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Some Natural Generalizations Of The Collatz Problem*

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Abstract

We define here simple natural generalizations of Collatz Problem, stating some corresponding conjectures and showing some first interesting computations. We also address the conceptual importance of this kind of problems and the expected difficulties into solving them.

1 Why Generalize an Apparently Intractable Problem?

The Collatz Conjecture or $3x + 1$ Conjecture, an elusive two-line algorithm simple to state and awfully hard to solve, is perhaps one of the most perplexing unsolved mathematical problems, challenging equally mathematicians, logicians and even philosophers. One of its generalizations is even undecidable (cf. [3], more on this below).

Lothar Collatz (1910-1990)[1] proposed the problem in 1928, originally stated as follows: consider the function which inputs a non-zero integer x and outputs $3x + 1$ if x is odd, and $x/2$ if x is even. The $3x + 1$ Conjecture asserts that, starting from any positive integer x, repeated iteration of this function eventually produces the value 1. In a more appropriate notation, the conjecture is usually rephrased by considering the function:

$$T_2(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \pmod 2, \\ \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod 2. \end{cases}$$

The (rephrased) conjecture states that every trajectory starting from a non-zero integer will end in an element of one of the four cycles:

$(1, 2)$;
(-1);
$(-5, -7, -10)$;
$(-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34)$.

Of course, the cycle is unique, i.e., $(1, 2)$, if inputs are restricted to positive integers x.

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[1] See [13] for a tribute to Lothar Collatz.

There is an extensive literature on the many attempts to settle the conjecture, as well as related questions, touching from number theory to Markov chains (see specially [4]) and dynamical systems, and there is a 47-page annotated bibliography in [10] and an excellent survey up to 1985 (cf. [9])

The problem is also known as the Syracuse problem, Kakutani's problem, Hasse's algorithm, Ulam's problem, Thwaites's problem and Hailstone Algorithm, and it is not known whether it is provable in Peano Arithmetic. But even if it is intractable, Lagarias in [9] offers a good reason to keep trying: "No problem is so intractable that something interesting cannot be said about it."

I present in the next section what I consider to be some of the most natural generalizations of Collatz Problem.

2 On Two Natural Generalizations

One of the most natural reasons why the original Collatz algorithm keeps running smoothly (whether or not it stops is another matter) is that it establishes a parity equilibrium in the sense that $x/2$ reaches 1 (in which case it enters a cycle) or reaches an odd number greater than 1, in which case $3x+1$ is even again. The present generalization simply widens the scope of such parity equilibrium to general congruences and explore its consequences.

Define a mapping $T_d : \mathbb{Z} \mapsto \mathbb{Z}$ by:

$$T_d(x) = \begin{cases} \frac{x}{d} & \text{if } x \equiv 0 \pmod{d}, \\ \frac{(d+1)x+d-i}{d} & \text{if } x \equiv i \pmod{d},\ 1 \le i \le d-1. \end{cases}$$

It is easy to prove that $x \equiv i \pmod{d}$ implies $T_d(x) \equiv 0 \pmod{d}$, thus the mapping $T_d(x)$ is well-defined over $\mathbb{Z}$. For instance, $d = 2$ gives the original $3x+1$ mapping in the form $T_2(n)$ above.

The first conjecture on the behavior of cycles is the following:

CONJECTURE 1 (The mapping $T_d(x)$ has finitely many finite cycles). The sequence of iterates

$$x, T_d(x), {T_d}^2(x), ..., {T_d}^k(x), ...$$

for each d always eventually enters a cycle, for finite k, and there are only finitely many such cycles.

2.1 Some Interesting Cycles

It is easy to prove[2] that (i) $T_d(x) = x$ for $x = -1, ..., -(d-1)$ and that (ii) $1, 2, ..., d$ is always a cycle; those are called *elementary cycles* (respectively, *positive* and *negative*. There are many non-elementary cycles, as for instance (for the first values of d and $x \le 50,000$):

[2] This observation is due to Keith R. Matthews, personal communication.

- For $d = 3$:

  $(7, 10, 14, 19, 26, 35, 47, 63, 21)$ (cycle length 9)

  $(-22, -29, -38, -50, -66)$ (cycle length 5)

- For $d = 4$: $(23, 29, 37, 47, 59, 74, 93, 117, 147, 184, 46, 58, 73, 92)$ (cycle-length 14)

  $(-18, -22, -27, -33, -41, -51, -63, -78, -97, -121, -151, -188, -47, -58, -72)$ (cycle-length 15)

- For $d = 5$:

  $(-57, -68, -81, -97, -116, -139, -166, -199, -238, -285)$ (cycle length 10)

- For $d = 6$:

  $(23, 27, 32, 38, 45, 53, 62, 73, 86, 101, 118, 138)$ (cycle length 12)

  $(88, 103, 121, 142, 166, 194, 227, 265, 310, 362, 423, 494, 577, 674, 787, 919, 1073,$
  $1252, 1461, 1705, 1990, 2322, 387, 452, 528)$ (cycle-length 25)

The first d for which apparently there are no cycles other than the elementary ones is $d = 7$; the same holds for $d = 14$, $d = 18$ and $d = 21$. An interesting point is that, differently from the original Collatz problem, several positive cycles arise. See Table 1 for more details.

A webpage and a CALC number theory program related to the present conjecture, developed by Keith Matthews, can be found at [6, 7]. The here explained notion of "balancing parity" has been used by Keith Matthews to generalize a mapping of Lu Pei (cf. [8]). The generalized Lu Pei's mapping is defined as follows:

$$L_d(x) = \begin{cases} \frac{x}{d} & \text{if } x \equiv 0 \pmod{d}, \\ \frac{(d+1)x-i}{d} & \text{if } x \equiv i \pmod{d}, \quad \frac{-d}{2} \le i \le \frac{d}{2},\ i \neq 0. \end{cases}$$

This also generalizes the $3x + 1$ mapping, which corresponds to $d = 3$ in Peis formulation. Notice that trajectories starting from nonzero x appear to meet 1 or -1, according as x is positive or negative. Notice also the trivial cycles $L_d(n) = n$, for $d/2 < n \le d/2$ (see Table 2).

It should be noticed that the above mappings $T_d(x)$ and $L_d(x)$ coincide with some particular cases of [11], which by its turn relates the $3x+1$ problem to 2-adic analysis. This suggests a more intimate connection between $T_d(x)$ and $L_d(x)$ and p-adic analysis, still to be clarified.

Some information on cycles with $d \le 150$ and $x \le 50,000$ provided by Keith Matthews can be found in Table 1. It is notorious in the studied cases that cycles are getting bigger and rarer, which itself suggests an obvious conjecture on the distribution of cycles and gaps on cycle lengths.

The corresponding conjecture is:

CONJECTURE 2 (Lower bounds for cycles). For each M there is a d such that the minimal cycle length of the mapping $T_d(x)$ is greater than M.

Another, more subtle, generalization of the $3x+1$ mapping concerns a two-parameter extension of T_d which enjoys some surprising properties. Define, for each $k \geq 3$ and $d \geq 2$, a mapping $T_{k,d} : \mathbb{Z} \mapsto \mathbb{Z}$ as follows:

$$T_{k,d}(x) = \begin{cases} \frac{x}{d} & \text{if } x \equiv 0 \pmod{d}, \\ \frac{kx+r(d-i)}{d} & \text{if } x \equiv i \pmod{d},\ 1 \leq i \leq d-1 \\ & \text{and } k \equiv r \pmod{d},\ 1 \leq r \leq d-1. \end{cases}$$

For $k = d+1$, $T_{d+1,d}$ gives the above defined mapping T_d, and of course $T_{3,2}$ gives the original Collatz mapping.

It is easy to see that d divides $(kx + r(di))$ as $x \equiv i (\bmod\, d)$ and $k \equiv r (\bmod\, d)$ so $T_{k,d}(x)$ is an integer ($T_{k,d}(x)$ is not defined for multiples of d, since $k \equiv r (\bmod\, d)$) for $1 \leq r \leq d$). An expected conjecture about $T_{k,d}(x)$, analogous to Conjecture 2, would be that the sequence of iterates

$$x, T_{k,d}(x), {T_{k,d}}^2(x), ..., {T_{k,d}}^k(x), ...$$

for each k and d always eventually enters a cycle, and that there are only finitely many such cycles.

Nothing is known, however, about this new hierarchy of generalizations of Collatz mapping, besides elementary facts, as for instance, that the next mapping in the hierarchy after $T_{3,2}(x)$ (the $3x+1$ mapping), namely, $T_{5,2}(x)$, has a non-elementary cycle at $x = 13$. Things get more chaotic with $T_{k,d}(x)$, as the noteworthy case of $T_{6,4}(x)$ illustrates. Consider $k = 6, d = 4$ and $r = 2$ and $gcd(k, d) > 1$:

$$T_{k,d}(x) = \begin{cases} \frac{x}{4} & \text{if } x \equiv 0 \ (\text{mod} 4), \\ \frac{6x+6}{4} & \text{if } x \equiv 1 \ (\text{mod} 4), \\ \frac{6x+4}{4} & \text{if } x \equiv 2 \ (\text{mod} 4), \\ \frac{6x+2}{4} & \text{if } x \equiv 3 \ (\text{mod} 4). \end{cases}$$

It can be proved, by elementary congruence class arguments, that:

1. The trajectory starting with $x = 1$ is divergent.
2. The only non-zero cycles are -1, -2 and -3.
3. Trajectories which start in the congruence classes $4N$ and $4N + 2$ (but not at $x = 2$) eventually end up in the union of the congruence classes $4N + 1$ and $4N + 3$, where they remain, and unless they hit the fixed points -1 or -3, they then diverge.

Hence the expected conjecture fails, albeit it can be easily modified to a second conjecture on the behavior of cycles, as follows:

CONJECTURE 3 ($T_{k,d}(x)$with finitely many finite cycles). The sequence of iterates

$$x, T_{k,d}(x), {T_{k,d}}^2(x), ..., {T_{k,d}}^k(x), ...$$

for infinitely many k and d eventually enters a cycle, and in each case there are only finitely many such cycles.

When $gcd(k, d) = 1$, the generalized mapping $T_{k,d}(x)$ coincides with the mappings defined in [11], and some particular conjectures therein will apply to $T_{k,d}(x)$. This is the case, for instance, when $k = d + 1$.

This kind of problem is of course still harder than any of the previous generalizations of the Collatz problems, and are naturally connected with the ergodic theory on the $p$adic integers and with Markov chains. In [5] Markov chain models for iterating generalized Collatz mappings and some heuristics are investigated, and it is patent that understanding some cases, even on a conjectural level, is a hard task. In our case, the mappings $T_{k,d}(x)$ satisfy the Condition C of [5] which makes them a potentially fruitful area for further research under a Markovian approach.

3 On Expected Difficulties: Final Remarks

It has been shown by Kurtz and Simon in [3] that a certain generalization of the Collatz problem is undecidable, building on previous work by J.H. Conway in [2]. It is easy to see that the definition of Collatz function given by Conway (definition 1.2. of [3] is a restriction of the functions $T_d(x)$ defined above: indeed, a function g is called a *Collatz function* if there is an integer n together with rational numbers $\{a_i : i < n\}$, $\{b_i : i < n\}$ such that if $x \equiv i \pmod p$, then $g(x) = a_i x + b_i$ is an integer.

The above mappings $T_d(x)$ define of course an infinite family of Collatz functions:

$$g(x) = \frac{1}{d}x + 0 \text{ if } x \equiv 0(\bmod d),$$

$$g(x) = \frac{d+1}{d}x + \frac{d-i}{d} \text{ if } x \equiv i(\bmod d), 1 \leq i \leq d-1,$$

$$\{a_i : i < n\} = \left\{\frac{1}{d}, \frac{d+1}{d}\right\} \text{ and } \{b_i : i < n\} = \left\{0, \frac{d-1}{d}, ..., \frac{1}{d}\right\}.$$

Conway had proved in [2], and the results is simplified in [3] (Theorem 1.4) that given a Collatz function g, it is undecidable whether or not for all integers x there exists an k such that $g_{(j)}(x) = 1$.

The undecidability result does not affect (at least directly) neither Collatz original problem nor our generalized problems concerning $T_d(x)$ or $T_{k,d}(x)$, but of course it gives a hint on the expected difficulties. It is encouraging, however, that such general problems may have many positive cycles, and they would not necessarily fall under intractable even if such undecidability result could be enhanced to values other than 1.

From a more philosophical standpoint, let me point out here that I completely disagree from van Bendegen [12] when he says at page 10 (albeit recognizing the conceptual importance and interest of Collatz Conjecture) that:

> Firstly, it is quite easy to “invent” similar problems, so why should this particular case attract our attention?

Not only it is not easy to "invent' sufficiently attractive problems to further research, but also beauty is an important fuel in many sciences, and especially in Mathematics. Erdős is reported to have offered US$ 500 for a solution of the original Collatz Conjecture, but he certainly was underestimating the problem, let alone its generalizations. Indeed, I would venture a bold meta-conjecture about such problems: mankind will disappear first than being able to solve all of them, and their difficulty will be part of human heritage.

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4 Appendix: Cycles for $T_d(x)$ and $L_d(x)$, $2 \le d \le 150$ and $x \le 50,000$

In this section, we present two tables: Table 1 and Table 2.

Table 1: $d : 2 \le d \le 150$ with extra cycles for T_d

d	Min cycle element	cycle–length
2	-5	3
	-17	11
3	7	9
	-22	5
4	23	14
	-18	15
5	-57	10
6	23	12
	88	25
8	-92	19
9	35	21
	-272	22
10	42	49
11	642	57
12	-25	35
	-63	33
	1348	32
	1010	96
13	-147	36
15	53	41
16	178	46
17	79	49
19	-954	117
20	71	60
	141	61
22	-426	71
23	82	72
	-800	75
24	335	78
	513	157
25	4064	83
26	-453	88
27	545	91
29	111	97
31	389	108
43	-634	166
48	136	181
	-137	198
51	-2385	204
54	3406	218
	3647	218
56	-157	240
57	3416	466
66	-5659	280
68	73734	290
72	46960	311
75	-81592	327
78	1494	341
84	-2119	377
85	4435	380
94	1308	427
109	-20146	515
111	2838	524
112	-34727	532
115	-434	566
	175060	549
118	21991	566
119	65413	572
123	195	564
127	13353	618
129	23653	630
134	1871	655
136	1356	665

Table 2: $d : 2 \leq d \leq 150$ with extra cycles for L_d

d	Min cycle element	cycle–length
2	-1	2
	5	3
	17	11
4	-9	7
5	-22	10
	22	10
9	-28	22
	28	22
12	86	32
	251	32
14	-20	39
	1453	157
23	-39	75
	39	75
24	105	79
	126	79
28	291	96
35	-235	127
	235	127
46	853	179
47	-142	368
	142	368
	-285	184
	285	184

d	Min cycle element	cycle–length
49	-178	387
	178	387
52	149	209
60	-139	249
62	1818	259
68	8384	290
78	-4324	343
	-31853	343
	-31774	343
	56789	343
	37371	343
	41154	343
105	-2082	492
	2082	492
108	-13281	509
	-10449	509
120	3698	578
125	-219	608
	219	608
	-1991	607
	1991	607
	-4103	607
	4103	607
150	-1973	755