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# The Evaluation Of A Quadratic And A Cubic Series With Trigamma Function*

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**Abstract**

The paper is about calculating the quadratic series

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = \sum_{n=1}^{\infty} \frac{1}{n}(\psi'(n+1))^2$$

and the cubic series

$$\sum_{n=1}^{\infty} n\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots\right)^3 = \sum_{n=1}^{\infty} n(\psi'(n))^3,$$

where $\psi$ denotes the digamma function.

## 1 Introduction and the Main Result

Throughout this paper, let $\mathbb{C}$, $\mathbb{Z}_0^-$, $\mathbb{N}$ denote the sets of complex numbers, nonpositive integers, positive integers respectively. The celebrated Riemann zeta function $\zeta$ is a function of a complex variable [9, p.265] defined by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots + \frac{1}{n^z} + \cdots \quad (\Re(z) > 1).$$

When $z = 2$ one has that the Riemann zeta function value $\zeta(2)$ is defined by the series formula

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots .$$

The trigamma function $\psi'$ is defined by [7, p.22]

$$\psi'(z) = \frac{d^2}{dz^2} \log \Gamma(z) = \frac{d}{dz}\psi(z) \quad \left(z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

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$\psi(z)$ being the $\psi$ (or digamma) function defined by $\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ or, in terms of the generalized (or Hurwitz) zeta function $\zeta(s,a)$ defined by $\zeta(s,a) := \sum\limits_{k=0}^{\infty} \frac{1}{(k+a)^s}$ $\left(\Re(s) > 1;\, a \in \mathbb{C} \setminus \mathbb{Z}_0^-\right)$,

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^2} = \zeta(2,z) \quad \left(z \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

This implies that

$$\psi'(n) = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{(n-1)^2} \quad (n \in \mathbb{N} \setminus \{1\}).$$

Closed form evaluation of series involving $\zeta(k)$ are collected in [7] and, more recently, in [8]. Other series, linear or quadratic, involving the Riemann zeta function and harmonic numbers, which are evaluated in terms of special constants can be found in [4].

In this paper we evaluate a quadratic and a cubic series involving the tail of $\zeta(2)$. More precisely, we calculate the quadratic series

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = \sum_{n=1}^{\infty} \frac{1}{n}(\psi'(n+1))^2$$

and the cubic series

$$\sum_{n=1}^{\infty} n\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots\right)^3 = \sum_{n=1}^{\infty} n(\psi'(n))^3,$$

where $\psi$ denotes the digamma function.

The main result of this paper is the following theorem.

THEOREM 1 (A quadratic and a cubic series with the tail of $\zeta(2)$)**.** The following identities hold:

(a) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\right)^2 = 5\zeta(2)\zeta(3) - 9\zeta(5)$;

(b) $\sum_{n=1}^{\infty} n\left(\psi'(n)\right)^3 = \frac{9}{2}\zeta(3) - \frac{17}{8}\zeta(4) - \frac{25}{4}\zeta(5) + \frac{9}{2}\zeta(2)\zeta(3)$.

We need in our analysis Abel's summation formula [1, p.55], [4, p.258] which states that if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two sequences of real or complex numbers and $A_n = \sum_{k=1}^{n} a_k$, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} + \sum_{k=1}^{n} A_k(b_k - b_{k+1}) \quad (n \in \mathbb{N}).$$

We will also use, in our calculations, *the infinite version* of the preceding formula:

$$\sum_{k=1}^{\infty} a_k b_k = \lim_{n\to\infty} (A_n b_{n+1}) + \sum_{k=1}^{\infty} A_k (b_k - b_{k+1}), \tag{1}$$

provided the infinite series on the right hand side of (1) converges and the limit is finite.

A special function which is used in the proof of part (a) of Theorem 1 is the Dilogarithm function. Recall that the Dilogarithm function $\mathrm{Li}_2$ is defined, for $|z| \le 1$, by [7, p.106]

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\ln(1-t)}{t} dt.$$

In particular, $\mathrm{Li}_2(1) = \zeta(2)$.

A special identity involving the Dilogarithm function is the following Landen type formula:

$$\zeta(2) - \mathrm{Li}_2(1-z) - \ln z \ln(1-z) = \mathrm{Li}_2(z), \tag{2}$$

whose proof can be found in [7, p.107].

## 2 Proof of the Main Result

In this section we collect some results we need for proving Theorem 1.

LEMMA 1 (Some logarithm and polylogarithm integrals). The following equalities hold:

(a) $\int_0^1 \frac{x \ln x}{1-x} dx = 1 - \zeta(2)$;

(b) $\int_0^1 \frac{\ln x \ln(1-x)}{x} dx = \zeta(3)$;

(c) $\int_0^1 \frac{\mathrm{Li}_2(x)}{x} dx = \zeta(3)$;

(d) $\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x} dx = 2\zeta(2)\zeta(3) - 3\zeta(5)$;

(e) $\int_0^1 \frac{\ln x \ln(1-x) \mathrm{Li}_2(x)}{x} dx = \zeta(2)\zeta(3) - \frac{3}{2}\zeta(5)$.

PROOF. (a) We have

$$\begin{aligned} \int_0^1 \frac{x \ln x}{1-x} dx &= \int_0^1 x \ln x \left( \sum_{n=0}^{\infty} x^n \right) dx = \sum_{n=0}^{\infty} \int_0^1 x^{n+1} \ln x dx \\ &= -\sum_{n=0}^{\infty} \frac{1}{(n+2)^2} = 1 - \zeta(2). \end{aligned}$$

(b) We have

$$\begin{aligned}\int_0^1 \frac{\ln x \ln(1-x)}{x} dx &= \int_0^1 \frac{\ln x}{x}\left(-\sum_{n=1}^{\infty}\frac{x^n}{n}\right) dx \\ &= -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x^{n-1}\ln x dx = \zeta(3).\end{aligned}$$

(c) We have

$$\int_0^1 \frac{\mathrm{Li}_2(x)}{x} dx = \int_0^1 \frac{1}{x}\left(\sum_{n=1}^{\infty}\frac{x^n}{n^2}\right) dx = \sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^1 x^{n-1} dx = \zeta(3).$$

The integrals in parts (d) and (e) are recorded in [3, Entry 1, Table 2, p.1435, Entry 2, Table 6, p.1436].

The next lemma is about calculating two Euler series and a quadratic series involving the tail of $\zeta(2)$.

LEMMA 2. The following equalities hold:

(a) $\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) = -\frac{9}{2}\zeta(5)+3\zeta(2)\zeta(3)$;

(b) $\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) = \frac{7}{4}\zeta(4)$;

(c) $\sum_{n=1}^{\infty}\left(\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}\right)^2 = 3\zeta(3)-\frac{5}{2}\zeta(4)$.

PROOF. (a) This part of the lemma is a special case of a more general result concerning the evaluation of Euler type series [2, Theorem 3.1, p.22].

(b) We apply Abel's summation formula (1) with $a_n = \frac{1}{n^2}$ and $b_n = 1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}$. We have

$$\begin{aligned} s &= \sum_{n=1}^{\infty}\frac{1}{n^2}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) \\ &= \lim_{n\to\infty}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\left(1+\frac{1}{2^2}+\cdots+\frac{1}{(n+1)^2}\right) \\ &\quad -\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right) \\ &= \zeta^2(2)-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{(n+1)^2}\right)+\sum_{n=1}^{\infty}\frac{1}{(n+1)^4} \\ &= \zeta^2(2)-s+1+\zeta(4)-1 \\ &= \frac{7}{2}\zeta(4)-s,\end{aligned}$$

and part (b) of the lemma is proved.

We used that $\zeta^2(2) = \frac{5}{2}\zeta(4)$ since $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$ [6, p.605].

(c) The evaluation of this quadratic series involving the tail of $\zeta(2)$ can be found in [4, Problem 3.22, p.142], [5, Theorem 1, (a)].

Now we are ready to prove Theorem 1.

PROOF. (a) First we note that if $k > 0$ is a real number then

$$\int_0^1 x^{k-1} \ln x \, dx = -\frac{1}{k^2},$$

and this implies that

$$\begin{aligned} \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} &= \sum_{m=1}^{\infty} \frac{1}{(n+m)^2} \\ &= -\sum_{m=1}^{\infty} \int_0^1 x^{n+m-1} \ln x \, dx \\ &= -\int_0^1 x^n \ln x \left( \sum_{m=1}^{\infty} x^{m-1} \right) dx \\ &= -\int_0^1 \frac{x^n}{1-x} \ln x \, dx. \end{aligned}$$

It follows that

$$\begin{aligned} T &\equiv \sum_{n=1}^{\infty} \frac{1}{n} \left( \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right)^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \left( \int_0^1 \frac{x^n}{1-x} \ln x \, dx \right) \left( \int_0^1 \frac{y^n}{1-y} \ln y \, dy \right) \\ &= \int_0^1 \int_0^1 \frac{\ln x \ln y}{(1-x)(1-y)} \sum_{n=1}^{\infty} \frac{(xy)^n}{n} dx dy = -\int_0^1 \int_0^1 \frac{\ln x \ln y \ln(1-xy)}{(1-x)(1-y)} dx dy. \end{aligned}$$

We have

$$I = \int_0^1 \int_0^1 \frac{\ln x \ln y \ln(1-xy)}{(1-x)(1-y)} dx dy = \int_0^1 \frac{\ln x}{1-x} \left( \int_0^1 \frac{\ln y \ln(1-xy)}{1-y} dy \right) dx.$$

We calculate the inner integral by parts, with $f(y) = \ln(1-xy)$, $f'(y) = -\frac{x}{1-xy}$, $g'(y) = \frac{\ln y}{1-y}$, $g(y) = -\ln y \ln(1-y) - \mathrm{Li}_2(y)$, and we have

$$\int_0^1 \frac{\ln y \ln(1-xy)}{1-y} dy \quad = \quad -\ln(1-xy)(\ln y \ln(1-y) + \mathrm{Li}_2(y)) \Big|_{y=0}^{y=1}$$

$$
\begin{aligned}
& -\int_0^1 \frac{x}{1-xy}\left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right) dy \\
= \quad & -\zeta(2)\ln(1-x) - \int_0^1 \frac{x}{1-xy}\left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right) dy.
\end{aligned}
$$

It follows, based on part (b) of Lemma 1, that

$$
\begin{aligned}
I \quad = \quad & -\zeta(2)\int_0^1 \frac{\ln x \ln(1-x)}{1-x} dx \\
& -\int_0^1\int_0^1 \frac{x\ln x}{(1-x)(1-xy)}\left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right) dxdy \\
= \quad & -\zeta(2)\zeta(3) - \int_0^1\int_0^1 \frac{x\ln x}{(1-x)(1-xy)}\left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right) dxdy.
\end{aligned}
$$

We calculate the double integral as follows

$$
\begin{aligned}
\mathcal{J} \quad = \quad & \int_0^1\int_0^1 \frac{x\ln x}{(1-x)(1-xy)}\left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right) dxdy \\
= \quad & \int_0^1 \left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right)\left(\int_0^1 \frac{x\ln x}{(1-x)(1-xy)} dx\right) dy.
\end{aligned}
$$

Using part (a) of Lemma 1 the inner integral becomes

$$
\begin{aligned}
\int_0^1 \frac{x\ln x}{(1-x)(1-xy)} dx \quad = \quad & \int_0^1 \frac{x\ln x}{1-y}\left(\frac{1}{1-x} - \frac{y}{1-xy}\right) dx \\
= \quad & \frac{1}{1-y}\int_0^1 \frac{x\ln x}{1-x} dx - \frac{1}{1-y}\int_0^1 \frac{xy\ln x}{1-xy} dx \\
= \quad & \frac{1-\zeta(2)}{1-y} + \frac{1}{1-y}\left(\int_0^1 \ln x dx - \int_0^1 \frac{\ln x}{1-xy} dx\right) \\
= \quad & -\frac{\zeta(2)}{1-y} - \frac{1}{1-y}\int_0^1 \frac{\ln x}{1-xy} dx.
\end{aligned}
$$

Using the substitution $1 - xy = t$, we get that

$$
\begin{aligned}
\int_0^1 \frac{\ln x}{1-xy} dx \quad = \quad & -\frac{1}{y}\int_1^{1-y} \frac{\ln(1-t) - \ln y}{t} dt \\
= \quad & -\frac{1}{y}\left(\int_1^{1-y} \frac{\ln(1-t)}{t} dt - \ln y \ln(1-y)\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\int_1^{1-y} \frac{\ln(1-t)}{t} dt \quad = \quad & \int_0^{1-y} \frac{\ln(1-t)}{t} dt - \int_0^1 \frac{\ln(1-t)}{t} dt \\
= \quad & -\mathrm{Li}_2(1-y) + \mathrm{Li}_2(1)
\end{aligned}
$$

$$= \quad \zeta(2) - \mathrm{Li}_2(1-y),$$

and it follows, based on formula (2), that

$$\int_0^1 \frac{\ln x}{1-xy}dx = -\frac{1}{y}\left(\zeta(2) - \mathrm{Li}_2(1-y) - \ln y \ln(1-y)\right) = -\frac{\mathrm{Li}_2(y)}{y}.$$

Therefore

$$\int_0^1 \frac{x\ln x}{(1-x)(1-xy)}dx = -\frac{\zeta(2)}{1-y} + \frac{\mathrm{Li}_2(y)}{y(1-y)},$$

and this in turn implies that

$$\begin{aligned}
\mathcal{J} &= \int_0^1 \left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right)\left(\frac{\mathrm{Li}_2(y)}{y(1-y)} - \frac{\zeta(2)}{1-y}\right)dy \\
&= \int_0^1 \left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right)\left(\frac{\mathrm{Li}_2(y)}{y} + \frac{\mathrm{Li}_2(y)-\zeta(2)}{1-y}\right)dy \\
&= \int_0^1 \frac{\ln y \ln(1-y)\mathrm{Li}_2(y)}{y}dy + \int_0^1 \frac{\mathrm{Li}_2^2(y)}{y}dy \\
&\quad + \int_0^1 \left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right)\frac{\mathrm{Li}_2(y)-\zeta(2)}{1-y}dy. \qquad (3)
\end{aligned}$$

Using Lemma 1 combined to $\ln y \ln(1-y) + \mathrm{Li}_2(y) = \zeta(2) - \mathrm{Li}_2(1-y)$, we have that

$$\begin{aligned}
&\int_0^1 \left(\ln y \ln(1-y) + \mathrm{Li}_2(y)\right)\frac{\mathrm{Li}_2(y)-\zeta(2)}{1-y}dy \\
&= \int_0^1 \frac{\left(\zeta(2) - \mathrm{Li}_2(1-y)\right)\left(\mathrm{Li}_2(y)-\zeta(2)\right)}{1-y}dy \quad (y \to 1-y) \\
&= \int_0^1 \frac{\left(\zeta(2) - \mathrm{Li}_2(y)\right)\left(\mathrm{Li}_2(1-y)-\zeta(2)\right)}{y}dy \\
&= \int_0^1 \frac{\left(\zeta(2) - \mathrm{Li}_2(y)\right)\left(-\mathrm{Li}_2(y)-\ln y \ln(1-y)\right)}{y}dy \text{ by (2)} \\
&= -\zeta(2)\int_0^1 \frac{\mathrm{Li}_2(y)}{y}dy - \zeta(2)\int_0^1 \frac{\ln y \ln(1-y)}{y}dy \\
&\quad + \int_0^1 \frac{\mathrm{Li}_2^2(y)}{y}dy + \int_0^1 \frac{\mathrm{Li}_2(y)\ln y \ln(1-y)}{y}dy \\
&= -2\zeta(2)\zeta(3) + \int_0^1 \frac{\mathrm{Li}_2^2(y)}{y}dy + \int_0^1 \frac{\mathrm{Li}_2(y)\ln y \ln(1-y)}{y}dy. \qquad (4)
\end{aligned}$$

We obtain in view of (3), (4) and parts (d) and (e) of Lemma 1 that

$$\begin{aligned}
\mathcal{J} &= 2\int_0^1 \frac{\ln y \ln(1-y)\mathrm{Li}_2(y)}{y}dy + 2\int_0^1 \frac{\mathrm{Li}_2^2(y)}{y}dy - 2\zeta(2)\zeta(3) \\
&= 4\zeta(2)\zeta(3) - 9\zeta(5),
\end{aligned}$$

and hence

$$I = -\zeta(2)\zeta(3) - \mathcal{J} = 9\zeta(5) - 5\zeta(2)\zeta(3).$$

Since $T = -I$ we get that part (a) of the theorem is proved.

(b) We apply Abel's summation formula (1) with $a_n = n$ and $b_n = x_n^3$ where

$$x_n = \psi'(n) = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \cdots .$$

A calculation shows that

$$b_n - b_{n+1} = \left(\frac{1}{n^2} + x_{n+1}\right)^3 - x_{n+1}^3 = \frac{1}{n^6} + \frac{3}{n^4}x_{n+1} + \frac{3}{n^2}x_{n+1}^2,$$

and we have

$$\begin{aligned}
\sum_{n=1}^{\infty} n\left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots\right)^3 &= \lim_{n\to\infty}\frac{n(n+1)}{2}\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right)^3 \\
&\quad + \frac{1}{2}\sum_{n=1}^{\infty} n(n+1)\left(\frac{1}{n^6} + \frac{3}{n^4}x_{n+1} + \frac{3}{n^2}x_{n+1}^2\right) \\
&= \frac{1}{2}\zeta(4) + \frac{1}{2}\zeta(5) + \frac{3}{2}\sum_{n=1}^{\infty}\frac{x_{n+1}}{n^2} + \frac{3}{2}\sum_{n=1}^{\infty}\frac{x_{n+1}}{n^3} \\
&\quad + \frac{3}{2}\sum_{n=1}^{\infty} x_{n+1}^2 + \frac{3}{2}\sum_{n=1}^{\infty}\frac{x_{n+1}^2}{n}. \qquad (5)
\end{aligned}$$

The preceding limit is 0 since

$$\begin{aligned}
& n(n+1)\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right]^3 \\
< \; & n(n+1)\left[\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots\right]^3 < \frac{n+1}{n^2},
\end{aligned}$$

and the limit follows based on the Squeeze Theorem.

Since

$$x_{n+1} = \psi'(n+1) = \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2},$$

we have, based on parts (a) and (b) of Lemma 2, that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{x_{n+1}}{n^2} &= \sum_{n=1}^{\infty}\frac{\zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}}{n^2} \\
&= \zeta^2(2) - \sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) \\
&= \frac{5}{2}\zeta(4) - \frac{7}{4}\zeta(4)
\end{aligned}$$

$$= \frac{3}{4}\zeta(4) \tag{6}$$

and

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{x_{n+1}}{n^3} &= \sum_{n=1}^{\infty}\frac{\zeta(2)-1-\frac{1}{2^2}-\cdots-\frac{1}{n^2}}{n^3}\\
&= \zeta(2)\zeta(3)-\sum_{n=1}^{\infty}\frac{1}{n^3}\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\\
&= \zeta(2)\zeta(3)-\left(-\frac{9}{2}\zeta(5)+3\zeta(2)\zeta(3)\right)\\
&= -2\zeta(2)\zeta(3)+\frac{9}{2}\zeta(5). \qquad (7)
\end{aligned}$$

Combining (5), (6), (7), part (c) of Lemma 2 and part (a) of Theorem 1 we have

$$\sum_{n=1}^{\infty} n\left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots\right)^3 = \frac{9}{2}\zeta(3)-\frac{17}{8}\zeta(4)-\frac{25}{4}\zeta(5)+\frac{9}{2}\zeta(2)\zeta(3),$$

and the theorem is proved.

A challenging problem would be to evaluate the alternating versions of the series in Theorem 1. We leave this as an open problem to the interested reader.

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