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The Complementary Exponential-Geometric Distribution Based On Generalized Order Statistics*

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Abstract

This article addresses the problem of moment generating functions of the complementary exponential-geometric distributions using generalized order statistics. The relations for marginal and joint moment generating functions of generalized order statistics from complementary exponential-geometric distribution are derived. The corresponding results for order statistics and record values are deduced from the relations derived. Further, using conditional expectation of generalized order statistics, we obtain characterization of this distribution. Finally, we suggest some applications.

1 Introduction

Let $\{X_n,\ n \geq 1\}$ be a sequence of independent and identically distributed random variables with cumulative distribution function (*cdf*) $F(x)$ and probability density function (*pdf*) $f(x)$. Let $X_{j:n}$ denote the jth order statistic of a sample $X_1, X_2, \ldots, X_n$. Assume that $k > 0$, $n \in N$, $n \geq 2$, $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$, $M_r = \sum_{j=1}^{n-1} m_j$ such that

$$\gamma_r = k + n - r + M_r > 0 \ \ \forall\, r \in \{1, 2, \ldots, n-1\}\,.$$

If the random variables $U(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ possess a joint *pdf* of the form

$$\begin{aligned} & f^{U(1,n,\tilde{m},k),\ldots,U(n,n,\tilde{m},k)}(u_1, u_2, \ldots, u_n) \\ = \ & k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} (1-u_i)^{m_i} \right) (1-u_n)^{k-1}, \end{aligned} \tag{1}$$

on the cone $0 \leq u_1 \leq u_2 \leq \ldots u_n < 1$ of $\Re^n$, then they are called uniform generalized order statistics.

Generalized order statistics (*gos*) based on some distribution function F are defined by means of quantile transformation $X(r, n, \tilde{m}, k) = F^{-1}(U(n, n, \tilde{m}, k))$, $r = 1, 2, \ldots, n$. Ordered random variables such as, order statistics, kth upper record values, upper

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record values, progressively Type II censoring order statistics, Pfeifer records and sequential order statistics are seen to be particular cases of *gos*. These models can be effectively applied in many statistical applications, statistical modeling and inference involving data pertaining to economics, life testing studies, reliability theory and so on. Suppose $X(1,n,m,k),\ldots,X(n,n,m,k)$, ($k \geq 1$, m is a real number), are n *gos* from an absolutely continuous cumulative distribution function *cdf* $F(x)$ with probability density function *pdf* $f(x)$. If their joint *pdf* is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i}f(x_i)\right)[1-F(x_n)]^{k-1}f(x_n), \tag{2}$$

on the cone

$$F^{-1}(0)\leq x_1\leq x_2\leq\ldots\leq x_n\leq F^{-1}(1).$$

For convenience, let us define $X(0,n,m,k)=0$. It can be seen that for

$$m_1=m_2=\cdots=m_{n-1}=0 \text{ and } k=1,$$

i.e., $\gamma_i = n-i+1$ and $1\leq i\leq n-1$, this model reduces to the ordinary order statistic and (2) will be the joint *pdf* of n order statistics $X_{1:n}\leq X_{2:n}\leq\ldots\leq X_{n:n}$ from *cdf* $F(x)$. In a similar manner, choosing the parameter appropriately, some other models such as kth upper record values ($m_1=\cdots=m_n=-1$, $k\in N$, i.e., $\gamma_i=k$, $1\leq i\leq n-1$), sequential order statistics ($m_r=(n-r+1)\alpha_r-(n-r)\alpha_{r+1}-1$; $r=1,2,\ldots,n-1$, $k=\alpha_n$; $\alpha_1,\alpha_2,\ldots,\alpha_n>0$, i.e., $\gamma_i=(n-i+1)\alpha_i$; $i\leq i\leq n-1$), order statistics with non-integral sample size ($m_1=\cdots=m_{n-1}=0$, $k=\alpha-n+1$ with $n-1<\alpha\in\Re$, i.e., $\gamma_i=\alpha-i+1$; $1\leq i\leq n-1$) (Rohatgi and Saleh [1], Saleh et al. [2]), Preifer's record values and progressively type-II right censored order statistics can be obtained (cf. Kamps [3, 4], Kamps and Cramer [5]).

In view of (2), the marginal *pdf* of the r-th *gos*, $X(r,n,m,k)$, $1\leq r\leq n$, is

$$f_{X(r,n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x)). \tag{3}$$

The joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1\leq r<s\leq n$, is

$$\begin{aligned} & f_{X(r,n,m,k),X(s,n,m,k)}(x,y) \\ = & \frac{C_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m f(x)g_m^{r-1}(F(x)) \\ & \times[h_m(F(y))-h_m(F(x))]^{s-r-1}\left[\bar{F}(y)\right]^{\gamma_s-1}f(y) \end{aligned} \tag{4}$$

for $x<y$ where

$$C_{r-1}=\prod_{i=1}^{r}\gamma_i,\quad \bar{F}(x)=1-F(x),$$

$$h_m(x)=\begin{cases} -\frac{1}{m+1}(1-x)^{m+1} & \text{if } m\neq-1, \\ -\ln(1-x) & \text{if } m=-1, \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1),\ x \in [0, 1).$$

Several authors utilized the concept of *gos* in their work. References may be made to Kamps and Gather [6], Keseling [7], Cramer and Kamps [8], Ahsanullah [9], Al-Hussaini et al. [10, 11], Kulshrestha et al. [12] among others.

Kumar [13] has established recurrence relations for marginal and joint mgf of lower generalized order statistics from Marshall-Olkin extended logistic distribution. Kumar [14, 15] also established explicit expressions and some recurrence relations for mgf of kth record values from generalized logistic and extended type II generalized logistic distributions. Recurrence relations for moments of kth record values were investigated, among others, by Grudzień and Szynal [16], and Pawlas and Szynal [17, 18].

The exponential distribution is the most popular distribution for modeling many problems in life testing and reliability studies. Recently Adamidis and Loukas [19] introduced two-parameter complementary exponential-geometric (CEG) distribution lifetime distribution, which is complementary to the exponential-geometric model. For $\lambda > 0$ and $0 < \theta < 1$ the two-parameter CEG distribution has the *pdf* of the form

$$f(x; \theta, \lambda) = \frac{\lambda\theta e^{-\lambda x}}{[\theta + (1-\theta)e^{-\lambda x}]^2}, \quad x > 0 \tag{5}$$

and the corresponding *cdf* is

$$F(x; \theta, \lambda) = 1 - \frac{e^{-\lambda x}}{[\theta + (1-\theta)e^{-\lambda x}]}, \quad x > 0. \tag{6}$$

Here, λ and θ are the scale and shape parameters respectively. Plots of the *pdf* of CEG distribution for some combination of the values of the model parameters are given in Figure 1.

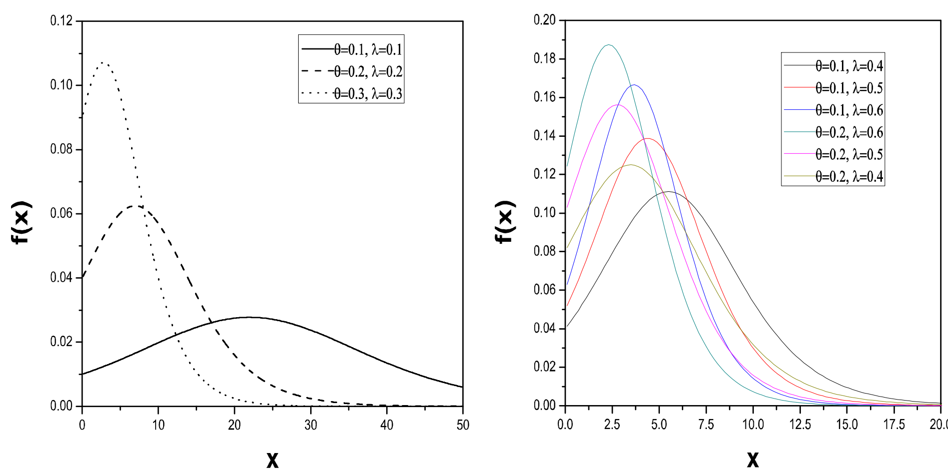

Figure 1. CEG Density Function.

The reliability function $R(x)$, which is the probability of an item not failing prior to some time t, is defined by $R(x) = 1 - F(x)$. The reliability function of a CEG

distribution is given by

$$R(x) = \frac{e^{-\lambda x}}{[\theta + (1-\theta)e^{-\lambda x}]}, \quad x > 0. \tag{7}$$

The basic tools for studying the ageing and reliability characteristics of the system are the hazard rate (HR) and the mean residual lifetime (MRL). The HR and the MRL deal with the residual lifetime of the system. The HR gives the rate of failure of the system immediately after time x, and the MRL measures the expected value of the remaining lifetime of the system, provided that it has survived up to time x. Thus the hazard rate function of the CEG distribution is given by

$$h(x) = \frac{f(x;\theta,\lambda)}{1 - F(x;\theta,\lambda)} = \left(\frac{\theta\lambda}{\theta + (1-\theta)e^{-\lambda x}}\right), \quad x > 0. \tag{8}$$

Plots of the hazard function of CEG distribution for some combination of the values of the model parameters are given in Figure 2.

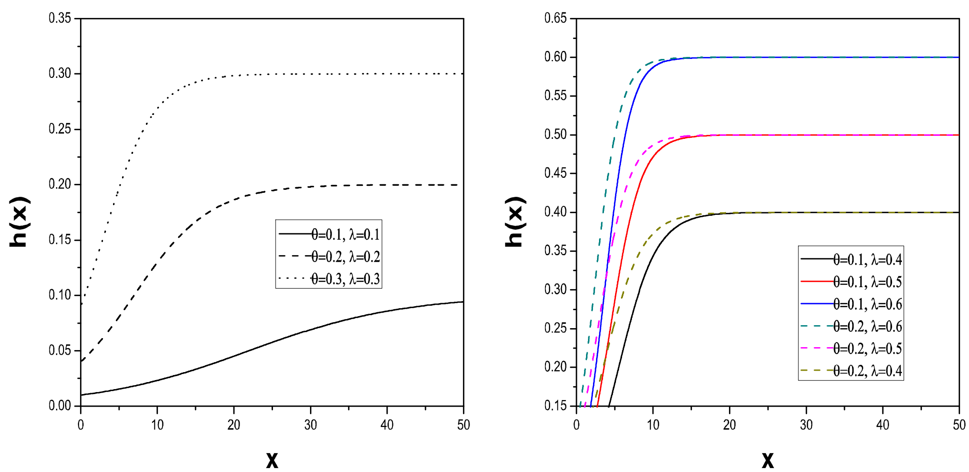

Figure 2. CEG reliability function.

A recurrence relation for moment generating functions of gos from the CEG distribution is obtained by making use of the following (obtained from (5) and (6))

$$\bar{F}(x;\theta,\lambda) = \left(\frac{\theta + (1-\theta)e^{-\lambda x}}{\theta\lambda}\right) f(x;\theta,\lambda). \tag{9}$$

Let us denote the marginal mgf of $X(r,n,m,k)$ by $M_{X(r,n,m,k)}(t)$ and its jth derivative by $M^{(j)}_{X(r,n,m,k)}(t)$. Similarly, let $M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)$ denote the joint mgf of $X(r,n,m,k)$ and $X(s,n,m,k)$ and its (i,j)th partial derivatives by

$$M^{(i,j)}_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2)$$

with respect to t_1 and t_2, respectively.

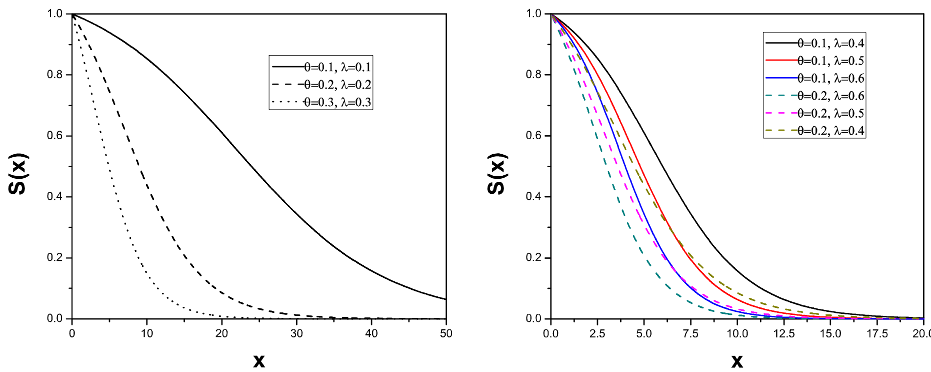

Figure 3. CEG reliability function.

The presentation of the content of this work is as follows: In Section 2, we present some explicit expressions and recurrence relations for marginal mgf of gos from CEG distribution. We obtain the relations for joint mgf of gos from this distribution in Section 3. We also present recurrence relations for the moments so that one can obtain the higher order moments from those of the lower order. In Section 4, we obtain a characterization result of this distribution by using conditional expectation of gos. In Section 5, three applications are demonstrated to illustrate the utility of the results derived in Sections 2 and 3. Section 6 ends with concluding remarks.

2 Relations for Marginal Moment Generating Functions

For the CEG distribution given in (5), the mgf of $X(r,n,m,k)$ is given as

$$\begin{aligned} M_{X(r,n,m,k)}(t) &= \int_{-\infty}^{\infty} e^{tx} f_{X(r,n,m,k)}(x)dx \\ &= \frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty} e^{tx}[\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))dx. \qquad (10) \end{aligned}$$

Further, by using the binomial expansion, we can rewrite (10) as

$$\begin{aligned} M_{X(r,n,m,k)}(t) &= \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} \\ &\quad \times \int_{-\infty}^{\infty} e^{tx}[\bar{F}(x)]^{\gamma_{r-u}-1} f(x)dx. \qquad (11) \end{aligned}$$

Now letting $z = \bar{F}(x)$ in (11), we get

$$M_{X(r,n,m,k)}(t) = \frac{\theta^{-t/\lambda} C_{r-1}}{(r-1)!(m+1)^r}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^{u+p}\binom{r-1}{u}\frac{\Gamma\left(1+\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)}$$

$$\times(1-\theta)^p \quad B\left(\frac{k}{m+1}+n-r+u+\frac{p-(t/\lambda)}{m+1},\quad 1\right). \tag{12}$$

Since

$$\sum_{a=0}^{b}(-1)^a \left(\begin{array}{c} b \\ a \end{array}\right) B(a+k,c)=B(k,c+b), \tag{13}$$

where $B(a,b)$ is the complete beta function, we have

$$M_{X(r,n,m,k)}(t)=\sum_{p=0}^{\infty}\frac{(-1)^p\ \theta^{-t/\lambda}\ (1-\theta)^p\ \ \Gamma\left(1+\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)\prod_{a=1}^{r}\left(1+\frac{p-(t/\lambda)}{\gamma_a}\right)}. \tag{14}$$

Special cases

i) Putting $m=0$ and $k=1$ in (14), the explicit formula for mgf of order statistics from the CEG distribution can be obtained as

$$M_{X_{r:n}(t)}=\frac{\theta^{-t/\lambda}\ n!}{(n-r)!}\sum_{p=0}^{\infty}\frac{(-1)^p\ \ (1-\theta)^p\ \ \Gamma\left(1+\frac{t}{\lambda}\right)\Gamma\left(n-r+1+p-\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)\Gamma\left(n+1+p-\frac{t}{\lambda}\right)},$$

and

$$M_{X_{1:n}(t)}=n\ \theta^{-t/\lambda}\sum_{p=0}^{\infty}\frac{(-1)^p\ \ (1-\theta)^p\ \ \Gamma\left(1+\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)\left(n+p-\frac{t}{\lambda}\right)}\ \text{for } r=1.$$

ii) Setting $m=-1$ in (14), we get the explicit expression for the marginal mgf of kth upper record values from the CEG distribution

$$M_{X(r,n,-1,k)(t)}=\sum_{p=0}^{\infty}\frac{(-1)^p\ \ (1-\theta)^p\ \ \theta^{-t/\lambda}\ \ \Gamma\left(1+\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)\left(1+\frac{p-(t/\lambda)}{k}\right)^r},$$

and

$$M_{X_{U(r)}}=\sum_{p=0}^{\infty}\frac{(-1)^p\ \ (1-\theta)^p\ \ \theta^{-t/\lambda}\ \ \Gamma\left(1+\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)\left(1+p-\frac{t}{\lambda}\right)^r}\ \text{for } r=1.$$

A recurrence relation for the marginal mgf for gos from (9) can be obtained in the following theorem.

THEOREM 1. For the distribution given in (5) and for $2\le r\le n$, $n\ge 2$, $k=1,2,\ldots,$

$$\begin{aligned}&\left(1-\frac{t}{\lambda\gamma_r}\right)M^{(j)}_{X(r,n,m,k)}(t)\\ =\ & M^{(j)}_{X(r-1,n,m,k)}(t)+\frac{j}{\lambda\gamma_r}M^{(j-1)}_{X(r,n,m,k)}(t)\\ &-\frac{(1-\theta)}{\lambda\theta\gamma_r}\left[tM^{(j)}_{X(r,n,m,k)}(t-\lambda)+jM^{(j-1)}_{X(r,n,m,k)}(t-\lambda)\right].\end{aligned} \tag{15}$$

PROOF. From (3), we have

$$M^{(j)}_{X(r,n,m,k)}(t)=\frac{C_{r-1}}{(r-1)!}\int_{-\infty}^{\infty}e^{tx}[\bar{F}(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{16}$$

Integrating by parts of (16) and by (9), we get

$$\begin{aligned}M_{(r,n,m,k)}(t) &= M_{X(r-1,n,m,k)}(t)+\frac{t}{\lambda\gamma_r}M_{(r,n,m,k)}(t)\\ &\quad +\frac{t(1-\theta)}{\lambda\theta\gamma_r}M_{(r,n,m,k)}(t-\lambda). \end{aligned}\tag{17}$$

Differentiating both sides of (17) j times with respect to t, we get the result given in (15). By differentiating both sides of equation (15) with respect to t and then setting $t=0$, we obtain the recurrence relations for moments of gos from CEG in the form

$$\begin{aligned}E[X^j(r,n,m,k)] &= E[X^j(r-1,n,m,k)]+\frac{j}{\lambda\gamma_r}E[X^{j-1}(r,n,m,k)]\\ &\quad +\frac{j(1-\theta)}{\lambda\theta\gamma_r}E[\phi(X(r,n,m,k))], \end{aligned}\tag{18}$$

where

$$\phi(x)=x^{j-1}e^{-\lambda x}.$$

REMARK 1. Putting $m=0$ and $k=1$ in (15) and (18), we can get the following relations for order statistics

$$\begin{aligned}&\left\{1-\frac{t}{\lambda(n-r+1)}\right\}M^{(j)}_{X_{r:n}}(t)\\ =\;& M^{(j)}_{X_{r-1:n}}(t)+\frac{j}{\lambda(n-r+1)}M^{(j-1)}_{X_{r:n}}(t)\\ &+\left\{\frac{(1-\theta)}{\lambda\theta(n-r+1)}\right\}\left[tM^{(j)}_{X_{r:n}}(t-\lambda)+jM^{(j-1)}_{X_{r:n}}(t-\lambda)\right]\end{aligned}$$

and

$$E[X^j_{r:n}]=E[X^j_{r-1:n}]+\frac{j}{\lambda(n-r+1)}\left\{E[X^{j-1}_{r:n}]+\frac{1-\theta}{\theta}E[\phi(X_{r;n})]\right\}.$$

REMARK 2. Setting $m=-1$ and $k\geq 1$ in (15) and (18), relations for record values can be obtained as

$$\begin{aligned}\left\{1-\frac{t}{\lambda k}\right\}M^{(j)}_{Z^{(k)}_r}(t) &= M^{(j)}_{Z^{(k)}_{r-1}}(t)+\frac{j}{\lambda k}M^{(j-1)}_{Z^{(k)}_r}(t)\\ &\quad +\left\{\frac{(1-\theta)}{\lambda\theta k}\right\}\left[tM^{(j)}_{Z^{(k)}_r}(t-\lambda)+jM^{(j-1)}_{Z^{(k)}_r}(t-\lambda)\right]\end{aligned}$$

and

$$E[(Z_r^{(k)})^j] = E[(Z_{r-1}^{(k)})^j] + \frac{j}{\lambda k}\left\{E[(Z_r^{(k)})^{j-1}] + \frac{1-\theta}{\theta}E[\phi(Z_r)]\right\},$$

and hence for upper records,

$$E[X_{U(r)}^j] = E[X_{U(r-1)}^j] + \frac{j}{\lambda}\left\{E[X_{U(r)}^{j-1}] + \frac{1-\theta}{\theta} - E[\phi(X_{U(r)})]\right\}.$$

REMARK 3. The relation in (18) can be used in a simple recursive process to obtain all the rth single moments of generalized order statistics for $j \in Z^+$, (Z^+ is the set of positive integer values). The computations of these moments can be done based on the rth single moment of the order statistics and record value.

3 Relations for Joint Moment Generating Functions

For CEG distribution, the joint mgf of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given as

$$M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) = \int_{-\infty}^{\infty}\int_x^{\infty} e^{t_1x+t_2y} f_{X(r,n,m,k)X(s,n,m,k)}(x,y)dxdy.$$

By (4) and binomial expansion, we have

$$\begin{aligned} & M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) \\ = \; & \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \\ & \times \sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v} \\ & \times \int_0^{\infty} e^{t_1x}[\bar{F}(x)]^{(s-r+u-v)(m+1)-1} f(x)G(x)dx, \end{aligned} \tag{19}$$

where

$$G(x) = \int_x^{\infty} e^{t_2y}[\bar{F}(y)]^{\gamma_{s-v}-1} f(y)dy. \tag{20}$$

By setting $z = \bar{F}(y)$ in (20), we obtain

$$G(x) = \theta^{-t_2/\lambda}\sum_{p=0}^{\infty}\frac{(-1)^p \;\; (1-\theta)^p \;\; \Gamma\left(1+\frac{t_2}{\lambda}\right)[\bar{F}(x)]^{\gamma_{s-v+p-(t_2/\lambda)}}}{p!\Gamma\left(1+\frac{t_2}{\lambda}-p\right)\left[\gamma_{s-v}+p-\frac{t_2}{\lambda}\right]}.$$

On substituting the above expression of $G(x)$ in (19) and simplifying the resulting equation, we get

$$\begin{aligned} & M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) \\ = \; & \frac{\theta^{-(t_1+t_2)/\lambda}\;C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \end{aligned}$$

$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{p+q} \quad (1-\theta)^{p+q} \quad \Gamma\left(1+\frac{t_2}{\lambda}\right)\Gamma\left(1+\frac{t_1}{\lambda}\right)}{p!q!\Gamma\left(1+\frac{t_2}{\lambda}-p\right)\Gamma\left(1+\frac{t_1}{\lambda}-q\right)}$$

$$\times \sum_{u=0}^{r-1}(-1)^u \binom{r-1}{u} B\left(\frac{k}{m+1}+n-r+u+\frac{p+q-(t_1+t_2)/\lambda}{m+1},1\right)$$

$$\times \sum_{v=0}^{s-r-1}(-1)^v \binom{s-r-1}{v} B\left(\frac{k}{m+1}+n-s+v+\frac{p-(t_2/\lambda)}{m+1},1\right). \quad (21)$$

By relation (13) in (21), and after simplification we get

$$\begin{aligned} & M_{X(r,n,m,k),X(s,n,m,k)}(t_1,t_2) \\ = & \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{p+q} \quad (1-\theta)^{p+q} \quad \Gamma\left(1+\frac{t_2}{\lambda}\right)\Gamma\left(1+\frac{t_1}{\lambda}\right)}{p!q!\Gamma\left(1+\frac{t_2}{\lambda}-p\right)\Gamma\left(1+\frac{t_1}{\lambda}-q\right)} \\ & \times \frac{\theta^{-(t_1+t_2)/\lambda}}{\prod_{a=1}^{r}\left(1+\frac{p+q-(t_1+t_2)/\lambda}{\gamma_a}\right)\prod_{b=r+1}^{s}\left(1+\frac{p-(t_2/\lambda)}{\gamma_b}\right)}. \end{aligned} \quad (22)$$

Special Cases

i) Putting $m=0$ and $k=1$ in (22), the explicit formula for joint mgf of order statistics can be obtained as

$$\begin{aligned} & M_{X_{r:n},X_{s:n}}(t_1,t_2) \\ = & \frac{\theta^{-(t_1+t_2)/\lambda} \; n!}{(n-s)!} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{p+q} \quad (1-\theta)^{p+q} \quad \Gamma\left(1+\frac{t_2}{\lambda}\right)\Gamma\left(1+\frac{t_1}{\lambda}\right)}{p!q!\Gamma\left(1+\frac{t_2}{\lambda}-p\right)\Gamma\left(1+\frac{t_1}{\lambda}-q\right)} \\ & \times \frac{\Gamma[n-r+1+p+q-(t_1+t_2)/\lambda]\Gamma[n-s+1+p-(t_2/\lambda)]}{\Gamma[n+1+p+q-(t_1+t_2)/\lambda]\Gamma[n-r+1+p-(t_2/\lambda)]}. \end{aligned}$$

ii) Setting $m=-1$ in (22), we deduce the explicit expression for joint mgf of upper record value in the form

$$\begin{aligned} M_{X_{U(r)},X_{U(s)}}(t_1,t_2) = & \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{(-1)^{p+q} \quad (1-\theta)^{p+q} \quad \Gamma\left(1+\frac{t_2}{\lambda}\right)\Gamma\left(1+\frac{t_1}{\lambda}\right)}{p!q!\Gamma\left(1+\frac{t_2}{\lambda}-p\right)\Gamma\left(1+\frac{t_1}{\lambda}-q\right)} \\ & \times \frac{\theta^{-(t_1+t_2)/\lambda}}{\left(1+\frac{p+q-(t_1+t_2)/\lambda}{k}\right)^{r}\left(1+\frac{p-(t_2/\lambda)}{k}\right)^{s-r}}. \end{aligned}$$

By (9), we can derive the recurrence relations for the joint mgf of gos.

THEOREM 2. Let $X(1,n,m,k),\ldots,X(n,n,m,k)$ be n gos formed from a random sample of size n from the *pdf* (5). Then for $1 \le r < s \le n$, $n \ge 2$ and $k \ge 1$ the following recurrence relation is satisfied

$$\left(1-\frac{t_2}{\lambda\gamma_s}\right) M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,\ t_2)$$

$$= M^{(i,j)}_{X(r,n,m,k)X(s-1,n,m,k)}(t_1,\ t_2) + \frac{(1-\theta)}{\lambda\theta\gamma_s}\Big[t_2 M^{(i,j)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,\ t_2-\lambda) + jM^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,\ t_2-\lambda)\Big] + \frac{j}{\lambda\gamma_s} M^{(i,j-1)}_{X(r,n,m,k)X(s,n,m,k)}(t_1,\ t_2). \quad (23)$$

PROOF. Using (4), the joint mgf of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given by

$$M_{X(r,n,m,k)X(s,n,m,k)}(t_1,\ t_2) = \frac{C_{s-1}}{(r-1)!(s-r-1)!} \times \int_0^\infty [\bar{F}(x)]^m f(x) g_m^{r-1}(F(x)) I(x) dx \quad (24)$$

and

$$I(x) = \int_x^\infty e^{t_1 x + t_2 y}[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(y) dy.$$

Solving the integral in $I(x)$ by parts and using (9), substituting the resulting expression in (24), we get

$$M_{X(r,n,m,k)X(s,n,m,k)}(t_1,\ t_2) = M_{X(r,n,m,k)X(s-1,n,m,k)}(t_1,\ t_2) + \frac{t_2}{\lambda\gamma_s}\Big\{M_{X(r,n,m,k)X(s,n,m,k)}(t_1,\ t_2) + \frac{(1-\theta)}{\theta} M_{X(r,n,m,k)X(s,n,m,k)}(t_1,\ t_2-\lambda)\Big\}. \quad (25)$$

Differentiating both sides of (25) i times with respect to t_1 and then j times with respect to t_2 and simplifying the resulting expression, we get the result given in (23).

One can also note that Theorem 1 can be deduced from Theorem 2 by letting t_1 tends to zero.

By differentiating both sides of equation (23) with respect to t_1, t_2 and then setting $t_1 = t_2 = 0$, we obtain the recurrence relations for product moments of gos from CEG in the form

$$E[X^i(r,n,m,k)X^j(s,n,m,k)] = E[X^i(r,n,m,k)X^j(s-1,n,m,k)] + \frac{j}{\lambda\gamma_s}\Big\{E[X^i(r,n,m,k)X^{j-1}(s,n,m,k)] + \frac{(1-\theta)}{\theta} E[\phi(X(r,n,m,k)X^j(s-1,n,m,k))]\Big\}, \quad (26)$$

where

$$\phi(x,y) = x^i y^{j-1} e^{-\lambda y}.$$

REMARK 4. Putting $m = 0$ and $k = 1$ in (23) and (26), we obtain the recurrence relations for joint mgf and single moments of order statistics in the form

$$\left(1 - \frac{t_2}{\lambda(n-s+1)}\right) M^{(i,j)}_{X_{r,s:n}}(t_1, t_2)$$

$$
\begin{aligned}
&= M^{(i,j)}_{X_{r,s-1:n}}(t_1,t_2) + \frac{j}{\lambda(n-s+1)} M^{(i,j-1)}_{X_{r,s:n}}(t_1,t_2) \\
&\quad + \frac{(1-\theta)}{\lambda\theta(n-s+1)} \left[t_2 M^{(i,j)}_{X_{r,s:n}}(t_1,t_2-\lambda) + j M^{(i,j-1)}_{X_{r,s:n}}(t_1,t_2-\lambda) \right]
\end{aligned}
$$

and

$$E[X^{i,j}_{r,s:n}] = E[X^{i,j}_{r,s-1:n}] + \frac{j}{\lambda(n-s+1)} \left\{ E[X^{i,j-1}_{r,s:n}] + \frac{1-\theta}{\theta} E[\phi(X_{r,s:n})] \right\}.$$

REMARK 5. Substituting $m=-1$ and $k \geq 1$ in (23) and (26), we get recurrence relation for joint mgf and product moments of the kth upper record values for CEG distribution.

4 Characterization

Let $X(r,n,m,k)$, $r=1,2,\ldots,n$ be *gos*. Then from a continuous population with *cdf* $F(x)$ and *pdf* $f(x)$, then the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k)=x$, $1 \leq r < s \leq n$, in view of (5) and (6), is

$$
\begin{aligned}
& f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) \\
= & \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \times \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}}{[\bar{F}(x)]^{\gamma_{r+1}}} f(y). \quad (27)
\end{aligned}
$$

THEOREM 3. Let $X(r,n,m,k)$, $r=1,2,\ldots,n$ be *gos* based on continuous distribution function $F(x)$ with $F(0)=0$ and $0 < F(x) < 1$ for all $x>0$. Then the conditional expectation of *gos* $X(s,n,m,k)$ given $X(r,n,m,k)=x$, is given as

$$
\begin{aligned}
& E[e^{tX(s,n,m,k)}|X(r,n,m,k)=x] \\
& = \theta^{-t/\lambda} \sum_{p=0}^{\infty} \frac{(-1)^p \ (1-\theta)^p \ \Gamma\left(1+\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)} \left(\frac{e^{-\lambda x}}{\theta+(1-\theta)e^{-\lambda x}} \right)^{p-(t/\lambda)} \\
& \times \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j}+p-(t/\lambda)} \right). \quad (28)
\end{aligned}
$$

if, and only if,

$$F(x;\theta,\lambda) = 1 - \frac{e^{-\lambda x}}{[\theta+(1-\theta)e^{-\lambda x}]}, \quad x>0.$$

PROOF. From (27), we have

$$
\begin{aligned}
& E[e^{tX(s,n,m,k)}|X(r,n,m,k)=x] \\
= & \frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}
\end{aligned}
$$

$$\times \int_x^\infty e^{ty} \left[1 - \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{m+1}\right]^{s-r-1} \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_s - 1} \frac{f(y)}{\bar{F}(x)} dy. \tag{29}$$

By setting $w = \frac{\bar{F}(y)}{\bar{F}(x)}$ from (6) in (29), we obtain

$$\begin{aligned} & E\left[e^{tX(s,n,m,k)}|X(r,n,m,k) = x\right] \\ = \; & \frac{\theta^{-t/\lambda} C_{s-1}}{(s-r-1)! C_{r-1}(m+1)^{s-r-1}} \\ & \times \sum_{p=0}^{\infty} \frac{(-1)^p \;\; (1-\theta)^p \;\; \Gamma\left(1+\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)} \left(\frac{e^{-\lambda x}}{\theta + (1-\theta)e^{-\lambda x}}\right)^{p-(t/\lambda)} \\ & \times \int_0^1 w^{\gamma_s + p - (t/\lambda) - 1}(1-w^{m+1})^{s-r-1} dw. \end{aligned} \tag{30}$$

Again by setting $z = w^{m+1}$ in (30) and simplyfying the resulting expression, we get the result given in (28). To prove sufficiency, we have from (27) and (28)

$$\begin{aligned} & \frac{C_{s-1}}{(s-r-1)! C_{r-1}(m+1)^{s-r-1}} \int_x^\infty e^{ty}[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-1} \\ & \times [\bar{F}(y)]^{\gamma_s - 1} f(y) dy = [\bar{F}(x)]^{\gamma_{r+1}} \phi_r(x), \end{aligned} \tag{31}$$

where

$$\begin{aligned} \phi_r(x) \;\; = \;\; & \theta^{-t/\lambda} \sum_{p=0}^{\infty} \frac{(-1)^p \;\; (1-\theta)^p \;\; \Gamma\left(1+\frac{t}{\lambda}\right)}{p!\Gamma\left(1+\frac{t}{\lambda}-p\right)} \\ & \times \left(\frac{e^{-\lambda x}}{\theta + (1-\theta)e^{-\lambda x}}\right)^{p-(t/\lambda)} \prod_{j=1}^{s-r} \left(\frac{\gamma_{r+j}}{\gamma_{r+j} + p - (t/\lambda)}\right). \end{aligned}$$

Differentiating (31) both sides with respect to x and rearranging the terms, we get

$$\begin{aligned} & -\frac{C_{s-1}[\bar{F}(x)]^m f(x)}{(s-r-2)! C_{r-1}(m+1)^{s-r-2}} \\ & \times \int_x^\infty e^{ty}[(\bar{F}(x))^{m+1} - (\bar{F}(y))^{m+1}]^{s-r-2}[\bar{F}(y)]^{\gamma_s - 1} f(y) dy \\ = \; & \phi_r'(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1}\phi_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x) \end{aligned}$$

or

$$-\gamma_{r+1}\phi_{r+1}(x)[\bar{F}(y)]^{\gamma_{r+2}+m} f(x) = \phi_r'(x)[\bar{F}(x)]^{\gamma_{r+1}} - \gamma_{r+1}\phi_r(x)[\bar{F}(x)]^{\gamma_{r+1}-1} f(x).$$

Therefore,

$$\frac{f(x)}{\bar{F}(x)} = -\frac{\phi_r'(x)}{\gamma_{r+1}[\phi_{r+1}(x) - \phi_r(x)]} = \frac{\lambda\theta}{[\theta + (1-\theta)e^{-\lambda x}]},$$

which proves that

$$F(x;\theta,\lambda) = 1 - \frac{e^{-\lambda x}}{[\theta + (1-\theta)e^{-\lambda x}]}, \quad x > 0.$$

REMARK 6. For $m = 0$, $k = 1$ and $k = 1$, $m = -1$, we obtain the characterization results of the CEG distribution based on order statistics and record values, respectively.

5 Numerical Results

In Tables 1–4, we have computed the values of means for $\lambda = 0.5(0.5)4$ and $\theta = 0.5, 1.0$. From Tables 1 and 2, one can see that the mean of order statistics is increasing with respect to θ but decreasing with respect to r, n and λ. Also from Tables 3 and 4 one can see that the means of record values are increasing with respect to θ and r but decreasing with respect to λ. In Tables 5–8, we have computed the variances of order statistics and record values for different values of r, s and n for different values of θ and λ. The numerical computation for the skewness, kurtosis and covariances of order statistics and record values are not presented here but they are available from the author on request.

6 Applications

- The recurrence relations for moments of ordered random variables are important because they reduce the amount of direct computations for moments, evaluate the higher moments and they can be used to characterize distributions.

- The recurrence relations of higher joint moments enable us to derive single, product, triple and quadruple moments which can be used in Edgeworth approximate inference.

- The explicit expressions given in Sections 2 and 3 can be used to calculate the means, variances, skewness, kurtosis and variance covariance matrix.

7 Concluding Remarks

In this paper, we considered the *gos* from CEG model and obtained exact explicit expressions as well as recurrence relations for the marginal and joint moment generating functions of *gos*. The recurrence relations obtained in the paper allow us to evaluate the means, variances and covariances of all order statistics and upper record values for all sample sizes in a simple recursive manner. However, we have only computed the means and variances of the order statistics and record values which are useful in determining best linear unbiased estimators (BLUEs) of location/scale parameters and best linear unbiased predictors (BLUPs) of censored failure times.

n	r	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 4$
1	1	1.188667	1.250000	1.382292	1.554845	1.761231	2.000591	2.274122	2.584052
2	1	0.961931	1.111111	1.322952	1.579761	1.880029	2.226001	2.621434	3.070969
	2	1.415403	1.388889	1.441631	1.529929	1.642434	1.775181	1.926811	2.097136
3	1	0.885451	1.071429	1.319792	1.617861	1.966329	2.369060	2.831299	3.359190
	2	1.114890	1.190476	1.329271	1.503560	1.707429	1.939883	2.201704	2.494528
	3	1.565659	1.488095	1.497812	1.543113	1.609936	1.692830	1.789364	1.898440
4	1	0.846188	1.052632	1.322300	1.645272	2.023573	2.462130	2.967211	3.546018
	2	1.003241	1.127820	1.312269	1.535631	1.794596	2.089847	2.423563	2.798705
	3	1.226540	1.253133	1.346273	1.471490	1.620261	1.789918	1.979845	2.190351
	4	1.678699	1.566416	1.548325	1.566988	1.606495	1.660467	1.725871	1.801136
5	1	0.822177	1.041667	1.325346	1.665063	2.063758	2.527120	3.062171	3.676912
	2	0.942235	1.096491	1.310115	1.566106	1.862833	2.202172	2.587367	3.022441
	3	1.094749	1.174812	1.315499	1.489918	1.692240	1.921361	2.177858	2.463100
	4	1.314401	1.305347	1.366788	1.459204	1.572275	1.702289	1.847836	2.008519
	5	1.769773	1.631683	1.593709	1.593934	1.61505	1.650012	1.695380	1.749290

Table 1: Means of order statistics for $\theta = 0.5$.

n	r	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 4$
1	1	0.953673	1.111111	1.315341	1.558618	1.841705	2.167586	2.540278	2.964495
2	1	0.820704	1.052632	1.339645	1.680008	2.07804	2.539851	3.072604	3.684331
	2	1.086641	1.169591	1.291037	1.437227	1.60537	1.795321	2.007953	2.244659
3	1	0.767948	1.034483	1.362406	1.752961	2.212526	2.74918	3.372145	4.09168
	2	0.926216	1.088929	1.294124	1.534103	1.809067	2.121193	2.473521	2.869631
	3	1.166854	1.209921	1.289493	1.388789	1.503522	1.632386	1.775169	1.932172
4	1	0.739125	1.025641	1.378092	1.799577	2.297864	2.882416	3.563929	4.354273
	2	0.85442	1.061008	1.315347	1.613114	1.956511	2.349473	2.796795	3.303902
	3	0.998012	1.11685	1.272901	1.455091	1.661622	1.892913	2.150248	2.435361
	4	1.223134	1.240945	1.295024	1.366689	1.450822	1.545543	1.650142	1.764443
5	1	0.720896	1.020408	1.389234	1.831757	2.356763	2.974742	3.697505	4.538145
	2	0.812039	1.046572	1.333525	1.670854	2.06227	2.513112	3.029625	3.618786
	3	0.917992	1.082661	1.28808	1.526504	1.797873	2.104015	2.44755	2.831577
	4	1.051359	1.139643	1.262782	1.407483	1.570787	1.752179	1.952046	2.171216
	5	1.266078	1.26627	1.303085	1.35649	1.420831	1.493884	1.574666	1.66275

Table 2: Means of order statistics for $\theta = 1.0$.

n	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 4.0$
1	1.052497	1.235210	1.363826	1.496204	1.693521	1.984294	2.173256	2.238103
2	1.653862	1.562500	1.565107	1.606407	1.671381	1.754250	1.852494	1.965039
3	2.163910	1.953125	1.868903	1.83252	1.821744	1.827416	1.844994	1.871963
4	2.759717	2.441406	2.288561	2.197241	2.137784	2.097798	2.071004	2.053820
5	3.480532	3.051758	2.834640	2.696082	2.597960	2.524277	2.466888	2.421114
6	4.367976	3.814697	3.528950	3.342564	3.207291	3.102857	3.018920	2.949526
7	5.469643	4.768372	4.403261	4.163082	3.987190	3.850085	3.738752	3.645691
8	6.842447	5.960464	5.499690	5.195513	4.971943	4.797017	4.654415	4.534732
9	8.556062	7.450581	6.872180	6.489780	6.208285	5.987694	5.807576	5.656160
10	10.69675	9.313226	8.588877	8.109671	7.756683	7.479881	7.253714	7.063456

Table 3: Means of record statistics for $\theta = 0.5$.

n	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 4.0$
1	1.04904	1.240132	1.337025	1.407328	1.642835	2.382046	2.629143	2.827416
2	1.194922	1.234568	1.312787	1.412352	1.528974	1.661473	1.809830	1.974579
3	1.401497	1.371742	1.38305	1.412203	1.452526	1.501324	1.557344	1.619987
4	1.596976	1.524158	1.497835	1.490128	1.492787	1.502308	1.516904	1.535566
5	1.795662	1.693509	1.644101	1.615365	1.59773	1.587077	1.581275	1.579082
6	2.006472	1.881676	1.816273	1.774041	1.744139	1.721883	1.704837	1.691575
7	2.235395	2.090752	2.012588	1.960348	1.921875	1.891913	1.86772	1.847691
8	2.486934	2.323057	2.23333	2.172524	2.127075	2.091113	2.061576	2.036668
9	2.764929	2.581175	2.479967	2.410964	2.359063	2.317728	2.283547	2.25452
10	3.073023	2.867972	2.754725	2.677301	2.618903	2.572259	2.533576	2.500628

Table 4: Means of record statistics for $\theta = 1.0$.

n	r	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 4$
1	1	0.509944	0.104167	0.291190	0.769530	1.383130	2.182631	3.225980	4.582961
2	1	0.387821	0.015432	0.435491	1.064220	1.951941	3.192420	4.901681	7.225310
	2	0.529249	0.154322	0.153932	0.476070	0.842552	1.274462	1.791540	2.414792
3	1	0.376531	0.005886	0.486213	1.212151	2.276760	3.807193	5.964452	8.953620
	2	0.375308	0.025075	0.334124	0.777083	1.347211	2.085670	3.040420	4.267142
	3	0.538492	0.189419	0.073321	0.326092	0.593490	0.889201	1.223781	1.607050
4	1	0.373941	0.003077	0.516120	1.306540	2.490832	4.221242	6.692970	10.15692
	2	0.365799	0.010073	0.396640	0.938321	1.673850	2.668971	4.000552	5.762712
	3	0.359884	0.032222	0.272142	0.618220	1.035331	1.547342	2.178691	2.956610
	4	0.546913	0.217281	0.017251	0.231324	0.446250	0.674012	0.921610	1.195073
5	1	0.373156	0.001887	0.536183	1.372260	2.642542	4.518630	7.222220	11.04010
	2	0.365546	0.005436	0.435970	1.051482	1.916331	3.116261	4.756320	6.966582
	3	0.352224	0.013351	0.337662	0.771271	1.327651	2.045363	2.967522	4.144610
	4	0.345690	0.037987	0.229521	0.516570	0.846210	1.234520	1.696364	2.247281
	5	0.555748	0.240807	0.025515	0.163232	0.346632	0.534430	0.732580	0.945450

Table 5: Variances of order statistics for $\theta = 0.5$.

n	r	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 4$
1	1	0.279175	0.123460	0.347830	0.874451	1.630651	2.697841	4.178892	6.204180
2	1	0.288376	0.280855	0.471720	1.242670	2.438220	4.224842	6.819460	10.50330
	2	0.234614	0.296510	0.225150	0.535690	0.934781	1.448301	2.105061	2.941360
3	1	0.295707	0.120321	0.536361	1.455012	2.928942	5.188930	8.540063	13.38270
	2	0.257014	0.302329	0.345490	0.849910	1.565290	2.559582	3.916620	5.740250
	3	0.204111	0.411280	0.164982	0.385623	0.650640	0.971853	1.361860	1.834850
4	1	0.299882	0.075881	0.576840	1.593210	3.256612	5.846190	9.734380	15.41370
	2	0.273207	0.127395	0.417871	1.066512	2.033341	3.430181	5.398501	8.117060
	3	0.230512	0.319062	0.274302	0.645810	1.140730	1.793210	2.643720	3.740630
	4	0.182642	0.498280	0.128760	0.300852	0.498391	0.728243	0.997103	1.312120
5	1	0.302486	0.055259	0.604631	1.690271	3.490570	6.321970	10.60940	16.91780
	2	0.282828	0.079499	0.468172	1.225653	2.390121	4.113562	6.591260	10.07320
	3	0.252040	0.133192	0.343650	0.840320	1.540110	2.505521	3.812641	5.554730
	4	0.209045	0.332897	0.227831	0.521801	0.895102	1.367843	1.962652	2.705660
	5	0.166819	1.603440	0.104322	0.246130	0.403710	0.581680	0.784190	1.015450

Table 6: Variances of order statistics for $\lambda = 1.0$.

n	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 4.0$
1	0.218182	0.028793	0.010534	0.022573	0.010768	0.001055	0.101827	0.400230
2	0.226023	0.038342	0.015830	0.038833	0.016638	0.001924	0.102299	0.044392
3	0.239716	0.071449	0.023920	0.461800	0.028809	0.002056	0.108033	0.054239
4	0.249385	0.118348	0.045051	0.063240	0.030630	0.002913	0.110099	0.060414
5	0.256130	0.183785	0.131572	0.086678	0.045219	0.005464	0.110354	0.072390
6	0.342046	0.273992	0.230102	0.195343	0.165270	0.137976	0.112451	0.088100
7	0.472652	0.397128	0.352751	0.320118	0.293586	0.270750	0.250374	0.231729
8	0.657606	0.563870	0.511927	0.475652	0.447495	0.424263	0.404319	0.386715
9	0.911230	0.788117	0.721944	0.677033	0.643107	0.615831	0.592989	0.573300
10	1.253280	1.087963	1.000367	0.941730	0.898030	0.863365	0.834707	0.810316

Table 7: Variance of record statistics for $\theta = 0.5$.

n	$\lambda = 0.5$	$\lambda = 1.0$	$\lambda = 1.5$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 3.0$	$\lambda = 3.5$	$\lambda = 4.0$
1	0.558917	0.093796	0.021201	0.221982	0.048280	0.499177	0.292868	0.408623
2	0.753036	0.336372	0.035760	0.246360	0.054090	0.56485	0.331170	0.519967
3	1.314017	0.814933	0.519670	0.289293	0.085532	0.610787	0.499730	0.405710
4	2.494502	1.755586	1.370353	1.104939	0.896828	0.720748	0.564076	0.419597
5	4.810347	3.546853	2.928756	2.529855	2.237340	2.006135	1.814041	1.648626
6	9.174486	6.881557	5.789232	5.103436	4.614814	4.240028	3.938192	3.686546
7	17.20157	12.98508	10.99684	9.761608	8.891106	8.230955	7.705583	7.272937
8	31.73006	24.01029	20.38415	18.14022	16.56524	15.37581	14.43327	13.66051
9	57.72040	43.71787	37.15032	33.09220	30.24814	28.10358	26.40688	25.01802
10	103.7977	78.64552	66.85519	59.57404	54.47417	50.63078	47.59183	45.10569

Table 8: Variance of record statistics for $\theta = 1.0$.

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