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Two-Steps Hybrid Iterative Schemes With Errors For Generalized Equilibrium Problems And Common Fixed Point Problems*

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Abstract

In this paper, we consider a two-steps hybrid iterative scheme with errors for generalized equilibrium problems and common fixed point problems, and prove the weak limits of sequences $\{x_n\}$ and $\{v_n\}$ obtained under the given scheme for N finite asymptotically k_i-strictly pseudo-contractive mappings $\{T_i\}_{i=1}^N$ and a firmly nonexpansive mapping S_r are the same and hence the point is a common fixed point of $\{T_i\}_{i=1}^N$ and S_r.

1 Introduction

The applications of equilibrium problems and fixed point theory to many branches have been well-known for a long time in nonlinear analysis including optimization theory, economics, etc. (see [2, 3, 5, 7, 9, 12]).

Recently there have been many researches on approximating convergence of fixed points under iteration schemes with errors concerning equilibrium problems and variational inequalities, etc. (see [2, 3, 4, 5, 9]).

On the other hand, Qin et al. [11] and Kumam et al. [6] considered equilibrium problems with fixed point problems under one-step hybrid iterative schemes and two-step hybrid iterative schemes, respectively, in Hilbert spaces.

Inspired by those results, we consider the following two-step hybrid iterative scheme with errors for generalized equilibrium problems and common fixed point problems, and obtain a result that the weak limits of sequences $\{x_n\}$ and $\{v_n\}$ obtained under the given scheme for N finite asymptotically k_i-strictly pseudo-contractive mappings $\{T_i\}_{i=1}^N$ and a firmly nonexpansive mapping S_r are the same and that the same point is a common fixed point of $\{T_i\}_{i=1}^N$ and S_r.

ALGORITHM 1.1. Let C be a closed convex subset of a Hilbert space $H, T_i, \psi : C \to C$ $(i = 1, 2, \cdots, N)$ be mappings and $\phi : C \times C \to \mathbb{R}$ be a bifunction. For any

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$x_0 \in C$, let $\{x_n\}$ and $\{v_n\}$ be sequences generated by

$$\begin{cases} \phi(v_{n-1}, y) + \langle \psi v_{n-1}, y - v_{n-1}\rangle + \frac{1}{r_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \geq 0, \\ x_n = a_{n-1}v_{n-1} + b_{n-1}T_{i(n)}^{h(n)}v_{n-1} + c_{n-1}u_{n-1}, \end{cases} \tag{1}$$

for all $y \in C$ and $n \in \mathbb{N}$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1)$ such that $a_n + b_n + c_n = 1$, $a_n \geq k + \varepsilon$, $b_n \geq \varepsilon$ for some $\varepsilon \in (0, 1)$, $\sum_{n=1}^{\infty} c_n < \infty$, $\{u_n\}$ is a bounded sequence in C, $\{r_n\}$ is a sequence in $(0, \infty)$ such that $\lim_{n\to\infty} \inf r_n > 0$ and $i(n) \equiv n(\operatorname{mod} N)$, $h(n) = \lceil \frac{n}{N} \rceil$ with a ceiling function $\lceil \cdot \rceil$.

REMARK 1.1. (a) Putting $\psi \equiv 0$ in (1), we obtain an algorithm

$$\begin{cases} \phi(v_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \geq 0 \quad \text{for all } y \in C, \\ x_n = a_{n-1}v_{n-1} + b_{n-1}T_{i(n)}^{h(n)}v_{n-1} + c_{n-1}u_{n-1} \quad \text{for each } n \in \mathbb{N}. \end{cases} \tag{2}$$

(b) Putting $c_n = 0$ for all $n \in \mathbb{N}$ in (2), we obtain the algorithm considered in [6]

$$\begin{cases} \phi(v_{n-1}, y) + \frac{1}{r_{n-1}}\langle y - v_{n-1}, v_{n-1} - x_{n-1}\rangle \geq 0 \quad \text{for all } y \in C, \\ x_n = a_{n-1}v_{n-1} + (1 - a_{n-1})T_{i(n)}^{h(n)}v_{n-1} \quad \text{or each } n \in \mathbb{N}. \end{cases} \tag{3}$$

(c) Putting $\phi \equiv 0$ and $v_n = x_n$ $(n \in \mathbb{N})$ in (3), we obtain the algorithm considered in [11]

$$x_n = a_{n-1}x_{n-1} + (1 - a_{n-1})T_{i(n)}^{h(n)}x_{n-1} \text{ for each } n \in \mathbb{N}. \tag{4}$$

2 Preliminaries

First of all, we recall some definitions and results needed in the main results.

DEFINITION 2.1. Let $\phi : C \times C \to \mathbb{R}$ be a function and $\psi : C \to C$ be a nonlinear mapping. (a) ϕ is said to be monotone if $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$. (b) ψ is said to be monotone if $\langle \psi x - \psi y, x - y\rangle \geq 0$ for all $x, y \in C$.

DEFINITION 2.2. A mapping $T : C \to C$ is asymptotically k-strictly pseudo-contractive if there exist $k \in [0, 1)$ and a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\|^2 \leq k_n^2\|x - y\|^2 + k\|(I - T^n)x - (I - T^n)y\|^2 \quad \text{for all } x, y \in C \text{ and } n \in \mathbb{N}.$$

LEMMA 2.1 ([7, 10]). Let H be a real Hilbert space. Then we have the following identities:

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for all $x, y \in H$.

(ii) For all $x, y \in H$ and $a, b, c \in [0,1]$ with $a+b+c=1$,

$$\|ax+by+cz\|^2 = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2.$$

(iii) If $\{x_n\}$ is a sequence in H weakly converging to z, then

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - z\|^2 + \|z-y\|^2 \quad \text{for all } y \in H.$$

LEMMA 2.2 ([10]). Let $\{a_n\}$, $\{c_n\}$ and $\{\delta_n\}$ be nonnegative real sequences satisfying the condition $a_{n+1} \leq (1+\delta_n)a_n + c_n$ for each $n \in \mathbb{N}$. If

$$\sum_{n=1}^{\infty} \delta_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} c_n < \infty,$$

then $\lim_{n\to\infty} a_n$ exists.

3 Main Results

We assume that the mapping $\phi : C \times C \to \mathbb{R}$ satisfies the following conditions:

(i) $\phi(x,x) = 0$ for all $x \in C$;

(ii) ϕ is monotone;

(iii) ϕ is upper hemi-continuous in the first variable;

(iv) ϕ is convex and lower semi-continuous in the second variable.

We have the following theorems.

THEOREM 3.1. Let C be a closed convex subset of a Hilbert space H. Let $\psi : C \to C$ be a monotone nonlinear mapping. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to 2^C$ by

$$S_r x = \left\{ z \in C : \phi(z,y) + \langle \psi z, y-z \rangle + \frac{1}{r}\langle y-z, z-x \rangle \geq 0 \text{ for all } y \in C \right\}.$$

Then the following statements (i)–(iii) hold.

(i) $S_r x$ is a singleton for each $x \in H$.

(ii) S_r is firmly nonexpansive, i.e.,

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x-y \rangle \quad \text{for } x, y \in H.$$

(iii) The set $F(S_r)$ of all fixed points of S_r is a closed and convex subset of C as a solution set of the following equilibrium problem considered in [9]: finding $x \in C$ such that $\phi(x,y) + \langle \psi x, y-x \rangle \geq 0$ for all $y \in C$.

PROOF. (i) We put $\zeta(x,y)=\phi(x,y)+\langle\psi x, y-x\rangle$ for all $x,y\in C$. By the conditions of ϕ and [1, Theorem 1], we see that $S_rx\neq\emptyset$ for any $x\in C$. Next, we show that S_rx is a singleton for $x\in C$. Suppose that $z_1,z_2\in S_rx$. Then

$$\begin{cases}\phi(z_1,y)+\langle\psi z_1,y-z_1\rangle+\frac{1}{r}\langle y-z_1,z_1-x\rangle\geq 0 \ \text{ for } y\in C,\\[2ex] \phi(z_2,y)+\langle\psi z_2,y-z_2\rangle+\frac{1}{r}\langle y-z_2,z_2-x\rangle\geq 0 \ \text{ for } y\in C.\end{cases}\tag{5}$$

Putting $y=z_2$ in the first inequality and $y=z_1$ in the second inequality (5), respectively and adding them, we have

$$\phi(z_1,z_2)+\phi(z_2,z_1)+\langle\psi z_1-\psi z_2,z_2-z_1\rangle\geq\frac{1}{r}\|z_1-z_2\|^2.$$

Since $\phi(z_1,z_2)+\phi(z_2,z_1)\leq 0$ and $\langle\psi z_1-\psi z_2,z_2-z_1\rangle\leq 0$, we have $z_1=z_2$. So we prove statement (i).

(ii) Let $z=S_rx$ and $z'=S_rx'$. Then

$$\begin{cases}\phi(z,z')+\langle\psi z,z'-z\rangle+\frac{1}{r}\langle z'-z,z-x\rangle\geq 0,\\ \phi(z',z)+\langle\psi z',z-z'\rangle+\frac{1}{r}\langle z-z',z'-x'\rangle\geq 0.\end{cases}$$

Adding two inequalities and applying the monotonicity of ϕ and ψ, we have

$$\langle S_rx-S_rx',x-x'\rangle=\langle z-z',x-x'\rangle\geq\|z-z'\|^2=\|S_rx-S_rx'\|^2.$$

Hence, S_r is a firmly nonexpansive mapping. So we prove statement (ii).

(iii) If $x\in F(S_r)$, then

$$\phi(x,y)+\langle\psi x,y-x\rangle=\phi(x,y)+\langle\psi x,y-x\rangle+\frac{1}{r}\langle y-x,x-x\rangle\geq 0$$

for all $y\in C$. So x is a solution of the equilibrium problem in [9]. Next, let $\{x_n\}$ be a convergent sequence in $F(S_r)$ with a limit $x\in H$. Since $F(S_r)\subset C$ and C is closed, we have $x\in C$. Also, S_r is continuous. Then we have

$$x=\lim_{n\to\infty}x_n=\lim_{n\to\infty}S_rx_n=S_rx.$$

It means that $x\in F(S_r)$, that is, $F(S_r)$ is closed.

To show that $F(S_r)$ is convex, we let $z=\lambda x+(1-\lambda)y$ for $x,y\in F(S_r)$ and $\lambda\in[0,1]$. By Lemma 2.1(ii) and the nonexpansiveness of S_r, we have

$$\begin{aligned}\|z-S_rz\|^2 &= \|\lambda(x-S_rz)+(1-\lambda)(y-S_rz)\|^2\\ &= \lambda\|x-S_rz\|^2+(1-\lambda)\|y-S_rz\|^2-\lambda(1-\lambda)\|x-y\|^2\\ &\leq \lambda\|x-z\|^2+(1-\lambda)\|y-z\|^2-\lambda(1-\lambda)\|x-y\|^2\\ &= \lambda\|x-(\lambda x+(1-\lambda)y)\|^2+(1-\lambda)\|y-(\lambda x+(1-\lambda)y)\|^2\\ &\quad -\lambda(1-\lambda)\|x-y\|^2\\ &= \lambda(1-\lambda)^2\|x-y\|^2+(1-\lambda)\lambda^2\|x-y\|^2-\lambda(1-\lambda)\|x-y\|^2=0.\end{aligned}$$

Hence, $S_r z = z$ and $z \in F(S_r)$. Therefore $F(S_r)$ is convex. So we prove statement (iii).

The proof of Theorem 3.1 is complete.

REMARK 3.1. By putting $\psi \equiv 0$ in Theorem 3.1, we obtain [4, Lemma 2.12].

Next, we consider our main result.

THEOREM 3.2. Assume that the mappings $T_i : C \to C$ for $i = 1, \cdots, N$ satisfy the following conditions:

(i) C is a closed convex subset of a Hilbert space H;

(ii) T_i is asymptotically k_i-strictly pseudo-contractive for $k_i \in [0, 1)$, $i = 1, 2, \cdots, N$ and for each $i \in \{1, 2, \cdots, N\}$, $\{k_{n,i}\}$ is a sequence in $[1, \infty)$ such that $\sum\limits_{n=1}^{\infty}(k_{n,i}^2 - 1) < \infty$;

(iii) $k = \max\{k_i : 1 \le i \le N\}$ and $k_n^{'} = \max\{k_{n,i} : 1 \le i \le N\}$ for each $n \in \mathbb{N}$.

Let $\psi : C \to C$ be a monotone nonlinear mapping with

$$F := \left(\bigcap_{i=1}^{N} F(T_i)\right) \bigcap F(S_r) \neq \emptyset.$$

For any $x_0 \in C$, let $\{x_n\}$ and $\{v_n\}$ be sequences generated by Algorithm 1.1. Then $\{x_n\}$ and $\{v_n\}$ converge weakly to the unique same element of F.

PROOF. Let $p \in F$. By Algorithm 1.1 and Theorem 3.1(i), we see that $v_{n-1} = S_{r_{n-1}} x_{n-1}$ and

$$\|v_{n-1} - p\| = \|S_{r_{n-1}} x_{n-1} - S_{r_{n-1}} p\| \le \|x_{n-1} - p\|$$

for each $n \in \mathbb{N}$. By Algorithm 1.1 and Lemma 2.1(ii), we have

$$\begin{aligned}
\|x_n - p\|^2 &= \left\|a_{n-1}(v_{n-1} - p) + b_{n-1}(T_{i(n)}^{h(n)} v_{n-1} - p) + c_{n-1}(u_{n-1} - p)\right\|^2 \\
&\le a_{n-1}\|v_{n-1} - p\|^2 + b_{n-1}\left\|T_{i(n)}^{h(n)} v_{n-1} - T_{i(n)}^{h(n)} p\right\|^2 + c_{n-1}\|u_{n-1} - p\|^2 \\
&\quad - a_{n-1} b_{n-1}\left\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\right\|^2 \\
&\le a_{n-1}\|v_{n-1} - p\|^2 + b_{n-1}\Big\{(k_{h(n)}^{'})^2\|v_{n-1} - p\|^2 + k\left\|(I - T_{i(n)}^{h(n)})v_{n-1}\right. \\
&\quad \left. -(I - T_{i(n)}^{h(n)})p\right\|^2\Big\} + c_{n-1}\|u_{n-1} - p\|^2 - a_{n-1}b_{n-1}\left\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\right\|^2 \\
&\le (k_{h(n)}^{'})^2\|v_{n-1} - p\|^2 - b_{n-1}(a_{n-1} - k)\left\|T_{i(n)}^{h(n)} v_{n-1} - v_{n-1}\right\|^2
\end{aligned}$$

$$+c_{n-1}\|u_{n-1}-p\|^2 \tag{6}$$
$$\leq \left[1+((k'_{h(n)})^2-1)\right]\|x_{n-1}-p\|^2+c_{n-1}\|u_{n-1}-p\|^2. \tag{7}$$

Since $\sum_{n=1}^{\infty}(k_{n,i}^2-1)<\infty$, and by Lemma 2.2, we see that $\lim_{n\to\infty}\|x_n-p\|$ exists. On the other hand, since $a_n \geq k+\varepsilon$ and $b_n \geq \varepsilon$ for $n\in\mathbb{N}$ and some $\varepsilon\in(0,1)$, we have

$$\begin{aligned}
& (k'_{h(n)})^2\|x_{n-1}-p\|^2-\|x_n-p\|^2+c_{n-1}\|u_{n-1}-p\|^2 \\
\geq\ & b_{n-1}(a_{n-1}-k)\|T_{i(n)}^{h(n)}v_{n-1}-v_{n-1}\|^2 \\
\geq\ & \varepsilon^2\|T_{i(n)}^{h(n)}v_{n-1}-v_{n-1}\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}k'_{h(n)}=1$ and $\lim_{n\to\infty}c_n=0$, taking the limits as $n\to\infty$ in the above inequality, we have

$$\lim_{n\to\infty}\left\|T_{i(n)}^{h(n)}v_{n-1}-v_{n-1}\right\|^2=0. \tag{8}$$

Observe that

$$\begin{aligned}
\|x_n-v_{n-1}\| &= \left\|a_{n-1}v_{n-1}+b_{n-1}T_{i(n)}^{h(n)}v_{n-1}+c_{n-1}u_{n-1}-v_{n-1}\right\| \\
&= \left\|-(1-a_{n-1})\left(v_{n-1}-T_{i(n)}^{h(n)}v_{n-1}\right)+c_{n-1}\left(u_{n-1}-T_{i(n)}^{h(n)}v_{n-1}\right)\right\| \\
&\leq (1-a_{n-1})\left\|v_{n-1}-T_{i(n)}^{h(n)}v_{n-1}\right\|+c_{n-1}\left\|u_{n-1}-T_{i(n)}^{h(n)}v_{n-1}\right\|.
\end{aligned}$$

By (8), we see that

$$\lim_{n\to\infty}\|x_n-v_{n-1}\|=0. \tag{9}$$

By the firm nonexpansiveness of $S_{r_{n-1}}$ and Lemma 2.1(i), we have

$$\begin{aligned}
\|v_{n-1}-p\|^2 &= \|S_{r_{n-1}}x_{n-1}-S_{r_{n-1}}p\|^2 \leq \langle S_{r_{n-1}}x_{n-1}-S_{r_{n-1}}p, x_{n-1}-p\rangle \\
&= \langle v_{n-1}-p, x_{n-1}-p\rangle = -\langle -(x_{n-1}-v_{n-1})-(x_{n-1}-p), x_{n-1}-p\rangle \\
&= -\frac{1}{2}\left(\|x_{n-1}-v_{n-1}\|^2-\|x_{n-1}-p\|^2-\|v_{n-1}-p\|^2\right),
\end{aligned}$$

and hence

$$\|v_{n-1}-p\|^2 \leq \|x_{n-1}-p\|^2-\|x_{n-1}-v_{n-1}\|^2.$$

Applying this inequality to (6), we have

$$\|x_n-p\|^2 \leq \left(k'_{h(n)}\right)^2(\|x_{n-1}-p\|^2-\|x_{n-1}-v_{n-1}\|^2)+c_{n-1}\|u_{n-1}-p\|^2.$$

Since $\lim_{n\to\infty}\|x_n-p\|$ exists and $\lim_{n\to\infty}k'_{h(n)}=1$, we see that

$$\lim_{n\to\infty}\|x_{n-1}-v_{n-1}\|=0. \tag{10}$$

Applying (9) and (10) to the triangle inequality, we have

$$\|v_n - v_{n-1}\| \le \|v_n - x_n\| + \|x_n - v_{n-1}\| \to 0 \text{ as } n \to \infty,$$

which implies that

$$\lim_{n\to\infty} \|v_n - v_{n+j}\| = 0 \quad \text{for } j \in \{1, \cdots, N\}. \tag{11}$$

Similarly, applying (10) and (11) to the triangle inequality, we obtain

$$\|x_n - x_{n-1}\| \le \|x_n - v_n\| + \|v_n - v_{n-1}\| + \|v_{n-1} - x_{n-1}\| \to 0 \text{ as } n \to \infty,$$

which implies that $\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0$ for $j \in \{1, \cdots, N\}$. On the other hand,

$$\begin{aligned} \|v_{n-1} - T_n v_{n-1}\| &\le \left\|v_{n-1} - T_{i(n)}^{h(n)} v_{n-1}\right\| + \left\|T_{i(n)} T_{i(n)}^{h(n)-1} v_{n-1} - T_{i(n)} v_{n-1}\right\| \\ &\le \left\|v_{n-1} - T_{i(n)}^{h(n)} v_{n-1}\right\| + L\left(\left\|T_{i(n)}^{h(n)-1} v_{n-1} - T_{i(n-N)}^{h(n)-1} v_{n-N}\right\|\right. \\ &\quad \left. + \left\|T_{i(n-N)}^{h(n)-1} v_{n-N} - v_{n-N-1}\right\| + \|v_{n-N-1} - v_{n-1}\|\right), \end{aligned} \tag{12}$$

where

$$L = \sup\left\{\frac{k + \sqrt{1 + (k_n^2 - 1)(1-k)}}{1-k} : n \in \mathbb{N}\right\}.$$

Since, for each $n > N$, $n = (h(n)-1)N + i(n)$, $i(n-N) = i(n)$ and $h(n-N) = h(n)-1$,

$$\begin{aligned} \left\|T_{i(n)}^{h(n)-1} v_{n-1} - T_{i(n-N)}^{h(n)-1} v_{n-N}\right\| &= \left\|T_{i(n)}^{h(n)-1} v_{n-1} - T_{i(n)}^{h(n)-1} v_{n-N}\right\| \\ &\le L\|v_{n-1} - v_{n-N}\| \end{aligned} \tag{13}$$

and

$$\begin{aligned} &\left\|T_{i(n-N)}^{h(n)-1} v_{n-N} - v_{n-N-1}\right\| \\ \le\ &\left\|T_{i(n-N)}^{h(n-N)} v_{n-N} - T_{i(n-N)}^{h(n-N)} v_{n-N-1}\right\| + \left\|T_{i(n-N)}^{h(n-N)} v_{n-N-1} - v_{n-N-1}\right\| \\ \le\ &L \cdot \|v_{n-N} - v_{n-N-1}\| + \left\|T_{i(n-N)}^{h(n-N)} v_{n-N-1} - v_{n-N-1}\right\|. \end{aligned} \tag{14}$$

So by (12)–(14), we see that

$$\begin{aligned} &\|v_{n-1} - T_n v_{n-1}\| \\ \le\ &\left\|v_{n-1} - T_{i(n)}^{h(n)} v_{n-1}\right\| + L \cdot \left\{\left\|T_{i(n)}^{h(n)-1} v_{n-1} - T_{i(n-N)}^{h(n)-1} v_{n-N}\right\|\right. \\ &\left. + \left\|T_{i(n-N)}^{h(n)-1} v_{n-N} - v_{n-N-1}\right\| + \|v_{n-N-1} - v_{n-1}\|\right\} \\ \le\ &\left\|v_{n-1} - T_{i(n)}^{h(n)} v_{n-1}\right\| + L \cdot \{L\,\|v_{n-1} - v_{n-N}\| + L \cdot \|v_{n-N} - v_{n-N-1}\| \\ &\left. + \left\|T_{i(n-N)}^{h(n-N)} v_{n-N-1} - v_{n-N-1}\right\| + \|v_{n-N-1} - v_{n-1}\|\right\}. \end{aligned}$$

By (8) and (11), we have that $\lim_{n\to\infty} \|v_{n-1} - T_n v_{n-1}\| = 0$. Since

$$\begin{aligned} \|v_n - T_n v_n\| &\leq \|v_n - v_{n-1}\| + \|v_{n-1} - T_n v_{n-1}\| + \|T_n v_{n-1} - T_n v_n\| \\ &\leq (1+L)\cdot\|v_n - v_{n-1}\| + \|v_{n-1} - T_n v_{n-1}\| \to 0 \text{ as } n\to\infty, \end{aligned}$$

for any $j = 1, \cdots, N$, we have

$$\begin{aligned} \|v_n - T_{n+j} v_n\| &\leq \|v_n - v_{n+j}\| + \|v_{n+j} - T_{n+j} v_{n+j}\| + \|T_{n+j} v_{n+j} - T_{n+j} v_n\| \\ &\leq (1+L)\cdot\|v_n - v_{n+j}\| + \|v_{n+j} - T_{n+j} v_{n+j}\| \to 0 \text{ as } n\to\infty, \end{aligned}$$

which gives that $\lim_{n\to\infty} \|v_n - T_l v_n\| = 0$ for $l \in \{1, \cdots, N\}$. Moreover, for each $l \in \{1, \cdots, N\}$, we have

$$\begin{aligned} \|x_n - T_l x_n\| &\leq \|x_n - v_n\| + \|v_n - T_l v_n\| + \|T_l v_n - T_l x_n\| \\ &\leq (1+L)\cdot\|x_n - v_n\| + \|v_n - T_l v_n\| \to 0 \text{ as } n\to\infty. \end{aligned}$$

Put

$$W(x_n) = \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Then $W(x_n) \neq \emptyset$ by the fact that $\{x_n\}$ is bounded in H. Next, we claim that $W(x_n) \subset F$. Let $w \in W(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Since $\lim_{n\to\infty} \|x_n - v_n\| = 0$, we can obtain that $v_{n_i} \rightharpoonup w$ as $i \to \infty$. By the fact that $\lim_{n\to\infty} \|v_n - T_l v_n\| = 0$, $T_l v_{n_i} \to w$ for $l \in \{1, \cdots, N\}$. Now, we show that w is a fixed point of S_r. Since $v_n = T_{r_n} v_n$ for each $n \in \mathbb{N}$, we have

$$\phi(v_n, y) + \langle \psi v_n, y - v_n\rangle + \frac{1}{r_n}\langle y - v_n, v_n - x_n\rangle \geq 0 \text{ for all } y \in C \text{ and } n \in \mathbb{N}.$$

By the monotonicity of ϕ, we have

$$\langle y - v_{n_i}, \frac{v_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq \phi(y, v_{n_i}) + \langle \psi v_{n_i}, v_{n_i} - y\rangle \text{ for } i \in \mathbb{N}.$$

Since $\frac{v_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $v_{n_i} \rightharpoonup w$ as $i \to \infty$, and by the condition (iv) of ϕ, we have

$$\phi(y, w) + \langle \psi w, w - y\rangle \leq 0 \quad \text{for } y \in C.$$

By the conditions (i) and (iv) of ϕ, we see that

$$\begin{aligned} 0 &= \phi(y_t, y_t) \leq t\phi(y_t, y) + (1-t)\phi(y_t, w) \\ &\leq t\phi(y_t, y) + (1-t)\langle \psi w, y_t - w\rangle = t\phi(y_t, y) + (1-t)t\langle \psi w, y - w\rangle \\ &\leq \phi(y_t, y) + (1-t)\langle \psi w, y - w\rangle, \end{aligned}$$

where $t \in (0, 1]$, $y \in C$, and $y_t = ty + (1-t)w$. By the condition (iii) of ϕ,

$$0 \leq \phi(w, y) + \langle \psi w, y - w\rangle \quad \text{for all } y \in C,$$

which shows that $w \in F(S_r)$. Moreover, $w \in \bigcap_{l=1}^{N} F(T_l)$. In fact, if $w \notin F(T_l)$ for some $l \in \{1, \cdots, N\}$, then from the Opial's condition and the fact that $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$,

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - w\| &< \liminf_{i\to\infty} \|x_{n_i} - T_l w\| \le \liminf_{i\to\infty} \{\|x_{n_i} - T_l x_{n_i}\| + \|T_l x_{n_i} - T_l w\|\} \\ &\le \liminf_{i\to\infty} L \cdot \|x_{n_i} - w\|,\end{aligned}$$

which derives a contradiction. Consequently, we have

$$w \in F = \left(\bigcap_{l=1}^{N} F(T_l)\right) \bigcap F(S_r).$$

Finally, we show that $\{x_n\}$ and $\{v_n\}$ converge weakly to the unique same element of F. Indeed, it is sufficient to show that $W(x_n)$ is a singleton. We take any $w_1, w_2 \in W(x_n)$ and let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w_1$ and $x_{n_j} \rightharpoonup w_2$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F$ and $w_1, w_2 \in F$, by Lemma 2.1(iii), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \|x_n - w_1\|^2 &= \limsup_{j\to\infty} \|x_{n_j} - w_1\|^2 = \limsup_{j\to\infty} \|x_{n_j} - w_2\|^2 + \|w_2 - w_1\|^2 \\ &= \limsup_{i\to\infty} \|x_{n_i} - w_2\|^2 + \|w_2 - w_1\|^2 \\ &= \limsup_{i\to\infty} \|x_{n_i} - w_1\|^2 + 2\|w_2 - w_1\|^2 \\ &= \limsup_{n\to\infty} \|x_n - w_1\|^2 + 2\|w_2 - w_1\|^2.\end{aligned}$$

Hence $w_1 = w_2$, which shows that $W(x_n)$ is a singleton. The proof of Theorem 3.2 is complete.

We have the following theorems in [6, 11] as corollaries of Theorem 3.2.

THEOREM 3.3 ([6]). Assume that the conditions (i)–(iii) in Theorem 3.2 hold and that ϕ satisfies

$$F := \left(\bigcap_{i=1}^{N} F(T_i)\right) \bigcap S(\phi) \neq \emptyset.$$

For any $x_0 \in C$, let $\{x_n\}$ and $\{v_n\}$ be sequences generated by (3), where $n = (h - 1)N + i (n \ge 1)$, $i = i(n) \in \{1, 2, \cdots, N\}$, $h = h(n) \ge 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. Let $\{a_n\}$ and $\{r_n\}$ be sequences satisfying $\{a_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in (k, 1)$, $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ and $\{v_n\}$ converge weakly to an element of F.

THEOREM 3.4 ([11]). Assume that the conditions (i)–(iii) in Theorem 3.2 hold and

$$F := \left(\bigcap_{i=1}^{N} F(T_i)\right) \neq \emptyset.$$

For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by (4), where $\{a_n\}$ is a sequence in $(0,1)$ such that $k+\varepsilon \leq a_n \leq 1-\varepsilon$ for some $\varepsilon \in (0,1)$, $n=(h-1)N+i(n \geq 1)$, where $i=i(n) \in \{1,2,\cdots,N\}$, $h=h(n) \geq 1$ is a positive integer and $h(n) \to \infty$ as $n \to \infty$. Then $\{x_n\}$ converges weakly to an element of F.

REMARK 3.2. Our result is a weak convergence under Algorithm 1.1 for a finite family of asymptotically k_i-strictly pseudo-contractive mappings in Hilbert spaces. The convergences, mappings and spaces need to be more weakened, for examples, strongly convergences, asymptotically nonexpansive mappings and $CAT(0)$-spaces, respectively. Till now, many kinds of strong convergence results are well-known, but the weak convergence results are few. So, we suggest the following open problem.

Open problem. Do $\{x_n\}$ and $\{v_n\}$ weakly converge for a finite family of asymptotically nonexpansive mappings with Algorithm 1.1 under suitable conditions?

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