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Open Neighborhood Chromatic Number Of An Antiprism Graph*

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Abstract

An open neighborhood k-coloring of a simple connected undirected graph $G(V, E)$ is a k-coloring $c : V \to \{1, 2, \cdots, k\}$, such that, for every $w \in V$ and for all $u, v \in N(w)$, $c(u) \neq c(v)$. The minimum value of k for which G admits an open neighborhood k-coloring is called the open neighborhood chromatic number of G denoted by $\chi_{onc}(G)$. In this paper, we obtain the open neighborhood chromatic number of the Petersen graph. Also, we determine this number for a family of graphs called antiprism graphs.

1 Introduction

All the graphs considered in this paper are simple, non-trivial, undirected, finite and connected. For standard terminologies, we refer [2] and [7]. A *vertex coloring*, or simply a *coloring*, of a graph $G = (V, E)$ is an assignment of colors to the vertices of G. A k-coloring of G is a surjection $c : V \to \{1, 2, \cdots, k\}$. A proper coloring of G is an assignment of colors to the vertices of G so that adjacent vertices are colored differently. A *proper k-coloring* of G is a surjection $c : V \to \{1, 2, \cdots, k\}$ such that $c(u) \neq c(v)$ if u and v are adjacent in G. The minimum k for which there is a proper k-coloring of G is called the *chromatic number* of G denoted by $\chi(G)$.

As seen in Fig. 3, the *Petersen graph* [10] is an undirected graph with 10 vertices and 15 edges and serves as a useful example and counterexample for many problems in graph theory. It is a cubic symmetric graph and is non-planar. The chromatic number and the domination number of the Petersen graph are both equal to 3. The generalized Petersen graph $GP(n, k), n \geq 3$ and $k < n/2$, is a graph consisting of an inner star polygon $\{n, k\}$ and an outer regular polygon C_n with corresponding vertices in the inner and outer polygons connected with edges. The Petersen graph can be obtained from this graph by choosing $n = 5$ and $k = 2$.

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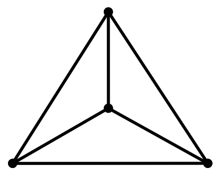

Figure 1: Tetrahedral graph

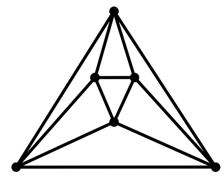

Figure 2: Octahedral graph

The graph obtained by replacing the faces of a polyhedron with its edges and vertices is called the *skeleton* [3] of the polyhedron. For example, the polyhedral graphs corresponding to the skeletons of tetrahedron and octahedron are illustrated in Fig. 1 and 2.

An n-*antiprism* [4], $n \geq 3$, is a semiregular polyhedron constructed with $2n$-gons and $2n$ triangles. It is made up of two n-gons on top and bottom, separated by a ribbon of $2n$ triangles, with the two n-gons being offset by one ribbon segment. The graph corresponding to the skeleton of an n-antiprism is called the n-*antiprism graph*, denoted by Q_n, $n \geq 3$ as shown in Fig. 4. As seen from the figure, Q_n has $2n$ vertices and $4n$ edges, and is isomorphic to the circulant graph $Ci_{2n}(1,2)$. In particular, the 3-antiprism graph Q_3 is isomorphic to the octahedral graph in Fig. 2.

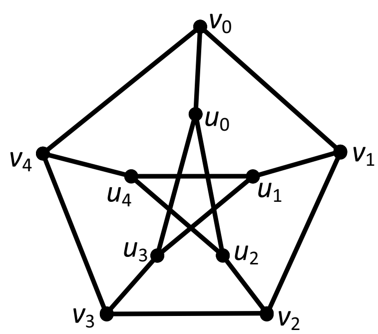

Figure 3: Petersen graph

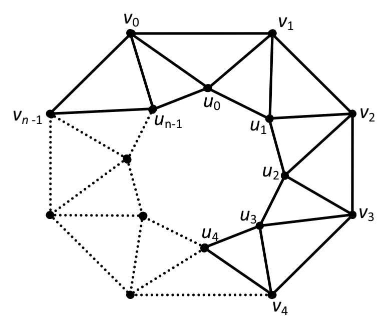

Figure 4: n-antiprism graph

An *open neighborhood coloring* [5] of a graph $G(V,E)$ is a coloring $c : V \rightarrow Z^+$, such that for each $w \in V$ and $\forall u, v \in N(w)$, $c(u) \neq c(v)$. An *open neighborhood k-coloring* of a graph $G(V,E)$ is a k-coloring $c : V \rightarrow \{1, 2, \cdots, k\}$ which admits the conditions of an open neighborhood coloring. The minimum value of k for which G admits an open neighborhood k-coloring is called the *open neighborhood chromatic number* of G denoted by $\chi_{onc}(G)$.

In [5], we have established some bounds on the open neighborhood chromatic number of a graph. We have also obtained this parameter for an infinite triangular lattice. Further, in [6], we have determined the open neighborhood chromatic number of prism graph which is obtained from the generalized Petersen graph $GP(n,k)$ by choosing $k = 1$ and $n \geq 3$.

We recall some of the definitions and results on the open neighborhood chromatic number discussed in [5].

THEOREM 1.1. If f is an open neighborhood k-coloring of $G(V,E)$ with $\chi_{onc}(G) = k$, then $f(u) \neq f(v)$ holds where u, v are the end vertices of a path of length 2 in G.

THEOREM 1.2. For any graph $G(V,E)$, $\chi_{onc}(G) \geq \Delta(G)$.

THEOREM 1.3. If H is a connected subgraph of G, then $\chi_{onc}(H) \leq \chi_{onc}(G)$.

THEOREM 1.4. The open neighborhood chromatic number of a connected graph G is 1 if and only if $G \cong K_1$ or K_2.

THEOREM 1.5. Let $G(V,E)$ be a connected graph on $n \geq 3$ vertices. Then $\chi_{onc}(G) = n$ if and only if $N(u) \bigcap N(v) \neq \emptyset$ holds for every pair of vertices $u, v \in V(G)$.

THEOREM 1.6. For a path $P_n,\ n \geq 2$,

$$\chi_{onc}(P_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n \geq 3. \end{cases}$$

THEOREM 1.7. For a cycle $C_n,\ n \geq 3$,

$$\chi_{onc}(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4, \\ 3 & \text{otherwise.} \end{cases}$$

DEFINITION 1.8. In a graph G, a subset V_1 of $V(G)$ such that no two vertices of V_1 are end vertices of a path of length two in G is called a P_3-independent set of G.

In this paper, we obtain the open neighborhood chromatic number of the Petersen graph. Also we determine this number for the n-antiprism graph Q_n.

2 Open Neighborhood Chromatic Number of Petersen Graph

OBSERVATION 2.1. For any graph G of order n, if $\chi_{onc}(G) = n$, then $diam(G) \leq 2$.

PROOF. Consider a graph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$ with $\chi_{onc}(G) = n$. Suppose $diam(G) \geq 3$. Without loss in generality, let $d(v_1, v_2) \geq 3$. We define a coloring $c : V(G) \to \{1, 2, \cdots, n-1\}$ as follows.

$$c(v_i) = \begin{cases} 1 & \text{if } i = 1 \text{ or } i = 2, \\ i-1 & \text{otherwise.} \end{cases}$$

Clearly, c is an open neighborhood $(n-1)$-coloring of G so that $\chi_{onc}(G) \leq n-1$, a contradiction.

THEOREM 2.2. If G is any graph of order n, $\chi_{onc}(G) = 2$ if and only if $G \cong P_n$, $n \geq 3$ or $G \cong C_n$, $n \equiv 0 (\text{mod}\, 4)$.

PROOF. Consider a graph G of order n. Suppose $\chi_{onc}(G) = 2$. By Theorem 1.2, we have $\chi_{onc}(G) \geq \Delta(G)$ so that $\Delta(G) \leq 2$. Thus, G is either a path or a cycle. However, by Theorem 1.7, we know that $\chi_{onc}(C_n) = 2$ only when $n \equiv 0(\text{mod}\, 4)$. Also by Theorem 1.6, $\chi_{onc}(P_n) = 2$ for any $n \geq 3$. Thus, if $\chi_{onc}(G) = 2$, then $G \cong P_n, n \geq 3$ or $G \cong C_n, n \equiv 0(\text{mod}\, 4)$. The converse is a direct consequence of Theorem 1.6 and Theorem 1.7.

THEOREM 2.3. The open neighborhood chromatic number of the Petersen graph $GP(5, 2)$ is 5.

PROOF. Let u be any vertex of $G = GP(n, 2)$. Then in any open neighborhood coloring c, $c(u) \neq c(v)$ for any $v \notin N(u)$ as every such vertex is connected by a path of length two from u. Further at most one vertex in $N(u)$ can be given the same color as that of u since there is a path of length two between every $v, w \in N(u)$. Thus, one color can be given to at most two vertices in any open neighborhood coloring c of G so that $\chi_{onc}(G) \geq 5$. To prove the reverse inequality, consider a coloring $c : V(G) \to \{1, 2, 3, 4, 5\}$ as

$$c(v) = \begin{cases} 1, & \text{if } v = v_0 \text{ or } v = v_4, \\ 2, & \text{if } v = v_1 \text{ or } v = v_2, \\ 3, & \text{if } v = u_3 \text{ or } v = v_3, \\ 4, & \text{if } v = u_0 \text{ or } v = u_2, \\ 5, & \text{otherwise.} \end{cases}$$

It is easy to verify that c is an open neighborhood 5-coloring of G so that $\chi_{onc}(G) \leq 5$. Hence, $\chi_{onc}(G) = 5$.

3 Open Neighborhood Chromatic Number of an Antiprism Graph

In this section, we determine the open neighborhood chromatic number of an n-antiprism graph Q_n.

OBSERVATION 3.1. Every integer $n \geq 8$ with $n \not\equiv 0(\text{mod}\, 5)$ can be expressed as $n = 3k + 5m$ for some integers $m \geq 0$ and $k \geq 1$.

LEMMA 3.2. For any integer $n \geq 3$, $\chi_{onc}(Q_n) \geq 5$.

PROOF. For each $n \geq 3$, Q_n contains a subgraph H as in Fig. 5. Further, in H, there is a path of length two between every pair of vertices so that $\chi_{onc}(H) = 5$. Hence by Theorem 1.3, $\chi_{onc}(Q_n) \geq 5$.

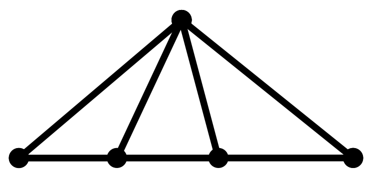

Figure 5: A subgraph of Q_n

OBSERVATION 3.3. In the antiprism graph Q_n, the only vertices that are connected to a vertex $u_i, 0 \leq i \leq n-1$ by a path of length two are $u_{i\pm1}, u_{i\pm2}, v_i, v_{i\pm1}, v_{i+2}$ where the suffix is under modulo n. Similarly, the vertices that are connected to a vertex $v_i, 0 \leq i \leq n-1$ by a path of length two are $u_i, u_{i\pm1}, u_{i-2}, v_{i\pm1}, v_{i\pm2}$ where the suffix is under modulo n.

LEMMA 3.4. Let Q_n be an antiprism graph and let

$$S_k = \{u_i, v_j |\ i \equiv (k+2)(\bmod 5) \text{ and } j \equiv k(\bmod 5)\} \text{ for } 0 \leq k \leq 4. \quad (1)$$

Then each S_k is a P_3- independent set if and only if $n \equiv 0(\bmod 5)$.

PROOF. Let $n \equiv 0(\bmod 5)$. We see that $i \equiv (k+2)(\bmod 5)$. It follows that $i+1 \equiv (k+3)(\bmod 5)$, $i-1 \equiv (k+1)(\bmod 5)$, $i+2 \equiv (k+4)(\bmod 5)$ and $i-2 \equiv k(\bmod 5)$ so that $u_{i\pm1}, u_{i\pm2}, v_i, v_{i\pm1}, v_{i+2} \notin S_k$. Also, $j \equiv k(\bmod 5)$ implies $j+1 \equiv (k+1)(\bmod 5)$, $j-1 \equiv (k+4)(\bmod 5)$, $j+2 \equiv (k+2)(\bmod 5)$ and $j-2 \equiv (k+3)(\bmod 5)$ so that $u_j, u_{j\pm1}, u_{j-2}, v_{j\pm1}, v_{j\pm2} \notin S_k$. Hence, by Observation 3.3, S_k is a P_3- independent set for $0 \leq k \leq 4$. Next, we assume that S_k is a P_3- independent set and we prove the converse by contraposition.

Case 1. Suppose $n \equiv 1(\bmod 5)$. Then $v_0, v_{n-1} \in S_0$. But v_0 and v_{n-1} are end vertices of a path of length 2 so that S_0 is not a P_3- independent set.

Case 2. Suppose $n \equiv 2(\bmod 5)$. Then $u_0, u_2 \in S_0$. But u_0 and u_2 are end vertices of a path of length 2 so that S_0 is not a P_3- independent set.

Case 3. Suppose $n \equiv 3(\bmod 5)$. Then $u_{n-1}, v_0 \in S_0$. But u_{n-1} and v_0 are end vertices of a path of length 2 so that S_0 is not a P_3- independent set

Case 4. Suppose $n \equiv 4(\bmod 5)$. Then $u_{n-2}, v_0 \in S_0$. But u_{n-2} and v_0 are end vertices of a path of length 2 so that S_0 is not a P_3- independent set.

So by Cases 1–4, we obtain $n \equiv 0(\bmod 5)$. Therefore, the proof of Lemma 3.4 is complete.

LEMMA 3.5. For any positive integer $n \geq 5$, $\chi_{onc}(Q_n) = 5$ if and only if $n \equiv 0(\bmod 5)$.

PROOF. Consider an n-antiprism graph Q_n as in Fig. 4 such that $n \equiv 0(\bmod 5)$. By Lemma 1.1, $\chi_{onc}(Q_n) \geq 5$. We recall the set S_k defined by (1). By Lemma 3.4, each S_k is a P_3-independent set which implies that every vertex in any S_k can be given the same color in any open neighborhood coloring of G. Thus, the coloring $c : V(Q_n) \to \{1, 2, 3, 4, 5\}$ defined by $c(v) = k + 1$ if $v \in S_k$ for $0 \leq k \leq 4$ is an open neighborhood 5-coloring of Q_n so that $\chi_{onc}(Q_n) = 5$.

We prove the converse by the method of contradiction. Let $\chi_{onc}(Q_n) = 5$. Suppose $n \not\equiv 0(\bmod 5)$. By Observation 3.3, each of the vertices v_0, v_1, v_2, u_0 and u_1 should be in different P_3-independent sets. Let S_0, S_1, S_2, S_3 and S_4 be mutually disjoint P_3-independent sets with $v_0 \in S_0$, $v_1 \in S_1$, $v_2 \in S_2$, $u_0 \in S_3$ and $u_1 \in S_4$. Now, v_3 cannot belong to any of the sets S_1, S_2 or S_4. However, it may be in S_0, S_3 or neither. Also, u_2 cannot belong to any of the sets S_1, S_2, S_3 or S_4. Based on this, we consider the following Cases 1–3.

Case 1. Suppose $v_3 \in S_0$. Then u_2 cannot be in S_k for any $0 \leq k \leq 4$ which means that $u_2 \in S$, a P_3-independent set different from the sets S_0, S_1, S_2, S_3 and S_4. Thus, at least six colors are needed to have an open neighborhood coloring of Q_n.

Case 2. Suppose $v_3 \in S_3$. Then u_2 may or may not be in S_0.

Subcase 2-1. Assume that $u_2 \notin S_0$. Then, u_2 is not in any of the sets $S_k, 0 \leq k \leq 4$,. Thus as in Case 1, at least six colors are needed to have an open neighborhood coloring of Q_n.

Subcase 2-2. Assume that $u_2 \in S_0$. Then, we see that $v_3 \in S_3$, $u_3 \in S_1$ and so on. However, proceeding further in this manner, we get $v \in S_0$ with v being one of v_{n-1}, v_{n-2}, u_{n-1} or u_{n-2} according as $n \equiv 1(\bmod 5)$, $n \equiv 2(\bmod 5)$, $n \equiv 3(\bmod 5)$ or $n \equiv 4(\bmod 5)$. In such a case, S_0 does not remain a P_3-independent set. To avoid this, we need to have $v \in S$, a P_3-independent set different from S_0, S_1, S_2, S_3 and S_4 so that at least six colors are needed to have an open neighborhood coloring of Q_n.

Case 3. Suppose $v_3 \notin S_0$ or S_3. Then, as in Case 1, at least six colors are needed to have an open neighborhood coloring of Q_n.

THEOREM 3.6. Let Q_n be an antiprism graph. Then

$$\chi_{onc}(Q_n) = \begin{cases} 5 & \text{if } n \equiv 0(mod\ 5), \\ 7 & \text{if } n = 7, \\ 8 & \text{if } n = 4, \\ 6 & \text{otherwise.} \end{cases} \qquad \text{for } n \geq 3.$$

PROOF. We prove the theorem by taking cases for various values of n.

Case 1. Suppose $n = 4$. Then we have the 4-antiprism graph Q_4 as in Fig. 6. Since each vertex is connected to every other vertex by a path of length 2, each vertex is to be colored by a different color in any open neighborhood coloring of Q_4 so that $\chi_{onc}(Q_4) = 8$.

Case 2. Suppose $n \geq 5$ with $n \equiv 0(\bmod 5)$. Then, by Lemma 3.5, $\chi_{onc}(Q_n) = 5$.

Case 3. Suppose $n = 7$, then we have the 7-antiprism graph Q_7 as in Fig. 7. As seen from the figure, in any open neighborhood coloring c, $c(v_0) \neq c(w)$ for any w with

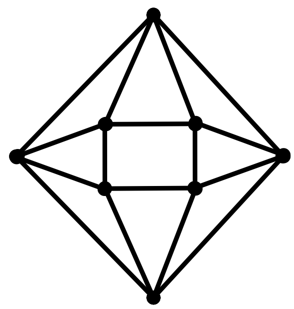

Figure 6: 4-antiprism graph Q_4

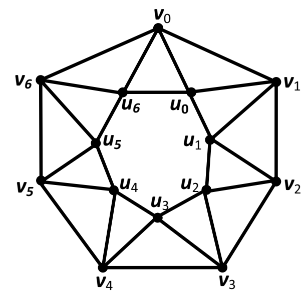

Figure 7: 7-antiprism graph Q_7

$w = u_i$, $i = 0, 1, 5, 6$ or $w = v_j$ with $j = 1, 2, 5, 6$. Further, at most one of the vertices u_2, u_3, u_4, v_3, v_4 can be given the same color as that of v_0. Thus, in general, not more than two vertices in Q_7 can be given the same color in any open neighborhood coloring of Q_7. This implies that $\chi_{onc}(Q_7) \geq 7$. To prove the reverse inequality, consider a coloring $c : V(Q_7) \to \{1, 2, 3, 4, 5, 6, 7\}$ as follows.

$$c(v) = \begin{cases} 1 & \text{if } v = v_0 \ or \ v = u_3, \\ 2 & \text{if } v = v_1 \ or \ v = u_4, \\ 3 & \text{if } v = v_2 \ or \ v = u_5, \\ 4 & \text{if } v = v_3 \ or \ v = u_6, \\ 5 & \text{if } v = v_4 \ or \ v = u_0, \\ 6 & \text{if } v = v_5 \ or \ v = u_1, \\ 7 & \text{otherwise.} \end{cases}$$

It is easy to verify that c is an open neighborhood 7-coloring of Q_7 so that $\chi_{onc}(Q_7) \leq 7$. Hence, $\chi_{onc}(Q_7) = 7$.

Case 4. Suppose n is any other integer, then we take up two subcases as follows.

Subcase 4-1. Suppose $n = 3$, we have the 3-antiprism graph Q_3 as in Fig. 2. Since each vertex is connected to every other vertex by a path of length 2, each vertex is to be colored by a different color in any open neighborhood coloring of Q_3 so that $\chi_{onc}(Q_3) = 6$.

Subcase 4-2. Suppose $n \geq 8$. Since $n \not\equiv 0 (\bmod 5)$, by Observation 3.1, $n = 3k + 5m$ for some integers $m \geq 0$ and $k \geq 1$. Also, $\chi_{onc}(Q_n) \geq 6$ by Lemma 1.1 and Lemma 3.5.

To prove the reverse inequality, consider a coloring $c : V(Q_n) \to \{1, 2, 3, 4, 5, 6\}$ as

$$c(v_i) = \begin{cases} 1, & \text{if } i \equiv 0 (\operatorname{mod} 3) \text{ and } 0 \leq i \leq 3k-1, \text{ or } i - 3k \equiv (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 2, & \text{if } i \equiv 1 (\operatorname{mod} 3) \text{ and } 0 \leq i \leq 3k-1, \text{ or } i - 3k \equiv 1 (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 3, & \text{if } i \equiv 2 (\operatorname{mod} 3) \text{ and } 0 \leq i \leq 3k-1, \text{ or } i - 3k \equiv 2 (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 4, & \text{if } i - 3k \equiv 3 (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 5, & \text{otherwise.} \end{cases}$$

and

$$c(u_i) = \begin{cases} 1, & \text{if } i - 3k \equiv 2 (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 2, & \text{if } i - 3k \equiv 3 (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 3, & \text{if } i - 3k \equiv 4 (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 4, & \text{if } i \equiv 0 (\operatorname{mod} 3) \text{ and } 0 \leq i \leq 3k-1, \text{ or } i - 3k \equiv 0 (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 5, & \text{if } i \equiv 1 (\operatorname{mod} 3) \text{ and } 0 \leq i \leq 3k-1, \text{ or } i - 3k \equiv 1 (\operatorname{mod} 5) \text{ and } 0 \leq 3k \leq 5m-1 \\ 6, & \text{otherwise.} \end{cases}$$

It can be easily seen that c is an open neighborhood coloring of Q_n so that $\chi_{onc}(Q_n) \leq 6$. Hence $\chi_{onc}(Q_n) = 6$.

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