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The Edge Fixing Edge-To-Vertex Monophonic Number Of A Graph*

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Abstract

The main purpose of this paper is to investigate the edge fixing edge-to-vertex monophonic number of certain classes of graphs and to study its general properties. In addition, we have shown that for every integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $m_{ev}(G) = a$ and $m_{efev}(G) = b$.

1 Introduction

In this paper G denotes $(G = (V, E))$ a finite undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by p and q respectively. We consider connected graphs with at least three vertices. For basic definitions and terminologies and for the concepts of distance in graph we refer to [1, 3].

A cord of a path $u_0, u_1, u_2, \cdots, u_h$ is an edge u_iu_j with $j \geq i+2$. A u-v path is called a *monophonic path* if it is a chordless path. The monophonic distance has been introduced and studied by [7, 8]. The *monophonic distance* $d_m(u, v)$ is the length of the longest u-v monophonic path in G. A u-v monophonic path of length $d_m(u, v)$ is called a *u-v monophonic*. The *monophonic eccentricity* $e_m(v)$ is the monophonic distance between v and a vertex farthest from v, where $v \in V(G)$. The minimum monophonic eccentricity among the vertices is the *monophonic radius*, $rad_m(G)$ and the maximum monophonic eccentricity is the *monophonic diameter*, $diam_m(G)$ of G.

A monophonic set of G is a set $M \subseteq V$ such that every vertex of G lies on a monophonic path joining some pair of vertices in M. The *monophonic number* $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a *m-set* of G. The monophonic number of a graph has been introduced and studied in [2, 4]. Later Santhakumaran and John introduced edge-to-vertex geodetic concepts in [9, 10]. John and Arul developed the concept the edge monophonic number of a graph in [5].

Although the edge monophonic number is greater than or equal to the monophonic number for an arbitrary graph, the properties of the edge monophonic sets and results regarding edge monophonic number are quite different from that of monophonic

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concepts. These concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts with regard to the problem of designing the route for a shuttle and communication network design. In the case of designing the route for a shuttle, although all the vertices are covered by the shuttle while considering monophonic sets, some of the edges may be left out. This drawback is rectified in the case of edge monophonic sets and hence considering edge monophonic sets is more advantageous to the real life application of routing problem. In particular, the edge monophonic sets are more useful than monophonic sets in the case of regulating and routing the goods vehicles to transport the commodities to important places. This is the motivation behind the introduction and study of edge-to-vertex monophonic concepts.

In [6], we define the edge-to-vertex monophonic number of a graph and have studied the edge-to-vertex monophonic number of standard graphs. Recently, Santhakumaran et al. developed the concept of edge-to-vertex detour monophonic number of a graph in [11].

1.1 Notation and Terminology

We consider finite and simple graphs and use standard terminology. For a graph G, the vertex set is denoted by $V(G)$ and the edge set by $E(G)$. If $e = \{u, v\}$ is an edge of a graph G, we write $e = uv$, we say that e joins the vertices u and v; u and v are adjacent vertices; u and v are incident with e. If two vertices are not joined, then we say that they are non-adjacent. If two distinct edges e and f are incident with a common vertex v, then e and f are said to be adjacent to each other. A set of vertices in a graph is independent if no two vertices in the set are adjacent. A vertex of degree 0 in G is called an end-vertex of G. A cut-vertex (cut-edge) of a graph G is a vertex and edges whose removal increases the number of components. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete.

2 Preliminaries

In this section, we give definition, example and a theorem which will be used in our main results.

DEFINITION 2.1. Let $e, f \in E(G)$. The *e-f monophonic path* is a u-v monophonic path where u is one end of e and v is one end of f. The vertex x is said to lie on a $e - f$ monophonic path if x is a vertex of e-f monophonic path. A set $S \subseteq E(G)$ is called an *edge-to-vertex monophonic set* if every vertex of G lies on a monophonic path between two vertices in $V(S)$. The edge-to-vertex monophonic number of G is the minimum cardinality of its edge-to-vertex monophonic sets and is denoted by $m_{ev}(G)$ and also any edge-to-vertex monophonic set of cardinality $m_{ev}(G)$ is a *m_{ev}-set* of G.

EXAMPLE 2.2. For the graph G given in Figure 2.1 with $e = v_1v_6$ and $f = v_3v_4$, the e-f monophonic paths are $P_1 : v_1, v_2, v_3$, $P_2 : v_6, v_1, v_2, v_3$, $P_3 : v_6, v_7, v_8, v_3$,$P_4 : v_6, v_5, v_4$,$P_5 : v_6, v_7, v_8, v_3, v_4$, $P_6 : v_6, v_5, v_4, v_3$, and $P_7 : v_6, v_1, v_4$. Since the vertices

v_2, v_5, v_7 and v_8 lies on the v_1v_6-v_3v_4 monophonic path, $S = \{e, f\}$ is a m_{ev}-*set* of G so that $m_{ev}(G) = 2$.

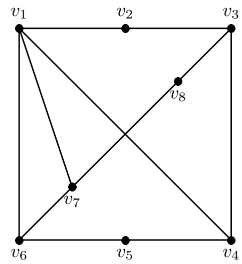

Figure 2.1

THEOREM 2.3 ([6]). Every end-edge of a connected graph G belongs to every edge-to-vertex monophonic set of G.

We have organized this paper in the following way. In section 3, edge fixing edge-to-vertex monophonic number of standard graphs are obtained. In Section 4, a lower bound and upper bound for edge fixing edge-to-vertex monophonic number of a graph is obtained. In Section 5, the realization result involving edge-to-vertex monophonic number and edge fixing edge-to-vertex monophonic number of a graph is obtained.

3 Edge Fixing Edge - To - Vertex Monophonic Number of a Graph

We begin with the following

DEFINITION 3.1. Let e be an edge of a graph G. A set $S_e \subseteq E(G) - \{e\}$ is called an *edge fixing edge-to-vertex monophonic set* of e of a connected graph G if every vertex of G lies on an e-f monophonic path of G where $f \in S_e$. The edge fixing edge-to-vertex monophonic number of G is the minimum cardinality of its edge fixing edge-to-vertex monophonic sets and is denoted by $m_{efev}(G)$ and also any edge fixing edge-to-vertex monophonic set of cardinality $m_{efev}(G)$ is an m_{efev}-*set* of G.

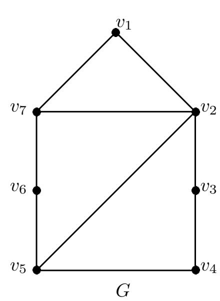

Figure 3.1

EXAMPLE 3.2. For the graph given in Figure 3.1, the edge fixing edge-to-vertex monophonic set is given in the following Table 3.1.

Fixing Edge (e)	Minimum Edge Fixing edge-to-vertex monophonic sets (S_e)	$m_{efev}(G)$
v_1v_7	$\{v_3v_4\}, \{v_4v_5\}$	1
v_6v_7	$\{v_3v_4, v_1v_2\}, \{v_2v_3, v_1v_2\}, \{v_6v_5, v_1v_2\}$, $\{v_1v_7, v_3v_4\}, \{v_4v_5, v_1v_7\}, \{v_2v_3, v_1v_7\}$	2
v_2v_5	$\{v_1v_7, v_3v_4\}, \{v_1v_2, v_3v_4\}$	2
v_2v_7	$\{v_1v_7, v_3v_4\}, \{v_1v_2, v_3v_4\}$	2
v_1v_2	$\{v_3v_4\}$	1
v_3v_4	$\{v_1v_7\}, \{v_1v_2\}$	1
v_6v_5	$\{v_1v_2, v_3v_4\}, \{v_1v_7, v_3v_4\}$	2
v_4v_5	$\{v_1v_7\}, \{v_1v_2\}$	1
v_2v_3	$\{v_1v_7\}$	1

Table 3.1

REMARK 3.3. For a connected graph G, the edge e of G does not belong to edge fixing edge-to-vertex monophonic set S_e.

THEOREM 3.4. Let e be an edge of G and let v be an extreme vertex of a connected graph G such that v is not incident with e. Then every edge fixing edge-to-vertex monophonic set of e of G contains at least one extreme edge that is incident with v irrespective of the fact whether e is an extreme edge or not.

PROOF. Let $e_1, e_2, \cdots, e_k$ be the edges incident with v and let S_e be any edge fixing edge-to-vertex monophonic set of e of G. We claim $e_i \in S_e$ for some $i\,(1 \leq i \leq k)$. Let us assume that $e_i \notin S_e$ for all $i(1 \leq i \leq k)$. Since S_e is an edge fixing edge-to-vertex monophonic set of e of G, the vertex v lies on the monophonic path joining a vertex, say x, incident with e and $y \in V(S_e)$. Since v is an internal vertex of a monophonic path x-y, v is not an extreme vertex of G, which is a contradiction. Hence $e_i \in S_e$ for some $i\,(1 \leq i \leq k)$.

COROLLARY 3.5. Every end-edge of a connected graph G belongs to every edge fixing edge-to-vertex monophonic set of an edge e of G.

THEOREM 3.6. Let G be a connected graph and S_e be an edge fixing edge-to-vertex monophonic set of e of G. Let f be a cut-edge of G, which is not an end-edge of G and let G_1 and G_2 be the two components of $G - \{f\}$.

(i) If $e = f$, then each of the two components of $G - \{f\}$ contains an element of S_e.

(ii) If $e \neq f$, then S_e contains at least one edge of components of $G - \{f\}$ where e does not lie.

PROOF. Let $f = uv$. Let G_1 and G_2 be the two components of $G - \{f\}$ such that $u \in V(G_1)$ and $v \in V(G_2)$.

(i). Let $e = f$. Suppose that S_e does not contain any element of G_1. Then $S_e \subseteq E(G_2)$. Let h be an edge of $E(G_1)$. Then h lies in e-$f^{'}$ monophonic path

$$P : v, v_1, v_2, \ldots, v_l, v, u, u_1, u_2, \ldots, u_s, u, v, v_1, v_2, \ldots, v^{'},$$

where $v_1, v_2, \ldots, v_l \in V(G_2)$, $u_1, u_2, \ldots, u_s \in V(G_1)$ and $v^{'}$ is end of $f^{'}$. Since u, v lies in P more than once, P is not a path, which is a contradiction. Hence each of the two components of $G - \{f\}$ contains an element of S_e.

(ii). By similar arguments, we can prove that if $e \neq f$, then S_e contains at least one edge of component of $G - \{f\}$ where e does not lie.

THEOREM 3.7. Let G be a connected graph and S_e be a minimum edge fixing edge-to-vertex monophonic set of an edge e of G. Then no cut-edge of G which is not an end-edge of G belongs to S_e.

PROOF. Let S_e be an edge fixing edge-to-vertex monophonic set of an edge $e = uv$ of G. Let $f = u'v'$ be a cut-edge of G which is not an edge of G such that $f \in S_e$. If $e = f$, then by definition of edge fixing edge-to-vertex monophonic set of an edge e of G, $f \notin S_e$. If $e \neq f$, let G_1 and G_2 be the two components of $G - \{f\}$ such that $u' \in V(G_1)$ and $v' \in V(G_2)$. By Theorem 3.6, G_1 contains an edge xy and G_2 contains an edge $x^{'}y^{'}$, where $xy, x^{'}y^{'} \in S_e$. Let $S_e^{'} = S_e - \{f\}$. We claim that $S_e^{'}$ is an edge fixing edge-to-vertex monophonic set of an edge e of G.

Case 1. Suppose that $e = xy$ is an edge in G_1 and $x^{'}y^{'}$ is an edge in G_2. Let z be any vertex of G. Assume without loss of generality that z belongs to G_1. Since $u'v'$ is a cut-edge of G_2, every path joining a vertex of G_1 with a vertex of G_2 contains the edge $u'v'$. Suppose that z is incident with $u'v'$ or the edge xy of S_e or that lies on a monophonic path joining xy and $u'v'$. If z is incident with $u'v'$, then $z = u'$. Let $P : y, y_1, y_2, \cdots, z = u'$ be a xy-u' monophonic path. Let $Q : v_1^{'}, v_2^{'}, \cdots, y^{'}$ be a $v^{'}y^{'}$ monophonic path. Then, it is clear that $P \cup \left\{u^{'}v^{'}\right\} \cup Q$ is a xy-$x^{'}y^{'}$ monophonic path .Thus z lies on the xy-$x^{'}y^{'}$ monophonic path. If z is incident with xy, then there is nothing to prove. If z lies on a xy-$x^{'}y^{'}$ monophonic path, say $y, v_1, v_2, \cdots, z, \ldots, u^{'}$, then let $v^{'}, v_1^{'}, v_2^{'}, \cdots, y^{'}$ be $v^{'}$-$x^{'}y^{'}$ monophonic path. Then clearly $y, v_1, v_2, \cdots, z, \cdots, u^{'}, v^{'}, v_1^{'}, v_2^{'}, \cdots, y^{'}$ is a xy-$x^{'}y^{'}$ monophonic path. Thus z lies on a monophonic path joining a pair of edges of $S_e^{'}$. Hence we have proved that a vertex that is incident with $u^{'}v^{'}$ or an edge of S_e or that lies on a monophonic path joining xy and $u^{'}v^{'}$ of S_e also is incident with an edge of $S_e^{'}$ or lies on a monophonic path joining a pair of edges of $S_e^{'}$. Therefore it follows that $S_e^{'}$ is an edge fixing edge-to-vertex monophonic set of an edge e of G such that $\left|S_e^{'}\right| = |S_e| - 1$, which is a contradiction to S_e a m_{efev}- set of G.

Case 2. Suppose that $e = xy \in G_2$.
The proof is similar to that of Case 1.

THEOREM 3.8. For any non-trivial tree T with k end-edges,

$$m_{efev}(G) = \begin{cases} k - 1 & \text{if } e \text{ is an end-edge of } G, \\ k & \text{if } e \text{ is an internal edge of } G. \end{cases}$$

PROOF. This follows from Corollary 3.5 and Theorem 3.7.

THEOREM 3.9. For the cycle graph $G = C_p\,(p \geq 4)$, $m_{efev}(G) = 1$, for every edge in $E(G)$.

PROOF. Let $G = C_p$ be a cycle graph and e be an edge of G. Let f be an edge of G, which is independent of e. Let $S_e = \{f\}$ be an edge fixing edge-to-vertex monophonic set of an edge e of G so that $m_{efev}(G) = 1$.

THEOREM 3.10. For the complete graph $K_p(p \geq 4)$ with p even, $m_{efev}(G) = \frac{p-2}{2}$, for every edge in $E(G)$.

PROOF. Let $G = K_p$ be a complete graph $K_p(p \geq 4)$ and e be an edge of G. Let S_e be any set of $\frac{p-2}{2}$ independent edges of K_p such that $e \notin S_e$. Since each vertex of K_p is either incident with an edge of S_e or incident with e, S_e is an edge fixing edge-to-vertex monophonic set of an edge e of G. Hence it follows that

$$m_{efev}(G) \leq \frac{p-2}{2}.$$

If $m_{efev}(G) < \frac{p-2}{2}$, then there exists an edge fixing edge-to-vertex monophonic set $S_e^{'}$ of e of K_p such that $\left|S_e^{'}\right| < \frac{p-2}{2}$. Therefore, there exists at least one vertex v of K_p such that v is not incident with any edge of $S_e^{'}$. Hence v is neither incident with any edge of $S_e^{'}$ nor lies on a monophonic path e-f where $f \in S_e^{'}$ and so $S_e^{'}$ is not an edge fixing edge-to-vertex monophonic set of an edge e of G, which is a contradiction. Thus S_e is an edge fixing edge-to-vertex monophonic set of an edge e of K_p. Hence $m_{efev}(G) = \frac{p-2}{2}$.

THEOREM 3.11. For the complete graph $G = K_p(p \geq 5)$ with p odd,

$$m_{efev}(K_p) = \frac{p-1}{2} \text{ for every edge in } E(G).$$

PROOF. Let $G = K_p$ be a complete graph $K_p(p \geq 5)$. Let e be an edge of G and let M_e consist of any set of $\frac{p-5}{2}$ independent edges of K_p such that $e \notin S_e$ and $M_e^{'}$ consist of 2 adjacent edges of K_p, each of which is independent with the edges of M_e. Let $S_e = M_e \cup M_e^{'}$. Since each vertex of K_p is either incident with an element of S_e or incident with e, S_e is an edge fixing edge-to-vertex monophonic set of an edge e of G. Hence it follows that

$$m_{efev}(G) \leq \frac{p-5}{2} + 2 = \frac{p-1}{2}.$$

If $m_{efev}(G) < \frac{p-1}{2}$, then there exists an edge fixing edge-to-vertex monophonic set $S_e^{'}$ of K_p such that $\left|S_e^{'}\right| < \frac{p-1}{2}$. Therefore, there exists at least one vertex v of K_p such that v is not incident with any edge of $S_e^{'}$. Hence the vertex v is neither incident with any edge of $S_e^{'}$ nor lies on a monophonic path e-f where $f \in S_e^{'}$ and so $S_e^{'}$ is not an edge fixing edge-to-vertex monophonic set of an edge e of G, which is a contradiction.

Thus S_e is an edge fixing edge-to-vertex monophonic set of an edge e of K_p. Hence $m_{efev}(K_p) = \frac{p-1}{2}$.

THEOREM 3.12. For the complete bipartite graph $G = K_{m,n}(2 \leq m \leq n)$, $m_{efev}(G) = 1$, for every edge in $E(G)$.

PROOF. Let $G = K_{m,n}(2 \leq m \leq n)$ be the complete bipartite graph. Let

$$X = \{x_1, x_2, \cdots, x_m\} \text{ and } Y = \{y_1, y_2, \cdots, y_n\}$$

be the bipartition of G. Let e be any edge of G and let $S_e = \{f\}$, where $e \neq f$. Let

$$e = x_i y_j \text{ and } f = x_k y_l,$$

where $1 \leq i$, $k \leq m$, and $1 \leq j, l \leq n$ such that $i \neq k$ and $j \neq l$. Let x be any vertex of G. If $x \in Y$, then x lies on the monophonic path x_i-x_k. If $x \in X$, then x lies on the monophonic path y_j-y_l. Hence S_e is an edge fixing edge-to-vertex monophonic set of an edge e of G so that $m_{efev}(G) = 1$.

4 Results on Edge Fixing Edge-to-Vertex Monophonic Number of a Graph

THEOREM 4.1. Let G be a connected graph with at least three vertices. Then $1 \leq m_{efev}(G) \leq q - 1$.

PROOF. For an edge e, an edge fixing edge-to-vertex monophonic set needs at least one edge of G so that $m_{efev}(G) \geq 1$. For an edge $e \in E(G)$, $E(G) - \{e\}$ is an edge fixing edge-to-vertex monophonic set of an edge e of G so that $m_{efev}(G) \leq q - 1$. Therefore $1 \leq m_{efev}G) \leq q - 1$.

REMARK.4.2. For the cycle $G = C_p(p \geq 4)$, for an edge e, any edge which is independent of e is its minimum edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) = 1$. For the star $G = K_{1,q}$, for an edge e, the set of edges $E(G) - \{e\}$ is the unique edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) = q - 1$. Thus the star $K_{1,q}$ has the largest possible edge fixing edge-to-vertex monophonic number $q - 1$ and the cycle $C_p(p \geq 4)$ has the smallest edge fixing edge-to-vertex monophonic number 1.

THEOREM 4.3. Let G be a connected graph so that G is neither a star nor a double star. Then $m_{efev}(G) \leq q - 2$ for every $e \in E(G)$.

PROOF. We consider two cases.

Case 1. Suppose that G is a tree such that G is neither a star nor a double star. Then by Theorem 3.8, $m_{efev}(G) \leq q - 2$ for every $e \in E(G)$.

Case 2. Suppose that G is not a tree. Then G contains at least one cycle, say, C. Let e be an edge of G and let $S_e = E(G) - \{e, f\}$, where f is an edge of C such that $f \neq e$. Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q-2$.

So by Cases 1–2, we prove Theorem 4.3.

THEOREM 4.4. Let G be a connected graph and $e \in E(G)$. Then $m_{efev}(G) = q-1$ if and only if e is an edge of $K_{1,q}$ or e is an internal edge of a double star.

PROOF. Let G be a connected graph. If e is an edge of $K_{1,q}$, then by Theorem 3.8, $m_{efev}(G) = q-1$. If e is an internal edge of a double star, then by Theorem 3.8, $m_{efev}(G) = q-1$. Conversely, let $m_{efev}(G) = q-1$ for an edge $e \in E(G)$. Suppose that e is neither an edge of $K_{1,q}$ nor an internal edge of a double star. Then by Theorem 4.3, $m_{efev}(G) \leq q-2$, which is a contradiction.

THEOREM 4.5. Let G be a connected graph with $q \geq 4$, which is not a cycle and not a tree and let $C(G)$ be the length of the longest cycle. Then $m_{efev}(G) \leq q - C(G) + 1$.

PROOF. Let $C(G)$ denote the length of the longest cycle in G and C the cycle of length of $C(G)$. We consider two cases.

Case 1. $C(G)$ is odd.

Subcase 1a. $C(G) = 3$. Let $C: v_1, v_2, v_3, v_1$ be a cycle of length three. Since G is not a cycle, there exists a vertex v in G such that v is not a vertex of C and which is adjacent to v_1. Let $e = v_1v$ be an edge of G such that e is not an edge of C and let

$$S_e = E(G) - \{v_1v_2, v_1v_3\} - \{e\}\,.$$

Clearly S_e is an edge fixing edge-to-vertex monophonic set of e of G so that

$$m_{efev}(G) \leq q-3 = q - C(G) < q - C(G) + 1.$$

Suppose that $e = v_2v_3$ is an edge of G such that $e \in E(C)$. Let

$$S_e = E(G) - \{v_1v_2, v_1v_3\} - \{e\}\,.$$

Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that

$$m_{efev}(G) \leq q-3 = q - C(G) < q - C(G) + 1.$$

Subcase 1b. $C(G) \geq 5$. Let

$$C: v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_{2k+1}, v_1$$

be a cycle of length $C(G) = 2k+1$, $k \geq 2$. Since G is not a cycle, there exists a vertex v in G such that v is not a vertex of C and which is adjacent to v_1, say. Let e be an edge of G but not in $E(C)$. Let

$$S_e = \{E(G) - E(C)\} \cup \{v_{k+1}v_{k+2}\}\,.$$

Clearly S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - C(G) + 1$. Suppose that e is an edge of both G and C, and which is incident with v_1. Let

$$S_e = \{E(G) - E(C)\} \cup \{v_{k+1}v_{k+2}\}\,.$$

Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that

$$m_{efev}(G) \leq q - C(G) + 1.$$

Caes 2. $C(G)$ is even.

Subcase 2a. $C(G) = 4$. Let $C : v_1, v_2, v_3, v_4, v_1$ be a cycle of length 4. Since G is not a cycle, there exists a vertex v in G but not a vertex of C and which is adjacent to v_1, say. Let $e = v_1v$ be an edge of G but not in $E(C)$. Let

$$S_e = E(G) - \{v_1v_2, v_1v_4, v_2v_3\} - \{e\}.$$

Clearly S_e is an edge fixing edge-to-vertex monophonic set of e of G so that

$$m_{efev}(G) \leq q - 4 = q - C(G) < q - C(G) + 1.$$

Suppose that e is an edge of G which is incident with v_1 and also an edge of C. Let

$$S_e = E(G) - \{v_2v_3, v_1v_4\} - \{e\}.$$

Clearly S_e is an edge fixing edge-to-vertex monophonic set of e of G so that

$$m_{efev}(G) \leq q - 3 = q - C(G) + 1.$$

Subcase 2b. $C(G) \geq 6$. Let

$$C : v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_{2k}, v_1$$

be a cycle of length $C(G) = 2k, k \geq 2$. Since G is not a cycle, then there exists a vertex v in G such that v is not a vertex of C and which is adjacent to v_1, say. Let e be an edge of G but not in $E(C)$. Let

$$S_e = \{E(G) - E(C)\} \cup \{v_kv_{k+1}\}.$$

Clearly S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - C(G) + 1$. Suppose that e is an edge of both G and C, and which is incident with v_1. Let

$$S_e = \{E(G) - E(C)\} \cup \{v_kv_{k+1}\}.$$

Then S_e is an edge fixing edge-to vertex monophonic set of e of G so that $m_{efev}(G) \leq q - C(G) + 1$.

Thus $m_{efev} \leq q - C(G) + 1$.

THEOREM 4.6. Let G be a connected graph which is not a double star with $m_{efev}(G) = q - 2$ for some edge e of G. Then G is unicycle.

PROOF. Suppose that G is not unicycle. Then G contains more than one cycle. Let C_1 and C_2 be the two cycles of G. By Theorem 4.5, $|C_1| = |C_2| = 3$.

Case 1. Suppose that C_1 and C_2 have exactly one vertex, say v in common. Let $C_1 : v, v_1, v_2$ and $C_2 : v, u_1, u_2$ be two cycles.

Subcase 1a. Let $e = v_1v_2$ be an edge of C_1 and let

$$S_e = E(G) - \{e, vv_1, vv_2, vu_1, vu_2\}.$$

Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - 5$, which is a contradiction.

Subcase 1b. Let $e = vv_1$ be an edge of C_1 and let

$$S_e = E(G) - \{e, vu_1, vu_2, f\}$$

where $f \in E(C_1)$ such that $f = v_1v_2$ or vv_2 .Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - 4$, which is a contradiction.

Subcase 1c. Let $e = u_1u_2$ be an edge of C_2 and let

$$S_e = E(G) - \{e, vv_1, vv_2, vu_1, vu_2\}.$$

Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - 5$, which is a contradiction.

Subcase 1d. Let $e = vu_1$ be an edge of C_2 and let

$$S_e = E(G) - \{e, vv_1, vv_2, g\}$$

where $g \in E(C_2)$ such that $g = u_1u_2$ or vu_2 .Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - 4$, which is a contradiction.

Case 2. Suppose that C_1 and C_2 have a common edge,say uv. Let $C_1 : u, v, v_1$ and $C_2 : u, v, u_1$ be two cycles.

Subcase 2a. Let $e = uv$ be an edge of G and let $S_e = E(G) - \{e, f, g\}$, where $f \in E(C_1)$ and $g \in E(C_2)$ such that $f = uu_1$ or vv_1 and $g = uu_1$ or vu_1. Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - 3$, which is a contradiction.

Subcase 2b. Let $e = vv_1$ be an edge of C_1 and let

$$S_e = E(G) - \{e, uv, uv_1, uv_2\}.$$

Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - 4$, which is a contradiction.

Subcase 2c. Let $e = vu_1$ be an edge of C_2 and let

$$S_e = E(G) - \{e, uv, uu_1, vv_1\}.$$

Then S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \leq q - 4$, which is a contradiction.

So by Cases 1–2, we have proved Theorem 4.6.

THEOREM 4.7. For a connected graph G, $m_{ev}(G) \leq m_{efev}(G) + 1$.

PROOF. Let e be an edge of G and S_e be the minimum edge fixing edge-to-vertex monophonic set of e of G. Then $S_e \cup \{e\}$ is an edge-to-vertex monophonic set of e of G so that

$$m_{ev}(G) \leq |S_e \cup \{e\}| = m_{efev}(G) + 1.$$

REMARK 4.8. The bounds in Theorem 4.7 is sharp. For the cycle $C_p, m_{efev}(C_p) = 1$ but $m_{ev}(C_p) = 2$. Also the inequality in the Theorem is strict. For the graph G, given in Figure 4.1, let $e = \{u_4u_7\}$ and $S_e = \{u_1u_2, u_5u_6\}$ be an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) = 2$ and $m_{ev}(G) = 2$. Hence $m_{ev}(G) < m_{efev}(G) + 1$.

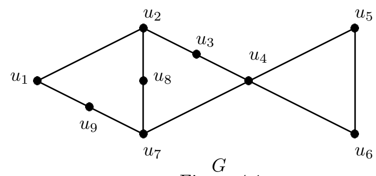

Figure 4.1

5 Realization Result

THEOREM 5.1. For positive integers r_m, d_m and $l \geq 2$ with $r_m < d_m \leq 2r_m$, there exists a connected graph G with $rad_m G = r_m$, $diam_m G = d_m$ and $m_{efev}(G) = l$ or $l-1$, for every $e \in E(G)$.

PROOF. When $r_m = 1$, $G = K_{1,l}$. Then the result follows from Corollary 3.8. Let $r_m \geq 2$. Let $C_{r+2} : v_1, v_2, \cdots, v_{r+2}$ be a cycle of length $r+2$ and let

$$P_{d_m - r+1} : u_0, u_1, u_2, \cdots, u_{d_m - r}$$

be a path of length $d_m - r_m + 1$. Let H be a graph obtained from C_{r+2} and P_{d_m-r+1} by identifying v_1 in C_{r+2} and u_o in P_{d_m-r+1}. Now add new vertices $l-1$, viz, $w_1, w_2, \cdots, w_{l-2}$ to H and join each $w_i (1 \leq i \leq l-2)$ to the vertex u_{d_m-r-1} and obtain the graph G of Figure 5.1. Then $rad_m G = r_m$ and $diam_m G = d_m$. Let

$$S_e = \{u_{d_m-r-1}w_1, u_{d_m-r-1}w_2, \cdots, u_{d_m-r-1}w_{l-2}, u_{d_m-r-1}u_{d_m-r}\}$$

be the set of all end-edges of G.

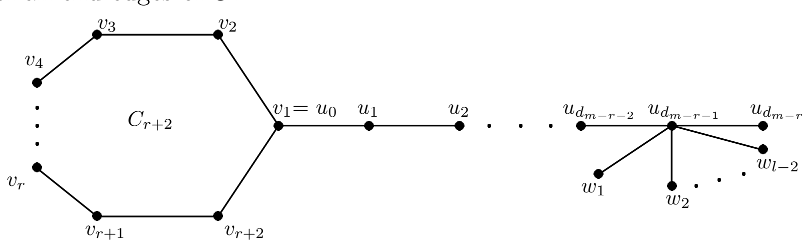

Figure 5.1

Case 1. Suppose that $e \in S_e$. By Corollary 3.5, $S_e - \{e\}$ is a subset of every edge fixing edge-to-vertex monophonic set of e of G. It is clear that $S_e - \{e\}$ is not an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \geq l-2$. However, the set $\{S_e - \{e\}\} \cup \{f\}$, where

$$f \in \{v_2v_3, v_3v_4, \ldots, v_{r+1}v_{r+2}\}$$

is an edge fixing edge-to-vertex monophonic set of e of G so that

$$m_{efev}(G) = l - 2 + 1 = l - 1.$$

Case 2. Suppose that $e \notin S_e$.
Subcase 2a. Let

$$e = u_i u_j \left(0 \leq i \leq u_{d_m - r - 1}, 1 \leq j \leq u_{d_m - r - 1}\right).$$

By Corollary 3.5, S_e is a subset of every edge fixing edge-to-vertex monophonic set of e of G. It is clear that S_e is not an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \geq l$. However $S_e \cup \{f\}$ where

$$f \in \{v_2 v_3, v_3 v_4, \ldots, v_{r+1} v_{r+2}\}$$

is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) = l - 1 + 1 = l$.
Subcase 2b. Suppose that

$$e \neq u_i u_j \left(0 \leq i \leq u_{d_m - r - 1}, 1 \leq j \leq u_{d_m - r - 1}\right).$$

If $e = v_1 v_2$, then by Corollary 3.5, S_e is a subset of every edge fixing edge-to-vertex monophonic set of e of G. It is clear that S_e is not an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \geq l$. However $S_e \cup \{f\}$ where

$$f \in \{v_3 v_4, v_4 v_5, \ldots, v_{r+1} v_{r+2}\}$$

is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) = l - 1 + 1 = l$. If $e = v_1 v_{r+2}$,then by Corollary 3.5, S_e is a subset of every edge fixing edge-to-vertex monophonic set of e of G. It is clear that S_e is not an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) \geq l$. However $S_e \cup \{f\}$ where

$$f \in \{v_2 v_3, v_3 v_4, \ldots, v_r v_{r+1}\}$$

is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) = l - 1 + 1 = l$. If $e \neq v_1 v_2$ and $v_1 v_{r+2}$, then by Corollary 3.5, S_e is a subset of every edge fixing edge-to-vertex monophonic set of e of G. It is clear that S_e is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) = l - 1$.

Hence $m_{efev}(G) = l$ or $l - 1$, for every $e \in E(G)$.

THEOREM 5.2. For any positive integer $a, 1 \leq a \leq q - 1$, there exists a connected graph G of size q such that $m_{efev}(G) = a$, for some edge $e \in E(G)$.

PROOF. Let G be a connected graph. We consider three cases.

Case 1. $a = q - 1$. For the star $G = K_{1,q}$, by Theorem 3.8, we see that $m_{efev} = q - 1 = a$ for every edge $e \in E(G)$.

Case 2. $a = 1$. Let G be a path of length q and e be an edge of G. Then by Theorem 3.8, $m_{efev} = 1 = a$.

Case 3. $1 < a < q - 1$. Let G be a tree with a end edges and $q - a$ internal edges and let e be an internal edge of G. Then by Theorem 3.8, $m_{efev}(G) = a$.

So by Cases 1–3, we have proved Theorem 5.2.

In view of Theorem 4.7, we have the following realization.

THEOREM 5.3. For every pair of positive integers with $2 \leq a \leq b$, there exists a connected graph G such that $m_{ev}(G) = a$ and $m_{efev}(G) = b$, for some edge $e \in E(G)$.

PROOF. Let G be a connected graph. We consider two cases.

Case 1. $a = b$. Let G be a double star with a end edges and let e be a cut - edge of G. Then by Theorem 3.8, $m_{efev}(G) = a$. Also by Theorem 2.3, $m_{ev}(G) = b$.

Case 2. $2 \leq a < b$. Let G be a graph obtained from the path on three vertices $P : u_1, u_2, u_3$ by adding $a - 2$ new vertices $z_1, z_2, \cdots, z_{a-2}$ and joining each $z_i (1 \leq i \leq a - 2)$ with u_2. And also adding $b - a + 2$ new vertices $v_1, v_2, \cdots, v_{b-a+2}$ and a new vertex x and also joining each $v_i (1 \leq i \leq b - a + 2)$ and x with u_1 and u_3. The graph G is shown in Figure 5.2.

First we show that $m_{ev}(G) = a$. Let $S = \{z_1u_2, z_2u_2, \cdots, z_{a-2}u_2\}$ be the set of all end-edges of G. By Theorem 2.3, S is a subset of every edge-to-vertex monophonic set of G. It is clear that S is not an edge-to-vertex monophonic set of G so that $m_{ev}(G) \geq a - 2$. It is easily verified that $S \cup \{f\}$ where $f \notin S$, is not an edge-to-vertex monophonic set of G. However, $S^{'} = S \cup \{u_1u_2, u_2u_3\}$ be the set of all edge-to-vertex monophonic set of G so that $m_{ev}(G) = a$. Let $e = xu_3$. By Corollary 3.5, $S_e = \{z_1u_2, z_2u_2, \cdots, z_{a-2}u_2\}$ is a subset of every edge fixing edge-to-vertex monophonic set of e of G. It is clear that S_e is not an edge fixing edge-to-vertex monophonic set of e of G. Let

$$W_i = \{u_1v_i, u_3v_i\} \, (1 \leq i \leq b - a + 2).$$

It can be observed that every edge fixing edge-to-vertex monophonic set S_e of G contains at least one edge from $W_i (1 \leq i \leq b - a + 2)$ and so

$$m_{efev}(G) \geq a - 2 + b - a + 2 = b.$$

Now

$$S_e^{'} = S_e \cup \{u_1v_1, u_1v_2, \cdots, u_1v_{b-a+2}\}$$

is an edge fixing edge-to-vertex monophonic set of e of G so that $m_{efev}(G) = b$.

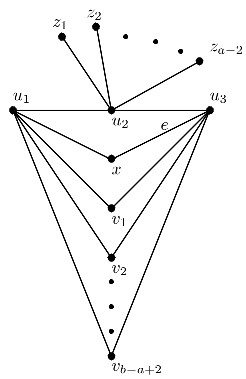

Figure 5.2

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