \lambda p(t). \end{cases} \tag{4}$$

Integrating the second equation of (4) from 0 to T, we have

$$\int_0^T g(t, u(t-\delta))dt = 0. \tag{5}$$

It follows from $[H_1](1)$ that there exist positive constants A_1, A_2 and $\tau \in [0, T]$ such that

$$A_1 \le u(\tau) \le A_2. \tag{6}$$

Then, we can have

$$\begin{aligned} \|u\|_0 &= \max_{t\in[0,T]} |u(t)| \le \max_{t\in[0,T]} \left| u(\tau) + \int_\tau^t u'(s)ds \right| \\ &\le A_2 + \int_0^T |u'(s)|ds \le A_2 + \sqrt{T} \|u'\|_2 . \end{aligned} \tag{7}$$

Multiplying the second equation of (4) by $u'(t)$ and integrating on the interval $[0, T]$, we have

$$\begin{aligned} 0 = \int_0^T v'(t)u'(t)dt = -\int_0^T c(u')^2 dt - \lambda \int_0^T g(t, u(t-\delta))u'(t)dt \\ + \lambda \int_0^T p(t)u'(t)dt. \end{aligned}$$

Combining with $[H_2]$, we get

$$\begin{aligned} |c| \int_0^T |u'^2 dt &\le \int_0^T (a|u(t-\delta)| + b) |u'(t)|dt + \int_0^T |p(t)||u'(t)|dt \\ &\le a\sqrt{T} \|u\|_0 \|u'\|_2 + b\sqrt{T} \|u'\|_2 + B \|u'\|_2 , \end{aligned}$$

which, combining with (7), gives

$$\begin{aligned} |c| \|u'\|_2^2 &\le a \|u\|_0 \sqrt{T} \|u'\|_2 + b\sqrt{T} \|u'\|_2 + B \|u'\|_2 \\ &\le a \left[A_2 + \sqrt{T} \|u'\|_2 \right] \sqrt{T} \|u'\|_2 + b\sqrt{T} \|u'\|_2 + B \|u'\|_2 \\ &= aT \|u'\|_2^2 + (aA_2\sqrt{T} + b\sqrt{T} + B) \|u'\|_2 . \end{aligned}$$

Then by $|c| > aT$, we obtain

$$\|u'\|_2 \le \frac{aA_2\sqrt{T} + b\sqrt{T} + B}{|c| - aT}. \tag{8}$$

Substituting (8) into (7), we obtain

$$\|u\|_0 \le A_2 + \frac{aA_2T + bT + B\sqrt{T}}{|c| - aT} := M_1. \tag{9}$$

From the second equation of (4), we can get

$$\int_0^T |v'(t)|dt \le \int_0^T |c||u'(t)|dt + \lambda \int_0^T |g(t, u(t-\delta))|dt + \lambda \int_0^T |p(t)|dt. \tag{10}$$

Write

$$I_+ = \{t \in [0, T] : g(t, u(t-\delta)) \ge 0\} \text{ and } I_- = \{t \in [0, T] : g(t, u(t-\delta)) \le 0\}.$$

Then, combining with (5) and $[H_2](1)$, we have

$$\begin{aligned}\int_0^T |g(t, u(t-\delta))|dt &= \int_{I_+} g(t, u(t-\delta))dt - \int_{I_-} g(t, u(t-\delta))dt \\ &= 2\int_{I_+} g(t, u(t-\delta))dt \\ &\le 2a \int_0^T u(t-\delta)dt + 2\int_0^T bdt \\ &\le 2aT \parallel u \parallel_0 + 2bT.\end{aligned} \tag{11}$$

Substituting (11) into (10) and in view of (8) and (9), we obtain

$$\begin{aligned}\int_0^T |v'(t)|dt &\le |c|\sqrt{T} \parallel u' \parallel_2 + \lambda(2aT \parallel u \parallel_0 + 2bT) + \lambda BT \\ &\le \frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B).\end{aligned} \tag{12}$$

Integrating the first equation of (4) on the interval $[0, T]$, we can get

$$\int_0^T \frac{v(t)}{\sqrt{1 - v^2(t)}}dt = 0.$$

Then we can see that there exists $\eta \in [0, T]$ such that $v(\eta) = 0$. It implies that

$$|v(t)| = \left|\int_\eta^t v'(s)ds + v(\eta)\right| \le \int_0^T |v'(s)|ds,$$

which, combining with (12), gives

$$\begin{aligned}|v(t)| &\le \int_0^T |v'(s)|ds \\ &\le \frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B) \\ &:= \rho.\end{aligned} \tag{13}$$

Since

$$\frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B) < 1,$$

we obtain

$$\|v\|_0 = \max_{t\in[0,T]} |v(t)| \le \rho < 1. \tag{14}$$

By (4), we can also have

$$\|u'\|_0 \le \lambda \max_{t\in[0,T]} \frac{|v(t)|}{\sqrt{1-v^2(t)}} \le \frac{\lambda\rho}{1-\rho^2}. \tag{15}$$

From the second equation of (4) and by $[H_2]$, we can have

$$v'(t+\delta) = -cu'(t+\delta) - \lambda[g_1(t+\delta, u(t)) + g_0(u(t))] + \lambda p(t+\delta). \tag{16}$$

Multiplying both sides of Eq.(16) by $u'(t)$, we can see that

$$\begin{aligned} v'(t+\delta)u'(t) = &-cu'(t+\delta)u'(t) - \lambda[g_1(t+\delta, u(t)) + g_0(u(t))]u'(t) \\ &+ \lambda p(t+\delta)u'(t). \end{aligned} \tag{17}$$

Let $\tau \in [0,T]$ be as in (6). For any $t \in [\tau, T]$, integrating Eq.(17) on the interval $[\tau, T]$, we obtain

$$\begin{aligned} \lambda \int_{u(\tau)}^{u(t)} g_0(u)du &= \lambda \int_\tau^t g_0(u(t))u'(t)dt \\ &= -\int_\tau^t v'(t+\delta)u'(t)dt - \int_\tau^t cu'(t+\delta)u'(t)dt \\ &\quad - \lambda \int_\tau^t g_1(t+\delta, u(t))u'(t)dt + \lambda \int_\tau^t p(t+\delta)u'(t)dt. \end{aligned}$$

Then from the inequality above and combining with (12), we get

$$\begin{aligned} \lambda \left| \int_{u(\tau)}^{u(t)} g_0(u)du \right| &= \lambda \left| \int_\tau^t g_0(u(t))u'(t)dt \right| \\ &\le \int_0^T |v'(t+\delta)||u'(t)|dt + \int_0^T |c||u'(t+\delta)||u'(t)|dt \\ &\quad +\lambda \int_0^T |g_1(t+\delta, u(t))||u'(t)|dt + \lambda \int_0^T |p(t+\delta)||u'(t)|dt \\ &\le \| u' \|_0 \left[\frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B) \right] \\ &\quad + \int_0^T |c||u'(t+\delta)||u'(t)|dt + \lambda \int_0^T |g_1(t+\delta, u(t))||u'(t)|dt \\ &\quad +\lambda \int_0^T |p(t+\delta)||u'(t)|dt. \end{aligned} \tag{18}$$

Set $G_{M_1} = \max_{|u|\le M_1} |g_1(t,u)|$, we have

$$\int_0^T c|u'(t+\delta)||u'(t)|dt \le |c|T \, \|u'\|_0^2 \tag{19}$$

and

$$\int_0^T |g_1(t+\delta, u(t))||u'(t)|dt \le G_{M_1} T \|u'\|_0 . \tag{20}$$

Substituting (19) and (20) into (18), we can obtain

$$\begin{aligned}
&\lambda \left| \int_{u(\tau)}^{u(t)} g_0(u)du \right| \\
&\le \|u'\|_0 \left[\frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B) \right] \\
&+ |c|T \|u'\|_0^2 + \lambda G_{M_1} T \|u'\|_0 + \lambda BT \|u'\|_0 \\
&\le \frac{\lambda\rho}{1-\rho^2} \left[\frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B) \right] \\
&+ |c|T \left(\frac{\lambda\rho}{1-\rho^2} \right)^2 + G_{M_1} T \frac{\lambda\rho}{1-\rho^2} + BT \frac{\lambda\rho}{1-\rho^2},
\end{aligned}$$

which, combining with (15), gives

$$\begin{aligned}
\left| \int_{u(\tau)}^{u(t)} g_0(u)du \right| &\le \frac{\rho}{1-\rho^2} \left[\frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B) \right] \\
&+ |c|T \left(\frac{\rho}{1-\rho^2} \right)^2 + \frac{G_{M_1} T\rho}{1-\rho^2} + \frac{BT\rho}{1-\rho^2} \\
&< +\infty.
\end{aligned}$$

According to $[H_2](2)$, for $t \in [\tau, T]$, we can see that there exists a constant $M_2 > 0$ such that

$$u(t) \ge M_2. \tag{21}$$

For the case $t \in [0, \tau]$, we can handle it similarly.

Define

$$0 < D_1 = \min\{A_1, M_2\} \text{ and } D_2 = \max\{A_2, M_1\}.$$

Then by (6), (9) and (21) we obtain

$$D_1 \le u(t) \le D_2. \tag{22}$$

Set

$$\Omega = \left\{ x = (u, v)^\top \in X : \frac{D_1}{2} < u(t) < D_2 + 1, \|v\|_0 < \rho_1 < \frac{\rho + 1}{2} \right\}.$$

Then the condition (1) of Lemma 1 is satisfied.

Suppose that there exists $x \in \partial\Omega \cap \ker L$ such that $QNx = \frac{1}{T}\int_0^T Nx(s)ds = (0, 0)^\top$, *i.e*,

$$\begin{cases} \frac{1}{T}\int_0^T \frac{v(t)}{\sqrt{1-v^2(t)}} dt = 0, \\ \frac{1}{T}\int_0^T \left[-c\frac{v(t)}{\sqrt{1-v^2(t)}} - g(t, u(t-\delta)) + p(t) \right] dt = 0. \end{cases} \tag{23}$$

Since $\ker L = \mathbb{R}^2$, and u,$v \in \mathbb{R}$ are constant, by the first equation of (23), we have

$$v = 0 < \rho_1.$$

Then from the second equation of (23), we get

$$\frac{1}{T}\int_0^T g(t, u(t-\delta))dt = 0.$$

It follows from $[H_1](1)$ that

$$\frac{D_1}{2} < D_1 < A_1 \leq u(t) \leq A_2 < D_2 < D_2 + 1,$$

which is contrary to the assumption $x \in \partial\Omega$. So for all $x \in \ker L \cap \partial\Omega$, we have $QNx \neq 0$. Then, we can see that the condition (2) of Lemma 1 is satisfied.

In the following, we prove that the condition (3) of Lemma 1 is also satisfied. Define

$$z = Kx = K\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - \frac{A_1+A_2}{2} \\ v \end{pmatrix}.$$

Then we have that

$$x = z + \begin{pmatrix} \frac{A_1+A_2}{2} \\ 0 \end{pmatrix}.$$

Define $J : \operatorname{Im} Q \to \ker L$ is a linear isomorphism with

$$J(u, v) = \begin{pmatrix} v \\ -u \end{pmatrix},$$

and define

$$H(\mu, x) = \mu Kx + (1-\mu)JQNx, \quad \forall (x, \mu) \in \Omega \times [0, 1],$$

Then,

$$H(\mu, x) = \begin{pmatrix} \mu u - \frac{\mu(A_1+A_2)}{2} \\ \mu v \end{pmatrix} + \frac{1-\mu}{T}\begin{pmatrix} \int_0^T [\frac{cv}{\sqrt{1-v^2}} + g(t, u)]dt \\ \int_0^T \frac{v}{\sqrt{1-v^2}}dt \end{pmatrix}. \tag{24}$$

Now we claim that $H(\mu, x)$ is a homotopic mapping. Assume, by way of contradiction, that there exist

$$\mu_0 \in [0, 1] \text{ and } x_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in \partial\Omega$$

such that $H(\mu_0, x_0) = 0$. Substituting μ_0 and x_0 into (24), we have

$$H(\mu_0, x_0) = \begin{pmatrix} \mu_0 u_0 - \frac{\mu_0(A_1+A_2)}{2} + (1-\mu_0)\frac{cv_0}{\sqrt{1-v_0^2}} + (1-\mu_0)\overline{g}(u_0) \\ \mu_0 v_0 + (1-\mu_0)\frac{v_0}{\sqrt{1-v_0^2}} \end{pmatrix}. \tag{25}$$

Since $H(\mu_0, x_0) = 0$, we can see that

$$\mu_0 v_0 + (1-\mu_0)\frac{v_0}{\sqrt{1-v_0^2}} = 0,$$

which combining with $\mu_0 \in [0,1]$, we obtain $v_0 = 0$. Thus $u_0 = A_1$ or A_2. If $u_0 = A_1$, it follows from $[H_1](2)$ that $g(u_0) < 0$. Then substituting $v_0 = 0$ into (25), we can have

$$\mu_0 u_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1-\mu_0)\overline{g}(u_0) < \mu_0(u_0 - \frac{A_1 + A_2}{2}) < 0. \tag{26}$$

If $u_0 = A_2$, it follows from $[H_1](2)$ that $g(u_0) > 0$, then substituting $v_0 = 0$ into (25), we can have

$$\mu_0 u_0 - \frac{\mu_0(A_1 + A_2)}{2} + (1-\mu_0)\overline{g}(u_0) > \mu_0(u_0 - \frac{A_1 + A_2}{2}) > 0. \tag{27}$$

Combining with (26) and (27), we can see that $H(\mu_0, x_0) \neq 0$, which contradicts the assumption. Therefore $H(\mu, x)$ is a homotopic mapping and

$$x^\top H(\mu, x) \neq 0, \quad \forall (x, \mu) \in (\partial\Omega \cap \ker L) \times [0,1].$$

Then

$$\begin{aligned}
\deg(JQN, \Omega \cap \ker L, 0) &= \deg(H(0,x), \Omega \cap \ker L, 0) \\
&= \deg(H(1,x), \Omega \cap \ker L, 0) \\
&= \deg(Kx, \Omega \cap \ker L, 0) \\
&= \sum_{x \in K^{-1}(0)} \operatorname{sgn}(\det K'(x)) \\
&= 1 \neq 0.
\end{aligned}$$

Thus, the condition (3) of Lemma 1 is also satisfied. Therefore, by applying Lemma 1, we can conclude that Eq.(2) has at least one positive T-periodic solution.

4 Example

In this section, we provide an example to illustrate results from the previous sections.

Example 4.1. As an application, we consider the following example:

$$\left(\frac{u'(t)}{\sqrt{1+(u')^2}}\right)' + 7u'(t) + \frac{1}{32}(1+\sin 8t)u(t-\delta) - \frac{1}{u(t-\delta)} = \frac{1}{64}\sin 8t. \tag{28}$$

Conclusion. The Problem (28) has at least one positive $\frac{\pi}{4}$-periodic solution. Corresponding to Theorem 1 and (2), we have

$$g(t, u(t-\delta)) = \frac{1}{32}(1+\sin 8t)u(t-\delta) - \frac{1}{u(t-\delta)}, \quad p(t) = \frac{1}{64}\sin 8t.$$

Then we can choose

$$T = \frac{\pi}{4}, \; a = \frac{1}{16}, \; b = \frac{1}{32}, \; c = 7, \; A_1 = 1, \; A_2 = 4,$$

and

$$B := \left(\int_0^T |p(t)|^2 dt\right)^{\frac{1}{2}} + \sup_{t\in[0,T]} |p(t)| < \frac{1}{32} < +\infty.$$

Then we can see that $[H_1]$ and $[H_2]$ hold. Moreover, $|c| > aT$ and

$$\frac{aA_2T + bT + B\sqrt{T}}{|c| - aT}(c + 2aT) + T(2aA_2 + 2b + B) \approx 0.7202 < 1.$$

Hence, by applying Theorem 1, we can see that Eq.(28) has at least one positive $\frac{\pi}{4}$-periodic solution.

REMARK 1. Since all the results in [1]-[17] and the references therein are not applicable to Eq.(28) for solving positive periodic solutions with periodic $\pi/4$, Theorem 1 in this paper is essentially new.

Acknowledgement. The authors express their thanks to the referee for his (or her) valuable suggestions. The work was supported by the National Natural Science Foundation of China (Grant No. 11271197).

References

[1] P. Benevieria, J. M. do Ó and E. S. de Medeiros, Periodic solutions for nonlinear systems with mean curvature-like operators, Nonlinear Anal., 65(2006), 1462–1475.

[2] D. Bonheure, P. Habets, F. Obersnel and P. Omari, Classical and non-classical solutions of a prescribed curvature equation, J. Differential Equations, 243(2007), 208–237.

[3] A. Fonda, R. Manásevich and F. Zanolin, Subharmonic solutions for some second order differential equations with singularities, SIAM J. Math. Anal., 24(1993), 1294–1311.

[4] M. Feng, Periodic solutions for prescribed mean curvature Liénard equation with a deviating argument, Nonlinear Anal, Real World Appl., 13(2012), 1216–1223.

[5] R. Gaines and J. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Lecture Notes in Mathematics, Vol. 568. Springer-Verlag, Berlin-New York, 1977.

[6] R. Hakl, P. Torres and M. Zamora, Periodic solutions to singular second order differential equations: the repulsive case, Nonlinear Anal. TMA., 74(2011), 7078–7093.

[7] A. Lazer and S. Solimini, On periodic solutions of nonlinear differential equations with singularities, Proc. Amer. Math. Soc., 88(1987), 109–114.

[8] X. Li and Z. Zhang, Periodic solutions for second order differential equations with a singular nonlinearity, Nonlinear Anal. TMA., 69(2008), 3866–3876.

[9] J. Li, Periodic solutions for prescribed mean curvature Rayleigh equation with a deviating argument, Adv. Difference Equ., 2013, 2013:88, 11 pp.

[10] S. Lu and F. Kong, Homoclinic solutions for n-dimensional prescribed mean curvature p-Laplacian equations, Bound. Value Probl., 2015, 2015:105, 16 pp.

[11] F. Obersnel, Existence, regularity and stability properties of periodic solutions of a capillarity equation in the presence of lower and upper solutions, Nonlinear Anal, Real World Appl., 13(2012), 2830–2852.

[12] H. Pan, One-dimensional prescribed mean curvature equation with exponential nonlinearity, Nonlinear Anal., 70(2009), 999–1010.

[13] Z. Wang, Periodic solutions of the second order differential equations with singularities, Nonlinear Anal. TMA., 58(2004), 319–331.

[14] Z. Wang and T. Ma, Existence and multiplicity of periodic solutions of semilinear resonant Duffing equations with singularities, Nonlinearity, 25(2012), 279–307.

[15] Z. Wang, Periodic solutions of Liénard equation with a singularity and a deviating argument, Nonlinear Anal., Real World Appl., 16(2014), 227–234.

[16] M. Zhang, Periodic solutions of Liénard equations with singular forces of repulsive type, J. Math. Anal. Appl., 203(1996), 254–269.

[17] M. Zheng and J. Li, Nontrivial homoclinic solutions for prescribed mean curvature Rayleigh equations, Adv. Difference Equ., 2015, 2015:77, 13 pp.