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The Exponentiated Gamma Distribution Based On Ordered Random Variable*

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Abstract

In this paper, we gave some new explicit expressions and recurrence relations for marginal and joint moment generating functions of dual generalized order statistics from exponentiated gamma distribution. The results for order statistics and lower record values are deduced from the relation derived. Further, characterizing result of this distribution on using a recurrence relation for marginal moment generating functions dual generalized order statistics is discussed.

1 Introduction

The concept of generalized order statistics (*gos*) was introduced by Kamps [1] as a general framework for models of ordered random variables. Moreover, many other models of ordered random variables, such as, order statistics, k-th upper record values, upper record values, progressively Type II censoring order statistics, Pfeifer records and sequential order statistics are seen to be particular cases of *gos*. These models can be effectively applied, e.g., in reliability theory. However, random variables that are decreasingly ordered cannot be integrated into this framework. Consequently, this model is inappropriate to study, e.g. reversed ordered order statistic and lower record values models. Burkschat et al. [2] introduced the concept of dual generalized order statistics (*dgos*). The *dgos* models enable us to study decreasingly ordered random variables like reversed order statistics, lower k record values and lower Pfeirfer records, through a common approach below: Suppose $X_d(1,n,m,k),\dots,X_d(n,n,m,k)$, $(k\geq 1$, m is a real number), are n *dgos* from an absolutely continuous cumulative distribution function *cdf* $F(x)$ with probability density function *pdf* $f(x)$, if their joint *pdf* is of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^m f(x_i)\right)[F(x_n)]^{k-1}f(x_n), \tag{1}$$

for $F^{-1}(1)>x_1\geq x_2\geq\ldots\geq x_n>F^{-1}(0)$, where $\gamma_j=k+(n-j)(m+1)>0$ for all j, $1\leq j\leq n$, k is a positive integer and $m\geq -1$. If $m=0$ and $k=1$, then this model reduces to the $(n-r+1)$-th order statistic, from the sample $X_1,X_2,\dots,X_n$ and (1)

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will be the joint *pdf* of n order statistics. If $k = 1$ and $m = -1$, then (1) will be the joint *pdf* of the first n record values of the identically and independently distributed (iid) random variables with *cdf* $F(x)$ and corresponding *pdf* $f(x)$.

In view of (1), the marginal *pdf* of the r-th *dgos*, is given by

$$f_{X_d(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)). \tag{2}$$

The joint *pdf* of r-th and s-th *dgos*, is

$$\begin{aligned} & f_{X_d(r,n,m,k),X_d(s,n,m,k)}(x,y) \\ = & \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x)) \\ & \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y), \end{aligned} \tag{3}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad h_m(x) = \begin{cases} -\frac{1}{m+1} x^{m+1} & \text{for } m \neq -1, \\ -lnx & \text{for } m = -1, \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1),$$

for $x \in [0, 1)$.

Ahsanullah and Raqab [3], Raqab and Ahsanullah [4, 5] have established recurrence relations for moment generating functions (mgf) of record values from Pareto and Gumble, power function and extreme value distributions. Recurrence relations for marginal and joint mgf of *gos* from power function distribution, Erlang-truncated exponential distribution and extended type II generalized logistic distribution are derived by Saran and Singh [6], Kulshrestha et al. [7] and Kumar [8] respectively. Kumar [9, 10, 11] have established recurrence relations for marginal and joint mgf of *dgos* from generalized logistic, Marshall-Olkin extended logistic and type I generalized logistic distribution respectively. Al-Hussaini et al. [12, 13] have established recurrence relations for conditional and joint mgf of *gos* based on mixed population. Kumar [14] have established explicit expressions and some recurrence relations for mgf of record values from generalized logistic distribution. Recurrence relations for single and product moments of *dgos* from the inverse Weibull distribution are derived by Pawlas and Szynal [15]. Ahsanullah [16] and Mbah and Ahsanullah [17] characterized the uniform and power function distributions based on distributional properties of *dgos* respectively. Characterizations based on *gos* have been studied by some authors, Keseling [18] characterized some continuous distributions based on conditional distributions of *gos*. Bieniek and Szynal [19] characterized some distributions via linearity of regression of *gos*. Cramer et al. [20] gave a unifying approach on characterization via linear regression of ordered random variables.

Rest of the paper is organized as follows: In Section 2 exact expressions and recurrence relations for marginal and joint mgf of *dgos* from exponentiated Gamma distribution (EGD) are presented, while in Section 3, the exact expressions and recurrence relations for joint mgf for $dgos$ from EGD are discussed In Section 4, a characterization of EGD is obtained by using the recurrence relation for marginal mgf of $dgos$. Some final comments in Section 5 conclude the paper.

Gupta et al. [21] introduced the EGD. This model is flexible enough to accommodate both monotonic as well as nonmonotonic failure rates. The *cdf* and *pdf* of EGD are given, respectively by

$$F(x) = [1 - e^{-x}(x+1)]^{\alpha}, x > 0, \quad \alpha > 0, \tag{4}$$

$$f(x) = \alpha x e^{-x}[1 - e^{-x}(x+1)]^{\alpha-1}, x > 0, \quad \alpha > 0. \tag{5}$$

Note that for FGD,

$$\alpha x F(x) = [e^{-x} - (x+1)] f(x). \tag{6}$$

For $\alpha = 1$, the above distribution corresponds to the gamma distribution $G(1,2)$.

2 Relations for Marginal Moments Generating Functions

In this Section the exact expressions and recurrence relations for marginal mgf of $dgos$ from EGD are considered. For the EGD when $m \neq -1$,

$$M_{X_d(r,n,m,k)}(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \frac{C_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} e^{tx} [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx. \tag{7}$$

On using (4) and (5) in (6) and simplification of the resulting equation we get

$$\begin{aligned} M_{X_d(r,n,m,k)}(t) &= \frac{\alpha C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1} \sum_{p=0}^{\infty} \sum_{q=0}^{p} (-1)^{u+p} \binom{r-1}{u} \\ &\times \binom{\alpha\gamma_{r-u} - 1}{p} \binom{p}{q} \frac{\Gamma(q+2)}{(p+1-t)^{q+2}}, \end{aligned} \tag{8}$$

and for $m = -1$

$$\begin{aligned} M_{X_d(r,n,-1,k)}(t) &= \frac{(\alpha k)^r}{(r-1)!} \sum_{p=0}^{\infty} \sum_{v=0}^{\infty} \sum_{w=0}^{r-1+v+p} (-1)^v \phi_p(r-1) \binom{\alpha k - 1}{v} \\ &\times \binom{r-1+v+p}{w} \frac{\Gamma(w+2)}{(r+v+p-t)^{w+2}}, \end{aligned} \tag{9}$$

where $\phi_p(r-1)$ is the coefficient of $e^{-(r-1+p)x}(x+1)^{r-1+p}$ in the expansion of

$$\left(\sum_{p=1}^{\infty} \frac{e^{-px}(x+1)^p}{p} \right)^{r-1},$$

see Balakrishnan and Cohan [22].

Differentiating both sides of (8) and (9) with respect to t, j times we get

$$M^{(j)}_{X_d(r,n,m,k)}(t) = \frac{\alpha C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1}\sum_{p=0}^{\infty}\sum_{q=0}^{p}(-1)^{u+p}\binom{r-1}{u} \times \binom{\alpha\gamma_{r-u}-1}{p}\binom{p}{q}\frac{\Gamma(j+q+2)}{(p+1-t)^{j+q+2}} \tag{10}$$

and

$$M^{(j)}_{X_d(r,n,-1,k)}(t) = \frac{(\alpha k)^r}{(r-1)!}\sum_{p=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{r-1+v+p}(-1)^v\phi_p(r-1)\binom{\alpha k-1}{v} \times \binom{r-1+v+p}{w}\frac{\Gamma(j+w+2)}{(r+v+p-t)^{j+w+2}}. \tag{11}$$

If α is a positive integer, the relations (10) and (11) then give

$$M^{(j)}_{X_d(r,n,m,k)}(t) = \frac{\alpha C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1}\sum_{p=0}^{\alpha\gamma_{r-u}-1}\sum_{q=0}^{p}(-1)^{u+p}\binom{r-1}{u} \times \binom{\alpha\gamma_{r-u}-1}{p}\binom{p}{q}\frac{\Gamma(j+q+2)}{(p+1-t)^{j+q+2}} \tag{12}$$

and

$$M^{(j)}_{X_d(r,n,-1,k)}(t) = \frac{(\alpha k)^r}{(r-1)!}\sum_{p=0}^{\infty}\sum_{v=0}^{\alpha k-1}\sum_{w=0}^{r-1+v+p}(-1)^v\phi_p(r-1)\binom{\alpha k-1}{v} \times \binom{r-1+v+P}{w}\frac{\Gamma(j+w+2)}{(r+v+p-t)^{j+w+2}}. \tag{13}$$

By differentiating both sides of equation (12) and (13) with respect to t and then setting $t=0$, we obtain the explicit expression for single moments of $dgos$ and k record values from EGD in the form

$$E[X_d^j(r,n,m,k)] = \frac{\alpha C_{r-1}}{(r-1)!(m+1)^{r-1}} \sum_{u=0}^{r-1}\sum_{p=0}^{\alpha\gamma_{r-u}-1}\sum_{q=0}^{p}(-1)^{u+p}\binom{r-1}{u} \times \binom{\alpha\gamma_{r-u}-1}{p}\binom{p}{q}\frac{\Gamma(j+q+2)}{(p+1)^{j+q+2}} \tag{14}$$

and

$$E[X_d^j(r,n,-1,k)] = \frac{(\alpha k)^r}{(r-1)!}\sum_{p=0}^{\infty}\sum_{v=0}^{\alpha k-1}\sum_{w=0}^{r-1+v+p}(-1)^v\phi_p(r-1)\binom{\alpha k-1}{v} \times \binom{r-1+v+p}{w}\frac{\Gamma(j+w+2)}{(r+v+p)^{j+w+2}}. \tag{15}$$

Special Cases:

(i) Putting $m = 0$, $k = 1$ in (12) and (14), relations for order statistics can be obtained as

$$
\begin{aligned}
M^{(j)}_{X_{r:n}}(t) &= \alpha C_{r:n} \sum_{u=0}^{n-r} \sum_{p=0}^{\alpha(r+u)-1} \sum_{q=0}^{p} (-1)^{u+p} \binom{n-r}{u} \\
&\quad \times \binom{\alpha(r+u)-1}{p} \binom{p}{q} \frac{\Gamma(j+q+2)}{(p+1-t)^{j+q+2}}
\end{aligned}
$$

and

$$
\begin{aligned}
E[X^{j}_{r;n}] &= \alpha C_{r:n} \sum_{u=0}^{n-r} \sum_{p=0}^{\alpha(r+u)-1} \sum_{q=0}^{p} (-1)^{u+p} \binom{n-r}{u} \\
&\quad \times \binom{\alpha(r+u)-1}{p} \binom{p}{q} \frac{\Gamma(j+q+2)}{(p+1)^{j+q+2}},
\end{aligned}
$$

where

$$
C_{r:n} = \frac{n!}{(r-1)!(n-r)!}.
$$

(ii) Putting $k = 1$ in (13) and (15), relations for record values can be obtained as

$$
\begin{aligned}
M^{(j)}_{X_{L(r)}}(t) &= \frac{\alpha^r}{(r-1)!} \sum_{p=0}^{\infty} \sum_{v=0}^{\alpha-1} \sum_{w=0}^{r-1+v+p} (-1)^v \phi_p(r-1) \binom{\alpha-1}{v} \\
&\quad \times \binom{r-1+v+p}{w} \frac{\Gamma(j+w+2)}{(r+v+p-t)^{j+w+2}}
\end{aligned}
$$

and

$$
\begin{aligned}
E[X^{j}_{L(r)}] &= \frac{\alpha^r}{(r-1)!} \sum_{p=0}^{\infty} \sum_{v=0}^{\alpha-1} \sum_{w=0}^{r-1+v+p} (-1)^v \phi_p(r-1) \binom{\alpha-1}{v} \\
&\quad \times \binom{r-1+v+P}{w} \frac{\Gamma(j+w+2)}{(r+v+p)^{j+w+2}}.
\end{aligned}
$$

A recurrence relation for mgf of *dgos* from *cdf* (4) can be obtained in the following theorem.

THEOREM 1. For $2 \leq r \leq n$ $n \geq 2$ and $k = 1, 2, \ldots,$

$$
\begin{aligned}
&\left(1 - \frac{t}{\alpha\gamma_r}\right) M^{(j)}_{X_d(r,n,m,k)}(t) \\
&= M^{(j)}_{X_d(r-1,n,m,k)}(t) + \frac{j}{\alpha\gamma_r} M^{(j-1)}_{X_d(r,n,m,k)}(t) \\
&\quad - \frac{1}{\alpha\gamma_r}\Big\{ tE[\phi(X_d(r,n,m,k))] + E[\psi(X_d(r,n,m,k))] \Big\}, \qquad (16)
\end{aligned}
$$

where
$$\phi(x) = x^{j-1}\left(e^{(t+1)x} - e^{tx}\right) \text{ and } \psi(x) = x^{j-2}\left(e^{(t+1)x} - e^{tx}\right).$$

PROOF. Integrating by parts of (7) and using (6), we get

$$\begin{aligned} & M_{X_d(r,n,m,k)}(t) \\ = \; & M_{X_d(r-1,n,m,k)}(t) + \frac{j}{\alpha\gamma_r}\left\{M_{X_d(r,n,m,k)}(t) - E[h(X_d(r,n,m,k))]\right\}, \end{aligned} \quad (17)$$

where
$$h(x) = \left(\frac{e^{(t+1)x}}{x} - \frac{e^{tx}}{x}\right).$$

Differentiating both the sides of (17) j times with respect to t, we get the result given in (16). By differentiating both sides of equation (17) with respect to t and then setting $t = 0$, we obtain the recurrence relations for moments of *dgos* from *EGD* in the form

$$\begin{aligned} E[X_d^j(r,n,m,k)] &= E[X_d^j(r-1,n,m,k)] + \frac{j}{\alpha\gamma_r}E[X_d^{j-1}(r,n,m,k)] \\ &\quad + \frac{j}{\alpha\gamma_r}\left\{E[X_d^{j-2}(r,n,m,k)] - E[\xi(X_d(r,n,m,k))]\right\}, \end{aligned} \quad (18)$$

where $\xi(x) = x^{j-2}e^x$.

REMARK 2.1. Putting $m = 0$, $k = 1$ in (16) and (18), relations for order statistics can be obtained as

$$\begin{aligned} M_{X_{r:n}}^{(j)}(t) &= \left(1 - \frac{t}{\alpha(r-1)}\right)M_{X_{r-1:n}}^{(j)}(t) - \frac{j}{\alpha(r-1)}M_{X_{r-1:n}}^{(j-1)}(t) \\ &\quad + \frac{1}{\alpha(r-1)}\left\{tE[\phi(X_{r-1:n})] + jE[\psi(X_{r-1:n})]\right\} \end{aligned}$$

and

$$E(X_{r:n}^j) = E(X_{r-1:n}^j) - \frac{j}{\alpha(r-1)}\left\{E(X_{r-1:n}^{j-1}) + E(X_{r-1:n}^{j-2}) - E(\xi(X_{r-1;n}))\right\}.$$

REMARK 2.2. Putting $k = -1$ in (16) and (18), relations for k record values can be obtained as

$$\begin{aligned} \left(1 - \frac{t}{\alpha k}\right)M_{X_{L(r)}}^{(j)}(t) &= M_{X_{L(r-1)}}^{(j)}(t) + \frac{j}{\alpha k}M_{X_{L(r)}}^{(j-1)}(t) \\ &\quad - \frac{1}{\alpha k}\left\{tE[\phi(X_{L(r)})] + jE[\psi(X_{L(r)})]\right\} \end{aligned}$$

and
$$E[X_{L(r)}^j] = E[X_{L(r-1)}^j] + \frac{j}{\alpha k}\left\{E[X_{L(r)}^{j-1}] + E[X_{L(r)}^{j-2}] - E[\xi(X_{L(r)})]\right\}.$$

3 Relations for Joint Moment Generating Functions

In this Section exact moments and recurrence relations for joint *mgf* of *dgos* from *EGD* are considered. For the *EGD* when $m \neq -1$,

$$\begin{aligned}
& M_{X_d(r,n,m,k),X_d(s,n,m,k)}(t_1,t_2) \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{t_1x+t_2y} f_{X_d(r,n,m,k)X_d(s,n,m,k)}(x,y)dxdy. \\
= & \frac{C_{s-1}}{(r-1)!(s-r-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{x} e^{t_1x+t_2y}[F(x)]^m f(x) g_m^{r-1}(F(x)) \\
& \times[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dydx. \qquad (19)
\end{aligned}$$

On using (4) and (5) in (19) and simplification of the resulting equation we get

$$\begin{aligned}
& M_{X_d(r,n,m,k),X_d(s,n,m,k)}(t_1,t_2) \\
= & \frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{a=0}^{\infty}\sum_{b=0}^{a}\sum_{u=0}^{r-1} \\
& \times\sum_{v=0}^{s-r-1}\sum_{l=0}^{\infty}\sum_{w=0}^{l}\sum_{p=0}^{w+1}(-1)^{a+l+u+v}\binom{\alpha\gamma_{s-v}-1}{a}\binom{a}{b}\binom{r-1}{u} \\
& \times\binom{s-r-1}{v}\binom{\alpha(s-r-v+u)(m+1)-1}{l}\binom{l}{w} \\
& \times\frac{\Gamma(w+2)\Gamma(p+a+2)}{p!(l+1-t_2)^{w+2-p}(a+l+2-t_1-t_2)^{p+a+2}}. \qquad (20)
\end{aligned}$$

For $m=-1$,

$$\begin{aligned}
& M_{X_d(r,n,-1,k),X_d(s,n,-1,k)}(t_1,t_2) \\
= & \frac{(\alpha k)^s}{(r-1)!(s-r-1)!}\sum_{a=0}^{s-r-1}\sum_{b=0}^{\infty}\sum_{c=0}^{s+b+p-a-2} \\
& \times\sum_{d=0}^{c+1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{a+q+v}(-1)^{s-r-1+a+v}\phi_p(s-a-2)\phi_q(a) \\
& \times\binom{s-r-1}{a}\binom{s+b+p-a-2}{c}\binom{\alpha k-1}{v}\binom{a+q+v}{w} \\
& \times\frac{\Gamma(c+2)\Gamma(w+d+2)}{d!(s+b+p-1-t_2)^{c-d+2}(s+b+p+q+v-t_1-t_2)^{w+d+2}}. \qquad (21)
\end{aligned}$$

Differentiating both side of (20) and (21) i times with respect to t_1 and then j times with respect to t_2, we get

$$
\begin{aligned}
& M^{(i,j)}_{X_d(r,n,m,k),X_d(s,n,m,k)}(t_1,t_2) \\
= & \frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{a=0}^{\infty}\sum_{b=0}^{a}\sum_{u=0}^{r-1} \\
& \times \sum_{v=0}^{s-r-1}\sum_{l=0}^{\infty}\sum_{w=0}^{l}\sum_{p=0}^{i+w+1} (-1)^{a+l+u+v} \binom{\alpha\gamma_{s-v}-1}{a}\binom{a}{b}\binom{r-1}{u} \\
& \times \binom{s-r-1}{v}\binom{\alpha(s-r-v+u)(m+1)-1}{l}\binom{l}{w} \\
& \times \frac{\Gamma(i+w+2)\Gamma(j+p+a+2)}{p!(l+1-t_2)^{i+w+2-p}(a+l+2-t_1-t_2)^{j+p+a+2}}
\end{aligned} \tag{22}
$$

and

$$
\begin{aligned}
& M^{(i,j)}_{X_d(r,n,-1,k),X_d(s,n,-1,k)}(t_1,t_2) \\
= & \frac{(\alpha k)^s}{(r-1)!(s-r-1)!} \sum_{a=0}^{s-r-1}\sum_{b=0}^{\infty}\sum_{c=0}^{s+b+p-a-2} \\
& \times \sum_{d=0}^{i+c+1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{\infty}\sum_{w=0}^{a+q+v} (-1)^{s-r-1+a+v}\phi_p(s-a-2)\phi_q(a) \\
& \times \binom{s-r-1}{a}\binom{s+b+p-a-2}{c}\binom{\alpha k-1}{v}\binom{a+q+v}{w} \\
& \times \frac{\Gamma(i+c+2)\Gamma(j+w+d+2)}{d!(s+b+p-1-t_2)^{i+c-d+2}(s+b+p+q+v-t_1-t_2)^{j+w+d+2}}.
\end{aligned} \tag{23}
$$

If α is a positive integer, the relations (22) and (23) then give

$$
\begin{aligned}
& M^{(i,j)}_{X_d(r,n,m,k),X_d(s,n,m,k)}(t_1,t_2) \\
= & \frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{a=0}^{\alpha\gamma_{s-v}-1}\sum_{b=0}^{a}\sum_{u=0}^{r-1} \\
& \times \sum_{v=0}^{s-r-1}\sum_{l=0}^{\alpha(s-r-v+u)(m+1)-1}\sum_{w=0}^{l}\sum_{p=0}^{i+w+1} (-1)^{a+l+u+v} \binom{\alpha\gamma_{s-v}-1}{a}\binom{a}{b} \\
& \times \binom{r-1}{u}\binom{s-r-1}{v}\binom{\alpha(s-r-v+u)(m+1)-1}{l}\binom{l}{w} \\
& \times \frac{\Gamma(i+w+2)\Gamma(j+p+a+2)}{p!(l+1-t_2)^{i+w+2-p}(a+l+2-t_1-t_2)^{j+p+a+2}}
\end{aligned} \tag{24}
$$

and

$$
\begin{aligned}
& M^{(i,j)}_{X_d(r,n,-1,k),X_d(s,n,-1,k)}(t_1,t_2) \\
= & \frac{(\alpha k)^s}{(r-1)!(s-r-1)!}\sum_{a=0}^{s-r-1}\sum_{b=0}^{\infty}\sum_{c=0}^{s+b+p-a-2} \\
& \times \sum_{d=0}^{i+c+1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{\alpha k-1}\sum_{w=0}^{a+q+v}(-1)^{s-r-1+a+v}\phi_p(s-a-2)\phi_q(a) \\
& \times \binom{s-r-1}{a}\binom{s+b+p-a-2}{c}\binom{\alpha k-1}{v}\binom{a+q+v}{w} \\
& \times \frac{\Gamma(i+c+2)\Gamma(j+w+d+2)}{d!(s+b+p-1-t_2)^{i+c-d+2}(s+b+p+q+v-t_1-t_2)^{j+w+d+2}}. \qquad (25)
\end{aligned}
$$

By differentiating both sides of equation (24) and (25) with respect to t_1, t_2 and then setting $t_1 = t_2 = 0$, we obtain the explicit expression for product moments of *dgos* and k record values from *EGD* in the form

$$
\begin{aligned}
& E[X^i_d(r,n,m,k), X^j_d(s,n,m,k)] \\
= & \frac{\alpha^2 C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{a=0}^{\alpha\gamma_{s-v}-1}\sum_{b=0}^{a}\sum_{u=0}^{r-1} \\
& \times \sum_{v=0}^{s-r-1}\sum_{l=0}^{\alpha(s-r-v+u)(m+1)-1}\sum_{w=0}^{l}\sum_{p=0}^{i+w+1}(-1)^{a+l+u+v}\binom{\alpha\gamma_{s-v}-1}{a}\binom{a}{b} \\
& \times \binom{r-1}{u}\binom{s-r-1}{v}\binom{\alpha(s-r-v+u)(m+1)-1}{l}\binom{l}{w} \\
& \times \frac{\Gamma(i+w+2)\Gamma(j+p+a+2)}{p!(l+1)^{i+w+2-p}(a+l+2)^{j+p+a+2}} \qquad (26)
\end{aligned}
$$

and

$$
\begin{aligned}
& E[X^i_d(r,n,-1,k), X^j_d(s,n,-1,k)] \\
= & \frac{(\alpha k)^s}{(r-1)!(s-r-1)!}\sum_{a=0}^{s-r-1}\sum_{b=0}^{\infty}\sum_{c=0}^{s+b+p-a-2} \\
& \times \sum_{d=0}^{i+c+1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{v=0}^{\alpha k-1}\sum_{w=0}^{a+q+v}(-1)^{s-r-1+a+v}\phi_p(s-a-2)\phi_q(a) \\
& \times \binom{s-r-1}{a}\binom{s+b+p-a-2}{c}\binom{\alpha k-1}{v}\binom{a+q+v}{w} \\
& \times \frac{\Gamma(i+c+2)\Gamma(j+w+d+2)}{d!(s+b+p-1)^{i+c-d+2}(s+b+p+q+v)^{j+w+d+2}}. \qquad (27)
\end{aligned}
$$

Special Cases:

(i) Putting $m = 0$, $k = 1$ in (24) and (26), relations for order statistics can be obtained as

$$\begin{aligned}
& M^{(i,j)}_{X_{r:n},X_{s:n}}(t_1,t_2) \\
= \; & \alpha^2 C_{r,s:n} \sum_{a=0}^{\alpha(r+v)-1} \sum_{b=0}^{a} \sum_{u=0}^{n-s} \sum_{v=0}^{s-r-1} \sum_{l=0}^{\alpha(s-r-v+u)-1} \\
& \times \sum_{w=0}^{l} \sum_{p=0}^{i+w+1} (-1)^{a+l+u+v} \binom{\alpha(r+v)-1}{a} \binom{a}{b} \\
& \times \binom{n-s}{u} \binom{s-r-1}{v} \binom{\alpha(s-r-v+u)-1}{l} \binom{l}{w} \\
& \times \frac{\Gamma(i+w+2)\Gamma(j+p+a+2)}{p!(l+1-t_2)^{i+w+2-p}(a+l+2-t_1-t_2)^{j+p+a+2}}
\end{aligned}$$

and

$$\begin{aligned}
E[X^i_{r:n}, X^j_{s:n}] \;=\; & \alpha^2 C_{r,s:n} \sum_{a=0}^{\alpha(r+v)-1} \sum_{b=0}^{a} \sum_{v=0}^{s-r-1} \sum_{l=0}^{\alpha(s-r-v+u)-1} \sum_{p=0}^{i+w+1} \sum_{u=0}^{n-s} \\
& \times \sum_{w=0}^{l} (-1)^{a+l+u+v} \binom{\alpha(r+v)-1}{a} \binom{a}{b} \binom{n-s}{u} \\
& \times \binom{s-r-1}{v} \binom{\alpha(s-r-v+u)-1}{l} \binom{l}{w} \\
& \times \frac{\Gamma(i+w+2)\Gamma(j+p+a+2)}{p!(l+1)^{i+w+2-p}(a+l+2)^{j+p+a+2}},
\end{aligned}$$

where

$$C_{r,s;n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

(ii) Putting $k = 1$ in (25) and (27), relations for record values in the form

$$\begin{aligned}
& M^{(i,j)}_{X_{L(r)},X_{L(s)}}(t_1,t_2) \\
= \; & \frac{\alpha^s}{(r-1)!(s-r-1)!} \sum_{a=0}^{s-r-1} \sum_{b=0}^{\infty} \sum_{c=0}^{s+b+p-a-2} \\
& \times \sum_{d=0}^{i+c+1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{v=0}^{\alpha-1} \sum_{w=0}^{a+q+v} (-1)^{s-r-1+a+v} \phi_p(s-a-2)\phi_q(a) \\
& \times \binom{s-r-1}{a} \binom{s+b+p-a-2}{c} \binom{\alpha-1}{v} \binom{a+q+v}{w} \\
& \times \frac{\Gamma(i+c+2)\Gamma(j+w+d+2)}{d!(s+b+p-1-t_2)^{i+c-d+2}(s+b+p+q+v-t_1-t_2)^{j+w+d+2}}
\end{aligned}$$

and

$$
\begin{aligned}
& E[X_{L(r)}^i, X_{L(s)}^j] \\
= & \frac{\alpha^s}{(r-1)!(s-r-1)!} \sum_{a=0}^{s-r-1} \sum_{b=0}^{\infty} \sum_{c=0}^{s+b+p-a-2} \\
& \times \sum_{d=0}^{i+c+1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{v=0}^{\alpha-1} \sum_{w=0}^{a+q+v} (-1)^{s-r-1+a+v} \phi_p(s-a-2)\phi_q(a) \\
& \times \binom{s-r-1}{a} \binom{s+b+p-a-2}{c} \binom{\alpha-1}{v} \binom{a+q+v}{w} \\
& \times \frac{\Gamma(i+c+2)\Gamma(j+w+d+2)}{d!(s+b+p-1)^{i+c-d+2}(s+b+p+q+v)^{j+w+d+2}}.
\end{aligned}
$$

Making use of (6), we can derive recurrence relations for joint mgf of $dgos$.

THEOREM 2. For $1 \leq r < s \leq n$ $n \geq 2$ and $k = 1, 2, \ldots,$

$$
\begin{aligned}
& \left(1 - \frac{t_2}{\alpha\gamma_s}\right) M_{X_d(r,n,m,k)X_d(s,n,m,k)}^{(i,j)}(t_1, t_2) \\
= & M_{X_d(r,n,m,k)X_d(s-1,n,m,k)}^{(i,j)}(t_1, t_2) + \frac{j}{\alpha\gamma_s} M_{X_d(r,n,m,k)X_d(s,n,m,k)}^{(i,j-1)}(t_1, t_2) \\
& - \frac{1}{\alpha\gamma_s}\Big\{ t_2 E[\phi(X_d(r,n,m,k)X_d(s,n,m,k))] \\
& + jE[\psi(X_d(r,n,m,k)X_d(s,n,m,k))] \Big\}, \qquad (28)
\end{aligned}
$$

where

$$\phi(x,y) = x^i y^{j-1}\left(e^{t_1 x + (t_2+1)y} - e^{t_1 x + t_2 y}\right)$$

and

$$\psi(x) = x^i y^{j-2}\left(e^{t_1 x + (t_2+1)y} - e^{t_1 x + t_2 y}\right).$$

PROOF. Integrating by parts of (19) and using (6), we get

$$
\begin{aligned}
& M_{X_d(r,n,m,k)X_d(s,n,m,k)}(t_1, t_2) \\
= & M_{X_d(r,n,m,k)X_d(s-1,n,m,k)}(t_1, t_2) + \frac{t_2}{\alpha\gamma_s}\Big\{ M_{X_d(r,n,m,k)X_d(s,n,m,k)}(t_1, t_2) \\
& - E[h(X_d(r,n,m,k)X_d(s,n,m,k))] \Big\} \qquad (29)
\end{aligned}
$$

and

$$h(x,y) = \left(\frac{e^{t_1 x + (t_2+1)y}}{y} - \frac{e^{t_1 x + t_2 y}}{y}\right).$$

Differentiating both the sides of above equation i times with respect to t_1 and then j times with respect to t_2 and simplifying the resulting expression, we get the result given in (28). By differentiating both sides of equation (28) with respect to t_1, t_2 and then setting $t_1 = t_2 = 0$, we obtain the recurrence relations for product moments of *dgos* from *EGD* in the form

$$\begin{aligned} & E[X_d^i(r,n,m,k)X_d^j(s,n,m,k)] \\ = \; & E[X_d^i(r,n,m,k)X_d^j(s-1,n,m,k)] \\ & + \frac{j}{\alpha\gamma_s}\Big\{E[X_d^i(r,n,m,k)X_d^{j-1}(s,n,m,k)] + E[X_d^i(r,n,m,k)X_d^{j-2}(s,n,m,k)]\Big\} \\ & - \frac{j}{\alpha\gamma_s}E[\xi(X_d(r,n,m,k)(X_d(s,n,m,k)], \end{aligned} \tag{30}$$

where

$$\xi(x,y) = x^i y^{j-2} e^y.$$

REMARK 3.1. Putting $m = 0$, $k = 1$ in (28) and (30), we obtain relations for order statistics

$$\begin{aligned} & M^{(i,j)}_{X_{r:n}X_{s:n}}(t_1,t_2) \\ = \; & \left(1 - \frac{t_{t_1}}{\alpha(r-1)}\right) M^{(i,j)}_{X_{r-1:n}X_{s:n}}(t_1,t_2) - \frac{i}{\alpha(r-1)} M^{(i,j)}_{X_{r-1:n}X_{s:n}}(t_1,t_2) \\ & + \frac{1}{\alpha(r-1)}\Big\{t_1 E[\phi(X_{r-1:n}X_{s:n})] + jE[\psi(X_{r-1:n}X_{s:n})]\Big\} \end{aligned}$$

and

$$\begin{aligned} E(X^i_{r:n}X^j_{s:n}) \;=\; & E(X^i_{r-1:n}X^j_{s:n}) - \frac{i}{\alpha(r-1)}\Big\{E(X^{i-1}_{r-1:n}X^j_{s:n}) + E(X^{i-2}_{r-1:n}X^j_{s:n}) \\ & - E(\phi(X_{r-1;n}X_{s;n}))\Big\}. \end{aligned}$$

REMARK 3.2. Putting $m = -1$ and $k \geq 1$in (28) and (30), we obtain relations for k record values in the form

$$\begin{aligned} & \left(1 - \frac{t_2}{\alpha\gamma_s}\right) M^{(i,j)}_{X_d(r,n,-1,k)X_d(s,n,-1,k)}(t_1,t_2) \\ = \; & M^{(i,j)}_{X_d(r,n,-1,k)X_d(s-1,n,-1,k)}(t_1,t_2) + \frac{j}{\alpha\gamma_s} M^{(i,j-1)}_{X_d(r,n,-1,k)X_d(s,n,-1,k)}(t_1,t_2) \\ & - \frac{1}{\alpha\gamma_s}\Big\{t_2 E[\phi(X_d(r,n,-1,k)X_d(s,n,-1,k))] \\ & + jE[\psi(X_d(r,n,-1,k)X_d(s,n,-1,k))]\Big\} \end{aligned}$$

and

$$\begin{aligned}
& E[X_d^i(r,n,-1,k)X_d^j(s,n,-1,k)] \\
= \; & E[X_d^i(r,n,-1,k)X_d^j(s-1,n,-1,k)] \\
& + \frac{j}{\alpha\gamma_s}\left\{E[X_d^i(r,n,-1,k)X_d^{j-1}(s,n,-1,k)] + E[X_d^i(r,n,-1,k)X_d^{j-2}(s,n,-1,k)]\right\} \\
& - \frac{j}{\alpha\gamma_s}E[\xi(X_d(r,n,-1,k)X_d(s,n,-1,k))].
\end{aligned}$$

4 Characterization

This Section contains characterization of EGD by using the recurrence relation for mgf of $dgos$. Let $L(a,\ b)$ stand for the space of all integrable functions on $(a,\ b)$. A sequence $(f_n) \subset L(a,\ b)$ is called complete on $L(a,\ b)$ if for all functions $g \in L(a,\ b)$ the condition

$$\int_a^b g(x)f_n(x)dx = 0, \qquad n \in N,$$

implies $g(x) = 0$ a.e. on $(a,\ b)$. We start with the following result of Lin [23].

PROPOSITION 1**.** Let n_0 be any fixed non-negative integer, $-\infty \leq a < b \leq \infty$ and $g(x) \geq 0$ an absolutely continuous function with $g'(x) \neq 0$ a.e. on $(a,\ b)$. Then the sequence of functions $\{(g(x))^n e^{-g(x)}, \quad n \geq n_0\}$ is complete in $L(a,\ b)$ iff $g(x)$ is strictly monotone on $(a,\ b)$.

Using the above Proposition we get a stronger version of Theorem 1.

THEOREM 3. A necessary and sufficient conditions for a random variable X to be distributed with *pdf* given by (5) is that

$$\begin{aligned}
& M^{(j)}_{X_d(r,n,m,k)}(t) \\
= \; & M^{(j)}_{X_d(r-1,n,m,k)}(t) + \frac{j}{\alpha\gamma_r}\left\{M^{(j-1)}_{X_d(r,n,m,k)}(t) - E[h(X_d(r,n,m,k))]\right\}. \qquad (31)
\end{aligned}$$

PROOF. The necessary part follows immediately from equation (17). On the other hand if the recurrence relation in equation (31) is satisfied, then on using equation (2), we have

$$\begin{aligned}
& \frac{C_{r-1}}{(r-1)!}\int_0^\infty e^{tx}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
= \; & \frac{C_{r-1}}{\gamma_r(r-2)!}\int_0^\infty e^{tx}[F(x)]^{\gamma_r+m}f(x)g_m^{r-2}(F(x))dx \\
& - \frac{tC_{r-1}}{\alpha\gamma_r(r-1)!}\int_0^\infty \frac{e^{(t+1)x}}{x}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx \\
& + \frac{tC_{r-1}}{\alpha\gamma_r(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx
\end{aligned}$$

and

$$\frac{tC_{r-1}}{\alpha\gamma_r(r-1)!}\int_0^{\infty 1}\frac{e^{tx}}{x}[F(x)]^{\gamma_r-1}f(x)g_m^{r-1}(F(x))dx. \tag{32}$$

Integrating the first integral on the right-hand side of the above equation by parts and simplifying the resulting expression, we get

$$\frac{tC_{r-1}}{(r-1)!}\int_0^{\infty} e^{tx}[F(x)]^{\gamma_r-1}g_m^{r-1}(F(x))\left\{F(x)-\frac{e^{tx}-(x+1)}{\alpha x}f(x)\right\}dx=0. \tag{33}$$

It now follows from Proposition 1, that

$$\alpha x F(x) = [e^{tx}-(x+1)]f(x)$$

which proves that $f(x)$ has the form (4).

5 Concluding Remarks

(i) In this paper, we proposed new explicit expressions and recurrence relations for marginal and joint moment generating functions of *dgos* from *EGD*. Further, characterization of this distribution has also been obtained on using recurrence relation for marginal moment generating functions of *dgos*. Special cases are also deduced.

(ii) The recurrence relations for moments of ordered random variables are important because they reduce the amount of direct computations for moments, evaluate the higher moments in terms of the lower moments and they can be used to characterize distributions.

(iii) The recurrence relations of higher joint moments enable us to derive single, product, triple and quadruple moments which can be used in Edgeworth approximate inference.

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