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Adapted Quadratic Approximation For Weakly Singular Integrals*

Mostefa Nadir†, Belkacem Lakehali‡

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Abstract

In this work, we present a new approximation to the weakly singular integral, for this goal, we use a modification of the quadratic spline function and replace it in the integral in order to eliminate the weak singularity. This approximation is destined to solve numerically the weakly singular integral equation on a smooth oriented curve or on an interval.

1 Introduction

Many physical and engineering problems, scattering theory, seismology, heat conduction and fluid flow lead to weakly singular integral equations based on the Abel's integral [2]. Various numerical approximations for Abel's integrals are treated, based on Legendre wavelet approximations [8], Bernstein polynomials [1] and Wavelet Galerkin method [3].

The idea is to replace the Abel kernel by its approximations in the weakly singular integral equation

$$\varphi(t_0) + b_0(t_0) \int_\Gamma \frac{\varphi(t)}{(t-t_0)^\alpha} dt + \int_\Gamma k(t,t_0)\varphi(t)dt = f(t_0), \tag{1}$$

where Γ designates an oriented smooth open curve, the points t and t_0 are on Γ and $0 \le \alpha < 1$, $b_0(t)$, $k(t,t_0)$ and $f(t)$ are a given functions on Γ.

The goal of this work is to present a new technical method based on the quadratic spline functions, in order to give a good and efficiency approximation to the weakly singular integral

$$F(t_0) = \int_\Gamma \frac{\varphi(t)}{(t-t_0)^\alpha} dt, \quad t, t_0 \in \Gamma, \tag{2}$$

where $\varphi(t)$ is a given function on Γ.

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†Department of Mathematics University of Msila 28000 Algeria
‡Department of Mathematics University of Msila 28000 Algeria

2 Quadrature

We denote by t the parametric complex function $t(s)$ of the curve Γ defined by

$$t(s) \ = x(s) + iy(s), \quad a \leq s \leq b,$$

where $x(s)$ and $y(s)$ are continuous functions on the finite interval of definition $[a, b]$ and have continuous first derivatives $x'(s)$ and $y'(s)$ never simultaneously null. Divide the interval $[a, b]$ into N equal subintervals $I_1, I_2, ..., I_N$ by the points

$$s_\sigma = a + \sigma \frac{l}{N}, \ l = b - a \ \text{ for } \sigma = 0, 1, 2, ..., N.$$

Further, we divide each of segments $[s_\sigma, s_{\sigma+1}]$ in two equals segments $[s_\sigma, s_{\sigma M}]$ and $[s_{\sigma M}, s_{\sigma+1}]$ where

$$s_{\sigma M} = s_\sigma + \frac{h}{2}, \quad h = \frac{l}{N}.$$

In other words, we have for each subinterval $[s_\sigma, s_{\sigma+1}]$ the following subdivision

$$[s_\sigma, s_{\sigma+1}] = \{s_\sigma < s_{\sigma M} < s_{\sigma+1}\}.$$

We introduce the notation

$$t_\sigma = t(s_\sigma), \ t_{\sigma M} = t(s_{\sigma M}), \quad t_{\sigma+1} = t(s_{\sigma+1}); \quad \sigma = 0, 1, 2, ..., N-1.$$

Assume that, for the indices $\sigma, \nu = 0, 1, 2, ..., N-1$, the points t and t_0 belong respectively to the arcs $t_\sigma \widehat{t_{\sigma+1}}$ and $t_\nu \widehat{t_{\nu+1}}$ where $t_\alpha \widehat{t_{\alpha+1}}$ designates the arc with ends t_α and $t_{\alpha+1}$ [4,5,6].

For an arbitrary number $\sigma = 0, 1, 2, ..., N-1$, we define the piecewise quadratic interpolation polynomial $S_2(\varphi; t, \sigma)$ dependent on φ, t and σ which represents the quadratic approximation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. We interpolate the function density $\varphi(t)$ with respect to the values $\varphi(t_\sigma), \ \varphi(t_{\sigma M})$ and $\varphi(t_{\sigma+1})$ at the points $t_\sigma, \ t_{\sigma M}$ and $t_{\sigma+1}$ respectively with a quadratic polynomial, given by the following formula.

For $t_\sigma \leq t \leq t_{\sigma+1}$,

$$\begin{aligned} S_2(\varphi; t, \sigma) = & \frac{(t - t_{\sigma M})(t - t_{\sigma+1})}{(t_{\sigma M} - t_\sigma)(t_{\sigma+1} - t_\sigma)} \varphi(t_\sigma) - \frac{(t - t_\sigma)(t - t_{\sigma+1})}{(t_{\sigma M} - t_\sigma)(t_{\sigma+1} - t_{\sigma M})} \varphi(t_{\sigma M}) \\ & + \frac{(t - t_\sigma)(t - t_{\sigma M})}{(t_{\sigma+1} - t_\sigma)(t_{\sigma+1} - t_{\sigma M})} \varphi(t_{\sigma+1}), \end{aligned} \tag{3}$$

this piecewise quadratic interpolating polynomial exists and is unique.

We define for an arbitrary numbers σ and ν, such that $0 \leq \sigma, \nu \leq N-1$, the following continuous function $\beta_{\sigma\nu}(\varphi; t, t_0)$, dependents on $\varphi, \ t$ and t_0

$$\beta_{\sigma\nu}(\varphi; t, t_0) = \begin{cases} U(\varphi; t, \sigma) - V(\varphi; t_0, \sigma, \nu) & \text{for} \quad t \neq t_0, \\ 0 & \text{for} \quad t = t_0. \end{cases} \tag{4}$$

The function $U(\varphi; t, \sigma)$ represents a modified quadratic interpolation of the function density $\varphi(t)$ on the subinterval $[t_\sigma, t_{\sigma+1}]$ of the curve Γ. Indeed, for $t_\sigma \leq t \leq t_{\sigma+1}$ we put

$$\begin{aligned} U(\varphi; t, \sigma) &= \frac{(t - t_{\sigma M})(t - t_{\sigma+1})}{(t_{\sigma M} - t_\sigma)(t_{\sigma+1} - t_\sigma)}\varphi(t_\sigma)\frac{(t - t_0)^\alpha}{(t_\sigma - t_0)^\alpha} \\ &- \frac{(t - t_\sigma)(t - t_{\sigma+1})}{(t_{\sigma M} - t_\sigma)(t_{\sigma+1} - t_{\sigma M})}\varphi(t_{\sigma M})\frac{(t - t_0)^\alpha}{(t_{\sigma M} - t_0)^\alpha} \\ &+ \frac{(t - t_\sigma)(t - t_{\sigma M})}{(t_{\sigma+1} - t_\sigma)(t_{\sigma+1} - t_{\sigma M})}\varphi(t_{\sigma+1})\frac{(t - t_0)^\alpha}{(t_{\sigma+1} - t_0)^\alpha}, \end{aligned}$$

and the function $V(\varphi; t_0, \sigma, \nu)$ is given by

$$\begin{aligned} V(\varphi; t_0, \sigma, \nu) &= S_2(\varphi; t_0, \nu)\frac{(t - t_{\sigma M})(t - t_{\sigma+1})}{(t_{\sigma M} - t_\sigma)(t_{\sigma+1} - t_\sigma)}\frac{(t - t_0)^\alpha}{(t_\sigma - t_0)^\alpha} \\ &- S_2(\varphi; t_0, \nu)\frac{(t - t_\sigma)(t - t_{\sigma+1})}{(t_{\sigma M} - t_\sigma)(t_{\sigma+1} - t_{\sigma M})}\frac{(t - t_0)^\alpha}{(t_{\sigma M} - t_0)^\alpha} \\ &+ S_2(\varphi; t_0, \nu)\frac{(t - t_\sigma)(t - t_{\sigma M})}{(t_{\sigma+1} - t_\sigma)(t_{\sigma+1} - t_{\sigma M})}\frac{(t - t_0)^\alpha}{(t_{\sigma+1} - t_0)^\alpha}. \end{aligned}$$

Denote by $\psi_{\sigma\nu}(\varphi; t, t_0)$ the cubic approximation of the density $\varphi(t)$ at the point $t \in [t_\sigma, t_{\sigma+1}],\ t_0 \in [t_\nu, t_{\nu+1}]$ and $0 \leq \sigma, \nu \leq N - 1$ by

$$\psi_{\sigma\nu}(\varphi; t, t_0) = \varphi(t_0) + \beta_{\sigma\nu}(\varphi; t, t_0). \tag{5}$$

Our idea is to replace the density $\varphi(t)$ by expansion (5) in the weakly singular integral (2)

$$F(t_0) = \int_\Gamma \frac{\varphi(t)}{(t - t_0)^\alpha}dt,$$

and obtain the following approximation noting by $F_n(t_0)$ given as

$$F_n(t_0) = \int_\Gamma \frac{\psi_{\sigma\nu}(\varphi; t, t_0)}{(t - t_0)^\alpha}dt = \int_\Gamma \frac{\varphi(t_0)}{(t - t_0)^\alpha}dt + \int_\Gamma \frac{\beta_{\sigma\nu}(\varphi; t, t_0)}{(t - t_0)^\alpha}dt. \tag{6}$$

3 Main Results

We have

THEOREM 1. Let Γ be an oriented smooth open curve and let φ be a function density defined on Γ. Then the following estimation

$$|F(t_0) - F_n(t_0)| \leq \frac{C}{(2N)^{1-\alpha}}$$

holds, where the constant C depends only on the curve Γ.

PROOF. Taking the points $t \in [t_\sigma, t_{\sigma+1}]$ and $t_0 \in [t_\nu, t_{\nu+1}]$, we write, for $t_\sigma \le t \le t_{\sigma+1}$ and $t_{\nu k} \le t_0 \le t_{\nu+1}$,

$$\begin{aligned}
\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0) &= \frac{\varphi(t) - \varphi(t_0) - \beta_{\sigma\nu}(\varphi; t, t_0)}{(t-t_0)^\alpha} \\
&= \frac{\varphi(t) - \varphi(t_0)}{(t-t_0)^\alpha} - \left\{ \frac{(t-t_{\sigma M})(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_\sigma)} \varphi(t_\sigma) \frac{(t-t_0)^\alpha}{(t_\sigma - t_0)^\alpha} \right. \\
&\quad - \frac{(t-t_\sigma)(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})} \varphi(t_{\sigma M}) \frac{(t-t_0)^\alpha}{(t_{\sigma M}-t_0)^\alpha} \\
&\quad + \frac{(t-t_\sigma)(t-t_{\sigma M})}{(t_{\sigma+1}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})} \varphi(t_{\sigma+1}) \frac{(t-t_0)^\alpha}{(t_{\sigma+1}-t_0)^\alpha} \\
&\quad - \frac{S_2(\varphi; t_0, \nu)(t-t_{\sigma M})(t-t_{\sigma+1})}{(t_{\sigma+1}-t_\sigma)(t_{\sigma M}-t_\sigma)} \frac{(t-t_0)^\alpha}{(t_\sigma-t_0)^\alpha} \\
&\quad + \frac{S_2(\varphi; t_0, \nu)(t-t_\sigma)(t-t_{\sigma+1})}{(t_{\sigma+1}-t_{\sigma M})(t_{\sigma M}-t_\sigma)} \frac{(t-t_0)^\alpha}{(t_{\sigma M}-t_0)^\alpha} \\
&\quad \left. - \frac{S_2(\varphi; t_0, \nu)(t-t_\sigma)(t-t_{\sigma M})}{(t_{\sigma+1)}-t_{\sigma M})(t_{\sigma+1}-t_\sigma)} \frac{(t-t_0)^\alpha}{(t_{\sigma+1}-t_0)^\alpha} \right\}. \qquad (7)
\end{aligned}$$

Taking into account the expression (7) we get

$$\int_\Gamma \frac{\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0)}{(t-t_0)^\alpha} dt = \sum_{\sigma=0}^{N-1} \int_{t_\sigma t_{\sigma+1}} \frac{\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0)}{(t-t_0)^\alpha} dt. \qquad (8)$$

Note that, the equalities $(t_\sigma - t_0)^\alpha = 0$, $(t_{\sigma M} - t_0)^\alpha = 0$ and $(t_{\sigma+1} - t_0)^\alpha = 0$ are possible only when $\sigma = \nu - 1, \nu + 1$ and ν. For these cases, it is easy to see that the integral (8) exists when t_σ tends to t_0 or $t_{\sigma M}$ tends to t_0 or $t_{\sigma+1}$ tends to t_0 as a weakly singular integral. For the other case $\sigma = \nu$, we can easily seeing that, the function $\beta_{\sigma\sigma}(\varphi; t, t_0)$ contains $(t_\sigma - t_0)$, $(t_{\sigma M} - t_0)$ and $(t_{\sigma+1} - t_0)$ as factors, so for all cases the function $\beta_{\sigma\nu}(\varphi; t, t_0)$ makes sense.

Indeed, for the points $t,\ t_0 \in [t_\sigma, t_{\sigma+1}]$ such that $t_\sigma \le t,\ t_0 \le t_{\sigma+1}$, we write

$$\beta_{\sigma\sigma}(\varphi; t, t_0) = U(\varphi; t, \sigma) - V(\varphi; t_0, \sigma, \sigma).$$

Hence

$$\begin{aligned}
&\beta_{\sigma\sigma}(\varphi; t, t_0) \\
= \; & \frac{(t-t_{\sigma M})(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_\sigma)} \frac{(t-t_0)^\alpha}{(t_\sigma-t_0)^\alpha} [\varphi(t_\sigma) - S_2(\varphi; t_0, \sigma)] \\
& - \frac{(t-t_\sigma)(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})} \frac{(t-t_0)^\alpha}{(t_{\sigma M}-t_0)^\alpha} [\varphi(t_{\sigma M}) - S_2(\varphi; t_0, \sigma)] \\
& + \frac{(t-t_\sigma)(t-t_{\sigma M})}{(t_{\sigma+1}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})} \frac{(t-t_0)^\alpha}{(t_{\sigma+1}-t_0)^\alpha} [\varphi(t_{\sigma+1}) - S_2(\varphi; t_0, \sigma)]. \qquad (9)
\end{aligned}$$

In other words, we write

$$\beta_{\sigma\sigma}(\varphi; t, t_0) = (t-t_0)^\alpha Q(\varphi; t, t_0),$$

where the expression $Q(\varphi;t,t_0)$ is given by

$$\begin{aligned} Q(\varphi;t,t_0) &= \frac{(t-t_{\sigma M})(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_\sigma)}\frac{1}{(t_\sigma-t_0)^\alpha}\left[\varphi(t_\sigma)-S_2(\varphi;t_0,\sigma)\right] \\ &\quad -\frac{(t-t_\sigma)(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})}\frac{1}{(t_{\sigma M}-t_0)^\alpha}\left[\varphi(t_{\sigma M})-S_2(\varphi;t_0,\sigma)\right] \\ &\quad +\frac{(t-t_\sigma)(t-t_{\sigma M})}{(t_{\sigma+1}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})}\frac{1}{(t_{\sigma+1}-t_0)^\alpha}\left[\varphi(t_{\sigma+1})-S_2(\varphi;t_0,\sigma)\right]. \end{aligned}$$

Passing now to the estimation of the expression (8), for $t_0 \in t_\nu \widehat{t_{\nu+1}}$ and $\sigma \neq \nu-1,\ \nu+1$ and ν we have

$$\begin{aligned} &\Bigg|\sum_{\sigma=0}^{N-1}\int_{t_\sigma t_{\sigma+1}}\frac{dt}{(t-t_0)^\alpha}\{(\varphi(t)-\varphi(t_0)) \\ &-\{\frac{(t-t_{\sigma M})(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_\sigma)}\varphi(t_\sigma)\frac{(t-t_0)^\alpha}{(t_\sigma-t_0)^\alpha} \\ &-\frac{(t-t_\sigma)(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})}\varphi(t_{\sigma M})\frac{(t-t_0)^\alpha}{(t_{\sigma M}-t_0)^\alpha} \\ &+\frac{(t-t_\sigma)(t-t_{\sigma M})}{(t_{\sigma+1}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})}\varphi(t_{\sigma+1})\frac{(t-t_0)^\alpha}{(t_{\sigma+1}-t_0)^\alpha} \\ &-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma M})(t-t_{\sigma+1})}{(t_{\sigma+1}-t_\sigma)(t_{\sigma M}-t_\sigma)}\frac{(t-t_0)^\alpha}{(t_\sigma-t_0)^\alpha} \\ &+\frac{S_2(\varphi;t_0,\nu)(t-t_\sigma)(t-t_{\sigma+1})}{(t_{\sigma+1}-t_{\sigma M})(t_{\sigma M}-t_\sigma)}\frac{(t-t_0)^\alpha}{(t_{\sigma M}-t_0)^\alpha} \\ &-\frac{S_2(\varphi;t_0,\nu)(t-t_\sigma)(t-t_{\sigma M})}{(t_{\sigma+1}-t_{\sigma M})(t_{\sigma+1}-t_\sigma)}\frac{(t-t_0)^\alpha}{(t_{\sigma+1}-t_0)^\alpha}\Bigg\}\Bigg| = O\left(\frac{1}{(2N)^{1-\alpha}}\right). \end{aligned}$$

Indeed, it is clear that

$$\max_{t_0\in t_\nu \widehat{t_{\nu+1}}}\left|\sum_{\sigma=0}^{N-1}\int_{t_\sigma}^{t_{\sigma+1}}\frac{(\varphi(t)-\varphi(t_0))}{(t-t_0)^\alpha}dt\right| = O\left(\frac{1}{(2N)^{1-\alpha}}\right)$$

and also we estimate the expression

$$\begin{aligned} &\Bigg|\sum_{\sigma=0}^{N-1}\int_{t_\sigma}^{t_{\sigma+1}} -\Bigg\{\frac{(t-t_{\sigma M})(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_\sigma)}\varphi(t_\sigma)\frac{(t-t_0)^\alpha}{(t_\sigma-t_0)^\alpha} \\ &-\frac{(t-t_\sigma)(t-t_{\sigma+1})}{(t_{\sigma M}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})}\varphi(t_{\sigma M})\frac{(t-t_0)^\alpha}{(t_{\sigma M}-t_0)^\alpha} \\ &+\frac{(t-t_\sigma)(t-t_{\sigma M})}{(t_{\sigma+1}-t_\sigma)(t_{\sigma+1}-t_{\sigma M})}\varphi(t_{\sigma+1})\frac{(t-t_0)^\alpha}{(t_{\sigma+1}-t_0)^\alpha} \\ &-\frac{S_2(\varphi;t_0,\nu)(t-t_{\sigma M})(t-t_{\sigma+1})}{(t_{\sigma+1}-t_\sigma)(t_{\sigma M}-t_\sigma)}\frac{(t-t_0)^\alpha}{(t_\sigma-t_0)^\alpha} \end{aligned}$$

$$+\frac{S_2(\varphi;t_0,\nu)(t-t_\sigma)(t-t_{\sigma+1})}{(t_{\sigma+1}-t_{\sigma M})(t_{\sigma M}-t_\sigma)}\frac{(t-t_0)^\alpha}{(t_{\sigma M}-t_0)^\alpha}$$
$$\left.\left.-\frac{S_2(\varphi;t_0,\nu)(t-t_\sigma)(t-t_{\sigma M})}{(t_{\sigma+1}-t_{\sigma M})(t_{\sigma+1}-t_\sigma)}\frac{(t-t_0)^\alpha}{(t_{\sigma+1}-t_0)^\alpha}\right\}\frac{1}{(t-t_0)^\alpha}dt\right| = O\left(\frac{1}{(2N)^{1-\alpha}}\right).$$

Naturally, the estimation given above is obtained by using expressions

$$\left|\frac{(t-t_0)^\alpha}{(t_\sigma-t_0)^\alpha}\right| = O(1),\ \left|\frac{(t-t_0)^\alpha}{(t_{\sigma M}-t_0)^\alpha}\right| = O(1),\ \left|\frac{(t-t_0)^\alpha}{(t_{\sigma+1}-t_0)^\alpha}\right| = O(1).$$

Further, for the cases where $\sigma=\nu-1,\ \nu+1$ and ν, using the condition of smoothness of Γ, we get

$$\left|\int_{t_\nu t_{\nu+1}}\frac{\varphi(t)-\varphi(t_0)}{(t-t_0)^\alpha}dt\right| \le A\int_{s_\nu}^{s_{\nu+1}}(t-t_0)^{1-\alpha}ds = O\left(\frac{1}{(2N)^{2-\alpha}}\right),$$

where A represents the bound of the derivative $\varphi'(t_0)$ of the density function, say $|\varphi'(t_0)| \le A$.

4 Numerical Experiments

Using our approximation, we apply the algorithm to weakly singular integrals and we present results concerning the accuracy of the calculations. In this numerical experiment each table F represents the exact value of the weakly singular integral and F_n corresponds to the approximate calculation produced by our approximation at points values interpolation.

EXAMPLE 1. Consider the Abel integral

$$I = F(t_0) = \int_0^{t_0}\frac{\varphi(t)}{\sqrt{t_0-t}}dt,$$

where the function $F(t_0)$ is calculated chosen so that the function $\varphi(t)$ is given

$$\varphi(t)=t,\quad F(t_0)=\frac{4}{3}t^{\frac{3}{2}}.$$

The approximate Abel integral $F_n(t_0)$ of $F(t_0)$ is obtained by the adapted quadratic approximation

TABLE 1. We present the exact and the approximate values of the Abel integral in the example 1 in some arbitrary points, the error for $N = 10$ is calculated.

Values of t	Exact integral F	Approx integral F_n	Error
0.000000	$0.000000e+000$	$0.000000e+000$	$0.000000e+000$
0.200000	$1.192570e-001$	$1.192570e-001$	$1.387779e-017$
0.400000	$3.373096e-001$	$3.373096e-001$	$5.551115e-017$
0.600000	$6.196773e-001$	$6.196773e-001$	$0.000000e+000$
0.800000	$9.540557e-001$	$9.540557e-001$	$1.110223e-016$
1.000000	$1.333333e+000$	$1.333333e+000$	$2.220446e-016$

EXAMPLE 2. Consider the Abel integral

$$I = F(t_0) = \int_0^{t_0} \frac{\varphi(t)}{\sqrt{t_0 - t}} dt,$$

where the function $F(t_0)$ is calculated chosen so that the function $\varphi(t)$ is given

$$\varphi(t) = t^2, \quad F(t_0) = \frac{16}{15} t^{\frac{5}{2}}.$$

The approximate Abel integral $F_n(t_0)$ of $F(t_0)$ is obtained by the adapted quadratic approximation

TABLE 2. We present the exact and the approximate values of the Abel integral in the example 2 in some arbitrary points, the error for $N = 10$ is calculated.

Values of t	Exact integral F	Approx integral F_n	Error
0.000000	$0.000000e+000$	$0.000000e+000$	$0.000000e+000$
0.200000	$1.908111e-002$	$1.908111e-002$	$0.000000e+000$
0.400000	$1.079391e-001$	$1.079391e-001$	$0.000000e+000$
0.600000	$2.974451e-001$	$2.974451e-001$	$0.000000e+000$
0.800000	$6.105956e-001$	$6.105956e-001$	$1.110223e-016$
1.000000	$1.066667e+000$	$1.066667e+000$	$2.220446e-016$

EXAMPLE 3. Consider the Abel integral

$$I = F(t_0) = \int_0^{t_0} \frac{\varphi(t)}{\sqrt{t_0 - t}} dt,$$

where the function $F(t_0)$ is calculated chosen so that the function $\varphi(t)$ is given

$$\varphi(t) = \frac{1}{\sqrt{1+t}}, \quad F(t_0) = \frac{\pi}{2} - \arcsin\left(\frac{1-t_0}{1+t_0}\right).$$

The approximate Abel integral $F_n(t_0)$ of $F(t_0)$ is obtained by the adapted quadratic approximation

TABLE 3. We present the exact and the approximate values of the Abel integral in the example 3 in some arbitrary points, the error for $N = 10$ is calculated.

Values of t	Exact integral F	Approx integral F_n	Error
0.000000	$0.000000e+000$	$0.000000e+000$	$0.000000e+000$
0.200000	$8.410687e-001$	$8.411104e-001$	$4.173975e-005$
0.400000	$1.127885e+000$	$1.128199e+000$	$3.137339e-004$
0.600000	$1.318116e+000$	$1.318987e+000$	$8.713448e-004$
0.800000	$1.459455e+000$	$1.461079e+000$	$1.623303e-003$
1.000000	$1.570796e+000$	$1.573241e+000$	$2.444518e-003$

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