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On Star Coloring Of Corona Graphs*

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Abstract

In this paper, we find the star chromatic number for the corona graph of path with complete graph on the same order $P_n \circ K_n$, path with cycle on the same order $P_n \circ C_n$, path on order n with star graph on order $n+1$ say $P_n \circ K_{1,n}$, path on order n with bipartite graph on order n_1+n_2 say $P_n \circ K_{n_1,n_2}$ and corona graph of star graph on order $n+1$ with complete graph on order n say $K_{1,n} \circ K_n$ respectively.

1 Introduction

The notion of star chromatic number was introduced by Branko Grünbaum in 1973. A star coloring [1, 4, 5] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. Equivalently, in a star coloring, the induced subgraphs formed by the vertices of any two colors has connected components that are star graphs. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

Guillaume Fertin et al. [5] gave the exact value of the star chromatic number of different families of graphs such as trees, cycles, complete bipartite graphs, outerplanar graphs, and 2-dimensional grids. They also investigated and gave bounds for the star chromatic number of other families of graphs, such as planar graphs, hypercubes, d-dimensional grids ($d \geq 3$), d-dimensional tori ($d \geq 2$), graphs with bounded treewidth, and cubic graphs.

Albertson et al. [1] showed that it is NP-complete to determine whether $\chi_s(G) \leq 3$, even when G is a graph that is both planar and bipartite. The problems of finding star colorings is NP-hard and remain so even for bipartite graphs [9, 10]. For some works related to the application and the algorithmic approach on star colorings we refer to [2].

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2 Preliminaries

Graph products are interesting and useful in many situations [8]. The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. This kind of product was introduced by Harary and Frucht in 1970 [7]. Additional graph theory terminology used in this paper can be found in [3, 6].

In the following section, we find the star chromatic number for the corona graph of path with complete graph on the same order $P_n \circ K_n$, path with cycle on the same order $P_n \circ C_n$, path on order n with star graph on order $n+1$ say $P_n \circ K_{1,n}$, path on order n with bipartite graph on order $n_1 + n_2$ say $P_n \circ K_{n_1,n_2}$ and corona graph of star graph on order $n+1$ with complete graph on order n say $K_{1,n} \circ K_n$ respectively.

In order to prove our results, we shall use the following Theorems by Guillaume et al. [5].

THEOREM 1 ([5]). If C_n is a cycle with $n \geq 3$ vertices, then

$$\chi_s(C_n) = \begin{cases} 4 & \text{when } n = 5, \\ 5 & \text{otherwise.} \end{cases}$$

THEOREM 2 ([5]). Let $K_{n,m}$ be a complete bipartite graph. Then

$$\chi_s(K_{n,m}) = \min\{m, n\} + 1.$$

3 Star Coloring on Corona Graphs

In this section, we prove our main theorems.

THEOREM 3. For any $n \geq 2$, $\chi_s(P_n \circ K_n) = n + 2$.

PROOF. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. Let

$$V(P_n \circ K_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}.$$

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of a copy of K_n i.e., every vertex $v_i \in V(P_n)$ is adjacent to every vertex from the set $\{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}$.

Assign the following $n+2$-coloring for $P_n \circ K_n$ as star chromatic:

(i) For $1 \leq i \leq n$, assign the color c_i to v_i.

(ii) For $1 \leq i \leq n; 1 \leq j \leq n$, assign the color c_{i+j} to u_{ij} $\forall\ i + j \leq n + 2$.

(iii) For $1 \leq i \leq n; 1 \leq j \leq n$, if $i + j > n + 2$ assign the coloring as follows:

  (a) c_1 to u_{ij} if $i + j \equiv 1 \bmod (n + 2)$.

(b) c_2 to u_{ij} if $i + j \equiv 2 \bmod (n+2)$.
....
....
(c) c_{n+1} to u_{ij} if $i + j \equiv (n+1) \bmod (n+2)$.

Therefore, $\chi_s(P_n \circ K_n) \leq n + 2$.

To prove $\chi_s(P_n \circ K_n) \geq n+2$. Let us assume that $\chi_s(P_n \circ K_n)$ is less than $n+2$ i.e., $\chi_s(P_n \circ K_n) = n + 1$. We must assign $n + 1$ colors for $\{v_1, u_{1i} : 1 \leq i \leq n\}$ for proper star coloring, since $\{v_1, u_{1i} : 1 \leq i \leq n\}$ induces a clique of order $n + 1$ (say K_{n+1}). If we assign the same $n + 1$ colors to the another clique induced by the second copy of K_n, $\{v_2, u_{2i} : 1 \leq i \leq n\}$ then an easy check shows that one of the path on 4 vertices between these cliques is bicolored. This is a contradiction, star coloring with $n + 1$ colors is impossible. Thus, $\chi_s(P_n \circ K_n) \geq n + 2$. Hence, $\chi_s(P_n \circ K_n) = n + 2$. This completes the proof of the theorem.

THEOREM 4. For any $n \geq 3$,

$$\chi_s(P_n \circ C_n) = \begin{cases} 6 & \text{if } n = 5, \\ 5 & \text{otherwise.} \end{cases}$$

PROOF. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, $V(C_n) = \{u_1, u_2, \ldots, u_n\}$, and

$$V(P_n \circ C_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i \leq n; 1 \leq j \leq n\}.$$

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of a copy of C_n i.e., every vertex of $V(P_n)$ is adjacent to every vertex from the set $V(C_n)$. Let $C_n^{(1)}, C_n^{(2)}, C_n^{(3)}, \ldots, C_n^{(n)}$ be the n copies of the cycle C_n. We consider the following cases.

Case(i): $n = 5$**.** Assign the following 6-coloring for $P_n \circ C_n$ as star-chromatic:

- For $1 \leq i \leq 5$, assign the color c_i to v_i.
- For $i \in \{2, 3, 4, 5\}$, assign the color c_i to u_{1i}.
- For $i \in \{3, 4, 5\}$, assign the color c_i to u_{2i}.
- For $i \in \{1, 4, 5\}$, assign the color c_i to u_{3i}.
- For $i \in \{1, 2, 5\}$, assign the color c_i to u_{4i}.
- For $i \in \{1, 2, 3\}$, assign the color c_i to u_{5i}.
- For $1 \leq i \leq 5$, assign the color c_6 to u_{ii}.

For the vertices $u_{21}, u_{32}, u_{43}, u_{54}$ assign the colors c_3, c_4, c_5, c_2 respectively. Thus $\chi_s(P_n \circ C_n) \leq 6$.

To prove $\chi_s(P_n \circ C_n) \geq 6$. Let us assume that $\chi_s(P_n \circ C_n)$ is less than 6 i.e., $\chi_s(P_n \circ C_n) = 5$. We must assign 5 colors for $\{v_1, u_{1i} : 1 \leq i \leq n\}$, since $\{u_{1i} : 1 \leq i \leq n\}$ is a cycle of order 5, by Theorem 1 it needs 4 distinct colors for proper star coloring and v_1 is adjacent to each $\{u_{1i} : 1 \le i \le n\}$. If we assign the same 5 colors for the another set of vertices $\{v_2, u_{2i} : 1 \le i \le n\}$, then an easy check shows that one of the path on 4 vertices between these two set of vertices is bicolored. This is a contradiction. Thus, $\chi_s(P_n \circ C_n) \ge 6$. Hence, $\chi_s(P_n \circ C_n) = 6$ for $n = 5$.

Case(ii): $n \ne 5$**.** Assign the following 5-coloring as star-chromatic for $P_n \circ C_n$:

- For $1 \le i \le 5$, assign the color c_i to v_i.
- For $i \in \{6, 7, \ldots, n\}$ assign the color c_k, $1 \le k \le 5$ to all such vertices v_i that $i \equiv k \bmod 5$.
- Color the vertices of $V(C_n^{(1)})$, $V(C_n^{(5)})$, $V(C_n^{(9)})$,... with colors c_2, c_3, c_4, alternatively.
- Color the vertices of $V(C_n^{(2)})$, $V(C_n^{(6)})$, $V(C_n^{(10)})$,... with colors c_3, c_4, c_5, alternatively.
- Color the vertices of $V(C_n^{(3)})$, $V(C_n^{(7)})$, $V(C_n^{(11)})$,... with colors c_4, c_5, c_1, alternatively.
- Color the vertices of $V(C_n^{(4)})$, $V(C_n^{(8)})$, $V(C_n^{(12)})$,... with colors c_5, c_1, c_2, alternatively.

Therefore $\chi_s(P_n \circ C_n) \le 5$. To prove $\chi_s(P_n \circ C_n) \ge 5$, let us suppose that $\chi_s(P_n \circ C_n)$ is less than 5 say $\chi_s(P_n \circ C_n) = 4$. We must assign 4 colors for $\{v_1, u_{1i} : 1 \le i \le n\}$, since $\{u_{1i} : 1 \le i \le n\}$ is a cycle, by Theorem 1 it needs 3 colors for proper star coloring and v_1 is adjacent to each $\{u_{1i} : 1 \le i \le n\}$. If we assign the same 4 colors to the another set of vertices $\{v_2, u_{2i} : 1 \le i \le n\}$ then an easy check shows that one of the path on 4 vertices between these set of vertices is bicolored. This is a contradiction, star coloring with 4 colors is impossible. Thus, $\chi_s(P_n \circ C_n) \ge 5$. Hence $\chi_s(P_n \circ C_n) = 5$, $n \ne 5$. This completes the proof of the theorem.

THEOREM 5. Let $n \ge 3$ be a positive integer. Then $\chi_s(P_n \circ K_{1,n}) = 4$.

PROOF. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$,

$$V(K_{1,n}) = \{u_i, u_{ij} : 1 \le i \le n; 1 \le j \le n\},$$

and

$$\begin{aligned} V(P_n \circ K_{1,n}) &= \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \\ &\quad \cup \{u_{ij} : 1 \le i \le n; 1 \le j \le n\}. \end{aligned}$$

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of a copy of $K_{1,n}$ i.e., every vertex from the set $V(P_n)$ is adjacent to every vertex from the set $V(K_{1,n})$.

- For $1 \le i \le n$, color the vertices v_i with colors c_1, c_2, c_3, c_4, alternatively.
- For $1 \le i \le n$, color the vertices u_i with colors c_2, c_3, c_4, c_1, alternatively.
- For $1 \le i \le 4$; $1 \le j \le n$, color the vertices u_{ij} with colors c_3, c_4, c_1, c_2, alternatively.
- For $5 \le i \le n$; $1 \le j \le n$, color the vertices u_{ij} with color c_3 if $i \equiv 1 \bmod 4$.
- For $5 \le i \le n$; $1 \le j \le n$, color the vertices u_{ij} with color c_4 if $i \equiv 2 \bmod 4$.
- For $5 \le i \le n$; $1 \le j \le n$, color the vertices u_{ij} with color c_1 if $i \equiv 3 \bmod 4$.
- For $5 \le i \le n$; $1 \le j \le n$, color the vertices u_{ij} with color c_2 if $i \equiv 0 \bmod 4$.

Therefore $\chi_s(P_n \circ K_{1,n}) \le 4$. To prove $\chi_s(P_n \circ K_{1,n}) \ge 4$, let us assume that $\chi_s(P_n \circ K_{1,n})$ is less than 4 i.e., $\chi_s(P_n \circ K_{1,n}) = 3$. We must assign 3 colors for $\{v_1, u_1, u_{1i} : 1 \le i \le n\}$, since $\{u_1, u_{1i} : 1 \le i \le n\}$ is a star graph and needs 2 colors for proper star coloring and each $\{u_1, u_{1i} : 1 \le i \le n\}$ is adjacent to v_1 shows v_1 needs one distinct color. If we use the same 3 colors for the another set of vertices $\{v_2, u_2, u_{2i} : 1 \le i \le n\}$ then an easy check shows that one of the path on 4 vertices is bicolored. This is a contradiction, star coloring with 3 colors is impossible. Thus, $\chi_s(P_n \circ K_{1,n}) \ge 4$. Hence, $\chi_s(P_n \circ K_{1,n}) = 4$. This completes the proof of the theorem.

THEOREM 6. For $n \ge 2$ and $n = n_1$ or $n = n_2$, $\chi_s(P_n \circ K_{n_1,n_2}) = \min\{n_1, n_2\} + 3$.

PROOF. We consider the following cases.

Case(i): If $n_1 < n_2$. Let $n = \max\{n_1, n_2\} = n_2$. Let $V(P_n) = \{v_i : 1 \le i \le n_2\}$,

$$V(K_{n_1,n_2}) = \{u_{ij} : 1 \le i \le n_2; 1 \le j \le n_1\} \cup \{w_{ij} : 1 \le i \le n_2; 1 \le j \le n_2\}$$

and

$$\begin{aligned} V(P_n \circ K_{n_1,n_2}) &= \{v_i : 1 \le i \le n\} \cup \{u_{ij} : 1 \le i \le n_2; 1 \le j \le n_1\} \\ &\quad \cup \{w_{ij} : 1 \le i \le n_2; 1 \le j \le n_2\}. \end{aligned}$$

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of a copy of K_{n_1,n_2} i.e., every vertex from set $V(P_n)$ is adjacent to every vertex from the set K_{n_1,n_2}.

Assign the star coloring as follows:

- For $1 \le i \le n_2$, if $i \le n_1 + 3$ color the vertex v_i with color c_i.
- For $1 \le i \le n_2$, if $i > n_1 + 3$ color the vertex v_i with c_j if $i \equiv j \bmod (n_1 + 3)$, $1 \le j < n_1 + 3$.
- For $1 \le i \le n_2$, $1 \le j \le n_1$, color the vertex u_{ij} with c_{i+j} if $i + j \le n_1 + 3$.
- For $1 \le i \le n_2$, $1 \le j \le n_1$, color the vertex u_{ij} with c_i if $i + j \equiv i \bmod (n_1 + 3)$.

- For $1 \leq i \leq 2$, $1 \leq j \leq n_2$, if $i + j > n_1 + 3$ then color the vertex w_{ij} with c_{i+3}.
- For $3 \leq i \leq n_2$, $1 \leq j \leq n_2$, color the vertex w_{ij} with one of the colors existing such that $c(w_{ij}) \neq \{c(v_i), c(v_{i-1})\}$.

Thus, $\chi_s(P_n \circ K_{n_1,n_2}) \leq n_1 + 3$, if $n_1 < n_2$. To prove $\chi_s(P_n \circ K_{n_1,n_2}) \geq n_1 + 3$, let us assume that $\chi_s(P_n \circ K_{n_1,n_2}) < n_1 + 3$, say $n_1 + 2$. By Theorem 2, $\chi_s(K_{n_1,n_2}) = \min\{n_1, n_2\}+1$, so we need n_1+1 colors to star color $\{u_{1i} : 1 \leq i \leq n_1; w_{1j} : 1 \leq j \leq n_2\}$ for a copy of K_{n_1,n_2}.

The vertex v_1 is adjacent to each of the vertices $\{u_{1i} : 1 \leq i \leq n_1; w_{1j} : 1 \leq j \leq n_2\}$, so we need n_1+2 colors for proper star coloring of $\{v_1, u_{1i} : 1 \leq i \leq n_1; w_{1j} : 1 \leq j \leq n_2\}$. If we assign the same $n_1 + 2$ colors to the set

$$\{v_2, u_{2i} : 1 \leq i \leq n_1; w_{2j} : 1 \leq j \leq n_2\}$$

then one of the path on 4 vertices between these two set of vertices is bicolored, this is a contradiction. Thus, $\chi_s(P_n \circ K_{n_1,n_2}) \geq n_1 + 3$. Hence $\chi_s(P_n \circ K_{n_1,n_2}) = n_1 + 3$ if $n_1 < n_2$.

Case(ii): If $n_2 < n_1$. Let $n = \max\{n_1, n_2\} = n_1$. Let $V(P_n) = \{v_i : 1 \leq i \leq n_1\}$,

$$V(K_{n_1,n_2}) = \{u_{ij} : 1 \leq i \leq n_1; 1 \leq j \leq n_1\} \cup \{w_{ij} : 1 \leq i \leq n_1; 1 \leq j \leq n_2\},$$

and

$$\begin{aligned} V(P_n \circ K_{n_1,n_2}) &= \{v_i : 1 \leq i \leq n_1\} \cup \{u_{ij} : 1 \leq i \leq n_1; 1 \leq j \leq n_1\} \\ &\quad \cup \{w_{ij} : 1 \leq i \leq n_1; 1 \leq j \leq n_2\}. \end{aligned}$$

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of a copy of K_{n_1,n_2} i.e., every vertex from the set $V(P_n)$ is adjacent to every vertex from the set K_{n_1,n_2}.

Assign the star coloring as follows:

- For $1 \leq i \leq n_1$, if $i \leq n_2 + 3$ color the vertex v_i with color c_i.
- For $1 \leq i \leq n_1$, if $i > n_2+3$ color the vertex v_i with color c_j if $i \equiv j \bmod (n_2+3)$, $1 \leq j < n_2 + 3$.
- For $1 \leq i \leq n_1$, $1 \leq j \leq n_2$, if $i + j \leq n_2 + 3$ color the vertex w_{ij} with c_{i+j} if $i + j \leq n_2 + 3$.
- For $1 \leq i \leq n_1$, $1 \leq j \leq n_2$, if $i + j > n_2 + 3$ then color the vertex w_{ij} with c_i if $i + j \equiv i \bmod (n_2 + 3)$.
- For $1 \leq i \leq 2$, $1 \leq j \leq n_1$, color the vertex u_{ij} with c_{i+3}.
- For $3 \leq i \leq n_1$, $1 \leq j \leq n_1$, color the vertex u_{ij} with one of the colors existing such that $c(u_{ij}) \neq \{c(v_i), c(v_{i-1})\}$, $1 \leq i \leq n_1; 1 \leq j \leq n_1$.

Thus, $\chi_s(P_n \circ K_{n_1,n_2}) \leq n_2 + 3$, if $n_2 < n_1$. To prove $\chi_s(P_n \circ K_{n_1,n_2}) \geq n_2 + 3$, let us assume that $\chi_s(P_n \circ K_{n_1,n_2}) < n_2 + 3$, say $n_2 + 2$. By Theorem 2, $\chi_s(K_{n_1,n_2}) = \min\{n_1, n_2\}+1$, so we need n_2+1 colors to star color $\{u_{1i} : 1 \leq i \leq n_1; w_{1j} : 1 \leq j \leq n_2\}$ the vertices of K_{n_1,n_2}.

The vertex v_1 is adjacent to each of the vertices

$$\{u_{1i} : 1 \leq i \leq n_1; w_{1j} : 1 \leq j \leq n_2\},$$

so we need $n_2 + 2$ colors for the proper star coloring of

$$\{v_1, u_{1i} : 1 \leq i \leq n_1; w_{1j} : 1 \leq j \leq n_2\}.$$

If we assign the same $n_2 + 2$ colors to the another set of vertices

$$\{v_2, u_{2i} : 1 \leq i \leq n_1; w_{2j} : 1 \leq j \leq n_2\}$$

then one of the path on 4 vertices between these two set of vertices is bicolored, this is a contradiction. Thus, $\chi_s(P_n \circ K_{n_1,n_2}) \geq n_2 + 3$. Hence $\chi_s(P_n \circ K_{n_1,n_2}) = n_2 + 3$ if $n_2 < n_1$. This completes the proof of the theorem.

THEOREM 7. For any $n \geq 3$, $\chi_s(K_{1,n} \circ K_n) = n + 2$.

PROOF. Let $V(K_{1,n}) = \{v_1, v_2, \ldots, v_{n+1}\}$ and $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. By the definition of star graph, v_1 is adjacent to each vertex $\{v_i : 2 \leq i \leq n\}$. Let

$$V(K_{1,n} \circ K_n) = \{v_i : 1 \leq i \leq n+1\} \cup \{u_{ij} : 1 \leq i \leq n+1; 1 \leq j \leq n\}.$$

By the definition of corona graph, each vertex of $K_{1,n}$ is adjacent to every vertex of a copy of K_n i.e., every vertex $v_i \in V(K_{1,n})$ is adjacent to every vertex from the set

$$\{u_{ij} : 1 \leq i \leq n+1; 1 \leq j \leq n\}.$$

Assign the following $n + 2$-coloring for $K_{1,n} \circ K_n$ as star chromatic:

- For $1 \leq i \leq n+1; 1 \leq j \leq n$, assign the color c_j to u_{ij}.
- For $2 \leq i \leq n+1$, assign the color c_{n+1} to v_i.
- For the vertex v_1 assign color c_{n+2}

Thus, $\chi_s(K_{1,n} \circ K_n) \leq n + 2$. To prove $\chi_s(K_{1,n} \circ K_n) \geq n + 2$, let us assume that $\chi_s(K_{1,n} \circ K_n) < n+2$, say $n+1$. We must assign $n+1$ colors for $\{v_1, u_{1i} : 1 \leq i \leq n\}$ for proper star coloring, since $\{v_1, u_{1i} : 1 \leq i \leq n\}$ induces a clique of order $n+1$ say K_{n+1}. If we assign the same $n+1$ colors for the another clique $\{v_2, u_{2i} : 1 \leq i \leq n\}$, then an easy check shows that one of the path on 4 vertices between these two cliques is bicolored. This is a contradiction, star coloring with $n+1$ colors is impossible. Thus, $\chi_s(K_{1,n} \circ K_n) \geq n+2$. Hence, $\chi_s(K_{1,n} \circ K_n) = n+2$. This completes the proof of the theorem.

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References

[1] M. O. Albertson, G. G. Chappell, H. A. Kierstead, A. Kündgen and R. Ramamurthi, Coloring with no 2-Colored P_4's, Electron. J. Combin., 11(2004), Research Paper 26, 13 pp.

[2] A. H. Gebremedhin, A. Tarafdar, F. Manne and A. Pothen, New acyclic and star coloring algorithms with application to computing Hessians, SIAM J. Sci. Comput., 29(2007), 1042–1072

[3] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, London, MacMillan 1976.

[4] B. Grünbaum, Acyclic colorings of planar graphs, Israel J. Math., 14(1973), 390–408.

[5] G. Fertin, A. Raspaud and B. Reed, On star coloring of graphs, Journal of Graph Theory, 47(2004), 163–182.

[6] F. Harary, Graph theory, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London 1969.

[7] R. Frucht and F. Harary, On the corona of two graphs, Aequationes Math., 4(1970), 322–325.

[8] S. Klavzar, R. Hammack and W. Imrich, Handbook of Graph Products, CRC Press 2011.

[9] T. F. Coleman and J.-Y. Cai, The cyclic coloring problem and estimation of sparse Hessian matrices, SIAM J. Algebraic Discrete Methods, 7(1986), 221–235.

[10] T. F. Coleman and J. J. Moré, Estimation of sparse Hessian matrices and graph coloring problems, Math. Programming, 28(1984), 243–270.