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On The Connection Problem Between Two Classical Orthogonal Polynomial Sequences*

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Abstract

In this paper, we solve the following connection problem

$$\Phi(x)Q_n(x) = \sum_{k=0}^{n+\deg\Phi} \lambda_{n,k} P_k(x) \text{ for } n \geq 0,$$

where $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are two MOPS and Φ is a monic polynomial. We establish a method for computing the coefficient $\lambda_{n,k}$ step by step. As application, we apply this process for some continuous, discrete and quantum classical MOPS with the choice $\deg\Phi \leq 2$ and some new relationships are obtained. In particular, some well known formulas such as duplication, addition are derived.

1 Introduction and Preliminaries

Given two MPS $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ and a monic polynomial Φ, the so-called connection problem between them, i.e. the computation of coefficients $\lambda_{n,k}$ in the following expression

$$\Phi(x)Q_n(x) = \sum_{k=0}^{n+\deg\Phi} \lambda_{n,k} P_k(x) \text{ for } n \geq 0. \tag{1}$$

plays an important role in many problems in pure and applied mathematics (see for instance [6] for adequate references). The literature on this topic is extremely vast and a wide variety of methods, based on specific properties of the involved polynomials, have been developed using several techniques for $\Phi(x) = 1$ [1, 2, 6, 7, 8, 9]. In the context of the connection problem (1), we are dealing in this contribution with a numerical method to compute the coefficient $\lambda_{n,k}$ step by step. Some illustrative examples from the classical continuous, discrete and q-discrete case (Hermite, Meixner and Little q-Laguerre) are highlighted for some monic polynomials Φ with $\deg\Phi \leq 2$. As consequence, some new connections are obtained and some well known formulas such as duplication, addition are recovered.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote

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by $(u)_n := \langle u, x^n \rangle$, $n \geq 0$, the moments of the form u (linear functional). Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any polynomial g, and any $(A, B) \in \mathbb{C} - \{0\} \times \mathbb{C}$, let gu, $h_A u$, and $\tau_B u$ be the forms defined by duality

$$\langle gu, f \rangle := \langle u, gf \rangle, \quad \langle h_A u, f \rangle := \langle u, h_A f \rangle, \quad \langle \tau_B u, f \rangle = \langle u, \tau_{-B} f \rangle,$$

for all $f \in \mathcal{P}$ where $(h_A f)(x) = f(Ax)$ and $(\tau_{-B} f)(x) := f(x + B)$ [3, 5].

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n,\ n \geq 0$ (MPS) and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n \in \mathcal{P}'$ defined by $\langle u_n, P_m \rangle := \delta_{n,m}, n, m \geq 0$. The sequence $\{P_n\}_{n\geq 0}$ is called orthogonal (MOPS) if we can associate with it a form u (with $(u)_0 = 1$) and a sequence of numbers $\{r_n\}_{n\geq 0}$ ($r_n \neq 0,\ n \geq 0$) such that [3, 5]

$$\langle u, P_m P_n \rangle = r_n \delta_{n,m} \text{ for } n, m \geq 0.$$

The form u is then said to be regular. The MOPS $\{P_n\}_{n\geq 0}$ fulfils the three-term recurrence relation [3, 5]

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \xi_0, \\ P_{n+2}(x) = (x - \xi_{n+1}) P_{n+1}(x) - \alpha_{n+1} P_n(x) \text{ for } n \geq 0, \end{cases} \tag{2}$$

where

$$\xi_n = \frac{\langle u, x P_n^2 \rangle}{r_n} \text{ and } \alpha_{n+1} = \frac{r_{n+1}}{r_n} \neq 0 \text{ for } n \geq 0.$$

The regular form u is positive definite if and only if $\xi_n \in \mathbb{R}$ and $\alpha_{n+1} > 0$ for $n \geq 0$. cf. [3, 5].

If we consider the shifted monic polynomials $\widetilde{P}_n(x) = A^{-n} P_n(Ax + B)$ for $n \geq 0$, then $\{\widetilde{P}_n\}_{n\geq 0}$ is also a MOPS and its recurrence coefficients are [3, 5]

$$\widetilde{\xi}_n = \frac{\xi_n - B}{A} \text{ and } \widetilde{\alpha}_{n+1} = \frac{\alpha_{n+1}}{A^2} \text{ for } n \geq 0. \tag{3}$$

A form u is said to be symmetric if and only if $(u)_{2n+1} = 0$ for $n \geq 0$. A MPS $\{P_n\}_{n\geq 0}$ is symmetric if and only if $P_n(-x) = (-1)^n P_n(x)$ for $n \geq 0$. cf. [3, 5]. Let $\{P_n\}_{n\geq 0}$ be a MOPS with respect to u, then

$$u \text{ is symmetric} \iff \{P_n\}_{n\geq 0} \text{ is symmetric} \iff \xi_n = 0 \text{ for } n \geq 0.$$

cf. [3, 5].

In the sequel, let $\{P_n\}_{n\geq 0}$ be a MOPS with respect to u_0 and satisfying (2) and $\{Q_n\}_{n\geq 0}$ be a MOPS fulfilling

$$\begin{cases} Q_0(x) = 1,\ Q_1(x) = x - \beta_0, \\ Q_{n+2}(x) = (x - \beta_{n+1}) Q_{n+1}(x) - \gamma_{n+1} Q_n(x) \text{ for } n \geq 0. \end{cases} \tag{4}$$

2 The Method

The scope of this section is to give recurrence relations in order to be able to calculate by induction the coefficients $\lambda_{n,k}$ between $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ with respect to Φ ($t = \deg \Phi \geq 0$) given by the expansion of ΦQ_n in terms of the P_n basis. We may write (1) in the following way

$$\Phi(x)Q_n(x) = \sum_{k\in\mathbb{Z}} \lambda_{n,k} P_k(x) \text{ for } n \geq 0. \tag{5}$$

By virtue of (5), (4) and (2), we get the following formula

$$\begin{aligned}\lambda_{n,n+t-j} \quad = \quad & \lambda_{0,t-j} + \sum_{k=\nu_j}^{n-1} \Big\{ (\xi_{k+t+1-j} - \beta_k)\lambda_{k,k+t+1-j} + \alpha_{k+t+2-j}\lambda_{k,k+t+2-j} \\ & -\gamma_k \lambda_{k-1,k+t+1-j} \Big\}\end{aligned} \tag{6}$$

for $n \geq \max(1, \nu_j + 1)$, where $\nu_j = \max(0, j - t - 1)$, $0 \leq j \leq n + t$, and the initial conditions are reached in the following values $(\lambda_{0,k})_{0\leq k\leq t}$. Moreover,

$$\lambda_{n,k} = 0 \text{ for either } k \leq -1 \text{ or } k \geq n + t + 1,\ n \geq 0. \tag{7}$$

We are going to detail the process (6)–(7). For $j = 0$, we have

$$\lambda_{n,n+t} = 1 \text{ for } n \geq 0. \tag{8}$$

For $j = 1$, we have $\nu_1 = 0$. Taking (6)–(7) into account, we get

$$\lambda_{n,n+t-1} = \lambda_{0,t-1} + \sum_{k=0}^{n-1} (\xi_{k+t} - \beta_k) \text{ for } n \geq 1. \tag{9}$$

For $j = 2$ in (6)-(7), two cases arise:

(i) If $t \geq 1$, then $\nu_2 = 0$. Therefore, for $n \geq 2$,

$$\lambda_{n,n+t-2} = \lambda_{0,t-2} + \sum_{k=0}^{n-1} (\xi_{k+t-1} - \beta_k)\lambda_{k,k+t-1} + \alpha_t + \sum_{k=1}^{n-1} (\alpha_{k+t} - \gamma_k) \tag{10}$$

and, for $n = 1$,

$$\lambda_{1,t-1} = \lambda_{0,t-2} + (\xi_{t-1} - \beta_0)\lambda_{0,t-1} + \alpha_t. \tag{11}$$

(ii) If $t = 0$, then $\nu_2 = 1$. Therefore, for $n \geq 2$,

$$\lambda_{n,n-2} = \sum_{k=1}^{n-1} \Big\{ (\xi_{k-1} - \beta_k)\lambda_{k,k-1} + (\alpha_k - \gamma_k) \Big\}. \tag{12}$$

If we suppose that for an integer j satisfying $0 \leq j+1 \leq n+t$, all the coefficients $\lambda_{k,k+t-(j-1)}$ and $\lambda_{k,k+t-j}, 0 \leq k \leq n-1$ have been calculated, then using (6)-(7) with the change $j \leftarrow j+1$ yields

$$\begin{aligned}\lambda_{n,n+t-(j+1)} &= \lambda_{0,t-j-1} + \sum_{k=\nu_{j+1}}^{n-1} \Big\{(\xi_{k+t-j} - \beta_k)\lambda_{k,k+t-j} \\ &\qquad + \alpha_{k+t+1-j}\lambda_{k,k+t-(j-1)} - \gamma_k \lambda_{k-1,k-1+t-(j-1)}\Big\}. \end{aligned} \tag{13}$$

Hence, it is possible to determine $\lambda_{n,n+t-(j+1)}$ for $n \geq \max(1, \nu_{j+1}+1)$.

REMARK 1. On account of (4), we obtain

$$(x+c)Q_n(x) = Q_{n+1}(x) + (c+\beta_n)Q_n(x) + \gamma_n Q_{n-1}(x) \text{ for } n \geq 0,\ c \in \mathbb{C}. \tag{14}$$

REMARK 2. When $\Phi(x) = x^2 + cx + d,\ c, d \in \mathbb{C}$ and using the previous relation, the coefficients $\{\theta_{n,k}\}_{n,k\geq 0}$ between $\{Q_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ by respect to Φ are given by

$$\begin{cases} \theta_{n,n+2} = 1\ ,\ \theta_{n,n+1} = c + \beta_n + \beta_{n+1} & \text{for } n \geq 0, \\ \theta_{n,n} = d + c\beta_n + \beta_n^2 + \gamma_n + \gamma_{n+1} & \text{for } n \geq 0, \\ \theta_{n,n-1} = \gamma_n(c + \beta_n + \beta_{n-1}),\ n \geq 1\ ,\ \theta_{n,n-2} = \gamma_n\gamma_{n-1} & \text{for } n \geq 2, \\ \theta_{n,k} = 0,\ 0 \leq k \leq n-3 & \text{for } n \geq 3. \end{cases} \tag{15}$$

PROPOSITION 1. Let consider the following connection problems

$$Q_n(x) = \sum_{k=0}^{n} \mu_{n,k} P_k(x) \text{ and } \Phi(x)Q_n(x) = \sum_{k=0}^{n+t} \lambda_{n,k} P_k(x) \text{ for } n \geq 0.$$

Then the following two statements hold.

(i) If $\Phi(x) = x + c$, then $\lambda_{n,k} = \mu_{n+1,k} + (\beta_n + c)\mu_{n,k} + \gamma_n \mu_{n-1,k}$ for $n, k \geq 0$.

(ii) If $\Phi(x) = x^2 + cx + d$, then

$$\lambda_{n,k} = \mu_{n+2,k} + \theta_{n,n+1}\mu_{n+1,k} + \theta_{n,n}\mu_{n,k} + \theta_{n,n-1}\mu_{n-1,k} + \theta_{n,n-2}\mu_{n-2,k}$$

for $n, k \geq 0$ where $\theta_{n,k}$ is given in (15).

PROOF. (i)(respectively (ii)) is an immediate consequence of (14)(respectively (15)).

3 Applications

3.1 The Continuous Classical Hermite MOPS $\{H_n\}_{n\geq 0}$

Let $\{H_n\}_{n\geq 0}$ be the Hermite MOPS satisfying (2) with $\xi_n = 0$ and $\alpha_{n+1} = \frac{1}{2}(n+1)$ for $n \geq 0$ [3]. Let consider the two shifted MOPS $\{\widetilde{H}_n\}_{n\geq 0}$ and $\{\widehat{H}_n\}_{n\geq 0}$ defined by

$$\widetilde{H}_n(x) = (\tau_{-y} H_n)(x) = H_n(x+y) \text{ for } y \in \mathbb{C}$$

and

$$\widehat{H}_n(x) = a^{-n} H_n(ax) \text{ for } a \in \mathbb{C} \setminus \{0\}.$$

Accordingly to (3), we obtain

$$\widetilde{\xi}_n = -y \text{ and } \widetilde{\alpha}_{n+1} = \frac{1}{2}(n+1) \text{ for } n \geq 0, \tag{16}$$

and

$$\widehat{\xi}_n = 0 \text{ and } \widehat{\alpha}_{n+1} = \frac{1}{2a^2}(n+1) \text{ for } n \geq 0. \tag{17}$$

3.1.1 The Connection Problem $\widehat{H}_n(x) = \sum_{k=0}^{n} \mu_{n,k} H_k(x)$

Choosing $Q_n(x) = \widehat{H}_n(x)$, $P_n = H_n$ and $\Phi(x) = 1$ $(t = 0)$ in (1), (6)-(7) and by virtue of (17), then (9) and (12) lead to

$$\lambda_{n,n-1} = 0 \text{ for } n \geq 1 \text{ and } \lambda_{n,n-2} = \frac{1}{2}\left(1 - \frac{1}{a^2}\right)\binom{n}{n-2} \text{ for } n \geq 2.$$

By induction and (13), we get $\lambda_{n,n-(2j+1)} = 0$. Suppose that

$$\lambda_{k,k-2j} = \frac{\prod_{\nu=1}^{j}(2\nu-1)}{2^j}\left(1 - \frac{1}{a^2}\right)^j \binom{k}{2j} \text{ for } 0 < 2j \leq n-2 \text{ and } 2j \leq k \leq n.$$

On account of (13) an other time, we obtain

$$\lambda_{n,n-(2j+2)} = \sum_{k=2j+1}^{n-1} \{\alpha_{k-2j}\lambda_{k,k-2j} - \gamma_k \lambda_{k-1,k-1-2j}\}.$$

It's easy to verify that

$$\alpha_{k-2j}\lambda_{k,k-2j} = \frac{\prod_{\nu=0}^{j}(2\nu+1)}{2^{j+1}}\left(1 - \frac{1}{a^2}\right)^j \binom{k}{2j+1}$$

and

$$\gamma_k \lambda_{k-1,k-1-2j} = \frac{\prod_{\nu=0}^{j}(2\nu+1)}{2^{j+1}}\left(1 - \frac{1}{a^2}\right)^j \left(\frac{1}{a^2}\right)\binom{k}{2j+1}.$$

Then

$$\begin{aligned}\lambda_{n,n-(2j+2)} &= \frac{\prod_{\nu=0}^{j}(2\nu+1)}{2^{j+1}}\left(1-\frac{1}{a^2}\right)^{j+1}\sum_{k=2j+1}^{n-1}\binom{k}{2j+1}\\ &= \frac{\prod_{k=0}^{j}(2k+1)}{2^{j+1}}\left(1-\frac{1}{a^2}\right)^{j+1}\binom{n}{2j+2}.\end{aligned}$$

Lastly, we obtain

$$\begin{cases}\lambda_{n,n-j}=0 & \text{for } j=2k+1 \text{ and } k\leq[\frac{n-1}{2}],\\ \lambda_{n,n-j}=\frac{\prod_{k=0}^{j}(2k+1)}{2^k}\left(1-\frac{1}{a^2}\right)^k\binom{n}{2k} & \text{for } j=2k \text{ and } 1\leq k\leq[\frac{n}{2}],\\ \lambda_{n,n}=1 & \text{for } n\geq 0.\end{cases}$$

Hence,

$$a^{-n}H_n(ax)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(2k)!}{2^{2k}k!}\left(1-\frac{1}{a^2}\right)^k\binom{n}{2k}H_{n-2k}(x) \text{ for } n\geq 0.$$

Consequently, we recover again the so-called duplication formula for the Hermite polynomials [6].

3.1.2 The Connection Problem $\widetilde{H}_n(x)=\sum_{k=0}^{n}\mu_{n,k}H_k(x)$

On account of (9), (12) with (16), where $P_n=H_n$, $Q_n=\widetilde{H}_n$ and $t=0$, we get

$$\mu_{n,n-1}=\binom{n}{n-1}y,\ n\geq 1 \text{ and } \mu_{n,n-2}=\binom{n}{n-2}y^2 \text{ for } n\geq 2.$$

Suppose that

$$\mu_{k,k-i}=\binom{k}{k-i}y^i \text{ for } i\leq j\leq n-1 \text{ and } j\leq k\leq n.$$

By virtue of (13), we obtain

$$\mu_{n,n-(j+1)}=\sum_{k=j}^{n-1}\left\{y\mu_{k,k-j}+\alpha_{k+1-j}\mu_{k,k-(j-1)}-\alpha_k\mu_{k-1,k-1-(j-1)}\right\}.$$

But $\alpha_{k+1-j}\mu_{k,k-(j-1)}=\alpha_k\mu_{k-1,k-1-(j-1)}$. Hence,

$$\mu_{n,n-(j+1)}=y^{j+1}\sum_{k=j}^{n-1}\binom{k}{j}=y^{j+1}\binom{n}{j+1}=y^{j+1}\binom{n}{n-(j+1)}.$$

Consequently, we recover again the well known addition formula [6]

$$H_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}y^{n-k}H_k(x),\ n\geq 0.$$

3.1.3 The Connection Problem $(x^2+cx+d)\widetilde{H}_n(x)=\sum_{k=0}^{n+2}\lambda_{n,k}H_k(x),\ c,d\in\mathbb{C}$

Using the connection problem in 3.1.2 and applying Proposition 1., we get that, for $n\geq 0$ and $k\leq n$,

$$\lambda_{n,n+1}=c+ny,\ \lambda_{n,n+2}=1,$$

$$\begin{aligned}\lambda_{n,k} &= y^{n-2-k}\tbinom{n}{k}\left\{y^4\frac{(n+2)(n+1)}{(n+2-k)(n+1-k)}+(c-2y)y^3\frac{n+1}{n+1-k}\right.\\ &\left.+y^2(n+\frac{1}{2}+d+y^2-cy)+y\frac{(c-2y)(n-k)}{2}+\frac{(n-k)(n-k-1)}{4}\right\}.\end{aligned}$$

3.2 The Discrete Classical Meixner MOPS $\{M_n^{(\alpha,a)}\}_{n\geq 0}$

Let us consider the Meixner MOPS $\{M_n^{(\alpha,a)}\}_{n\geq 0}$ of parameters α,a. It satisfies (2) with [3]

$$\xi_n=\frac{a\alpha+n(1+a)}{1-a}\text{ for }n\geq 0\text{ and }\alpha_n=\frac{an(\alpha+n-1)}{(1-a)^2}\text{ for }n\geq 1. \tag{18}$$

3.2.1 The Connection Problem $M_n^{(\beta,a)}(x)=\sum_{k=0}^{n}\mu_{n,k}M_k^{(\alpha,a)}(x)$

Choosing $P_n=M_n^{(\alpha,a)}$ and $Q_n=M_n^{(\beta,a)}$ in (1). On account of (18), (9) gives

$$\mu_{n,n-1}=\sum_{k=0}^{n-1}(\xi_k-\beta_k)=n\left(\frac{a}{1-a}\right)(\alpha-\beta)=\frac{a}{1-a}\tbinom{n}{n-1}(\alpha-\beta)_1,$$

where

$$(x)_n:=\prod_{k=0}^{n-1}(x+k)=\frac{\Gamma(x+n)}{\Gamma(x)}\text{ for }n\geq 1,$$

being the well-known Pochhammer's symbol and Γ be the Gamma function [3]. Then

$$\mu_{n,n-1}=\tbinom{n}{n-1}\left(\frac{a}{a-1}\right)^{n-(n-1)}(\beta-\alpha)_{n-(n-1)}.$$

Likewise, taking (12) into account and by virtue of (18), we have

$$\begin{aligned}\mu_{n,n-2} &= \sum_{k=1}^{n-1}(\xi_{k-1}-\beta_k)\lambda_{k,k-1}+\sum_{k=1}^{n-1}(\alpha_k-\gamma_k)\\ &= \sum_{k=1}^{n-1}\left\{\frac{a(\alpha-\beta)-(1+a)}{1-a}\ \frac{a}{1-a}\,(\alpha-\beta)k\right\}+\sum_{k=1}^{n-1}\frac{ak(\alpha-\beta)}{(1-a)^2}\\ &= \left(\frac{a}{1-a}\right)^2(\alpha-\beta)(\alpha-\beta-1)\frac{(n-1)n}{2}\\ &= \tbinom{n}{n-2}\left(\frac{a}{a-1}\right)^{n-(n-2)}(\beta-\alpha)_{n-(n-2)}.\end{aligned}$$

Suppose that

$$\mu_{k,k-i} = \binom{n}{i}\left(\frac{a}{a-1}\right)^{i}(\beta-\alpha)_i \text{ for } i \leq j \text{ and } k \leq n.$$

Using (13) and by virtue of (18) an other time, we obtain

$$\begin{aligned}
&\mu_{n,n-(j+1)}\\
=\ & \sum_{k=j}^{n-1}\Big\{(\xi_{k-j}-\beta_k)\mu_{k,k-j}+\alpha_{k+1-j}\mu_{k,k-(j-1)}-\gamma_k\mu_{k-1,k-1-(j-1)}\Big\}\\
=\ & \frac{a^j}{(a-1)^{j+1}}(\beta-\alpha)_j\left\{[a(\beta-\alpha+j)+j]\sum_{k=j}^{n-1}\binom{k}{j}-\sum_{k=j}^{n-1}(k-j+1)\binom{k}{j-1}\right\}\\
=\ & \left(\frac{a}{a-1}\right)^{j+1}(\beta-\alpha)_{j+1}\sum_{k=j}^{n-1}\binom{k}{j}\\
=\ & \binom{n}{j+1}\left(\frac{a}{a-1}\right)^{j+1}(\beta-\alpha)_{j+1}.
\end{aligned}$$

Hence,

$$\mu_{n,n-j} = \binom{n}{j}\left(\frac{a}{a-1}\right)^{j}(\beta-\alpha)_j \text{ for } 0 \leq j \leq n \text{ and } n \geq 1.$$

3.2.2 The Connection Problem $(x+c)M_n^{(\beta,a)}(x) = \sum_{k=0}^{n}\lambda_{n,k}M_k^{(\alpha,a)}(x),\ c\in\mathbb{C}$

Taking into account Proposition 1. and the connection problem 3.2.1, the coefficients between $\{M_n^{(\alpha,a)}\}_{n\geq 0}$ and $\{M_n^{(\alpha,b)}\}_{n\geq 0}$ by respect to $\Phi(x)=x+c$ are given by $\lambda_{n,n+1}=1$ and

$$\begin{aligned}
\lambda_{n,k} \ =\ & \binom{n}{k}\left(\frac{a}{a-1}\right)^{n+1-k}(\beta-\alpha)_{n-k}\times\left\{\frac{(n+1)(\beta-\alpha+n-k)}{n+1-k}\right.\\
& \left.-\frac{(1-a)c+a\beta+n(1+a)}{a}+\frac{(\beta+n-1)(n-k)}{a(\beta-\alpha+n-k-1)}\right\}
\end{aligned}$$

for $n\geq 0$ and $0\leq k\leq n$.

3.3 The Quantum Classical Little q-Laguerre MOPS $\{L_n(.;a|q)\}_{n\geq 0}$

Let us consider the Little q-Laguerre MOPS $\{L_n(.;a|q)\}_{n\geq 0}$ of parameters $a\neq 0$. It satisfies (2) with [4]

$$\begin{cases}\xi_n = \{1+a-a(1+q)q^n\}q^n & \text{for } n\geq 0,\\ \alpha_n = a(1-q^n)(1-aq^n)q^{2n-1} & \text{for } n\geq 1.\end{cases} \tag{19}$$

On account of (3), its shifted MOPS $\{q^{-n}h_q L_n(.;a|q)\}_{n\geq 0}$ satisfies (4) with

$$\begin{cases} \beta_n = \{1+a-a(1+q)q^n\}q^{n-1} & \text{for } n\geq 0,\\ \gamma_n = a(1-q^n)(1-aq^n)q^{2n-3} & \text{for } n\geq 1. \end{cases} \tag{20}$$

3.3.1 The Connection Problem $q^{-n}L_n(qx;a|q)=\sum_{k=0}^{n}\mu_{n,k}L_k(x;a|q)$

Choosing $P_n(x)=L_n(x;a|q)$ and $Q_n(x)=q^{-n}L_n(qx;a|q)$ in (1). On account of (19)-(20), (9) gives

$$\mu_{n,n-1}=-q^{-1}(1-q^n)(1-aq^n) \text{ for } n\geq 1.$$

Moreover, after some calculations taking into account (19)-(20) and (12)-(13) we get

$$\mu_{n,n-2}=0 \text{ for } n\geq 2 \text{ and } \mu_{n,n-3}=0 \text{ for } n\geq 3.$$

Consequently, formula (13) an other time yields $\mu_{n,n-k}=0$ for $0\leq k\leq n-2$. Therefore,

$$q^{-n}L_n(qx;a|q)=L_n(x;a|q)-q^{-1}(1-q^n)(1-aq^n)L_{n-1}(x;a|q) \text{ for } n\geq 0 \tag{21}$$

with $L_{-1}(x;a|q):=0$.

3.3.2 The Connection Problem $(x+c)q^{-n}L_n(qx;a|q)=\sum_{k=0}^{n+1}\lambda_{n,k}L_k(x;a|q),\ c\in\mathbb{C}$

Taking into account Proposition 1. and the relationship (21) in the connection problem 3.3.1, the coefficients between $\{q^{-n}h_q L_n(.;a|q)\}_{n\geq 0}$ and $\{L_n(.;a|q)\}_{n\geq 0}$ by respect to $\Phi(x)=x+c,\ c\in\mathbb{C}$ and

$$(x+c)q^{-n}L_n(qx;a|q)=\sum_{k=n-2}^{n+1}\lambda_{n,k}L_k(x;a|q)$$

are given by

$$\lambda_{n,n+1}=1,\ \lambda_{n,n}=c-q^{-1}+\big\{(1+a)(1+q)-aq^n(1+q+q^2)\big\}q^{n-1},$$

$$\lambda_{n,n-1}=-q^{-1}(1-q^n)(1-aq^n)\big\{c+q^{n-1}-aq^{n-1}\big(1-q^{n-1}(1+q+q^2)\big)\big\},$$

and

$$\lambda_{n,n-2}=-a(1-q^{n-1})(1-q^n)(1-aq^{n-1})(1-aq^n)q^{2n-4}.$$

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