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Some Properties Of $(p, q; r)$-Convex Sequences*

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Abstract

In this paper we introduce an essential class of real sequences named as $(p, q; r)$-convex sequences. Employing this class we generalize two different results proved previously by others.

1 Introduction

Let $(a_n)_{n=1}^{\infty}$ be a real sequence and let the difference of order k of the sequence $(a_n)_{n=1}^{\infty}$ be defined by

$$\triangle^0 a_n = a_n, \quad \triangle^k a_n = \triangle^{k-1} a_{n+1} - \triangle^{k-1} a_n, \quad n = 1, 2, \dots,$$

and throughout the paper we shall write $\triangle a_n$ instead of $\triangle^1 a_n$.

Next definition introduces the well-known notion of a convex sequence of order k, $(k = 1, 2, \dots)$.

DEFINITION 1. A sequence $(a_n)_{n=1}^{\infty}$ is said to be convex of order k if $\triangle^k a_n \geq 0$ for all n. In particular, a convex sequence of order $k = 2$ is said to be convex.

In 1965 N. Ozeki [1] (see also [2], page 202) has proved the following theorem, relevant to convex sequences.

THEOREM 1. Let $(a_n)_{n=1}^{\infty}$ be a real sequence and let the sequences be defined by

$$A_n = \frac{1}{n} \sum_{k=1}^{n} a_k, \quad B_n = \triangle^2 A_n, \quad (n = 1, 2, \dots). \tag{1}$$

If the sequence $(a_n)_{n=1}^{\infty}$ is convex, then:

(i) $B_n \geq \frac{n-1}{n+2} B_{n-1}$ for $n = 2, 3, \dots$,

(ii) Sequence $(A_n)_{n=1}^{\infty}$ is convex, i.e. $\triangle^2 (A_n) \geq 0$ for all $n = 1, 2, \dots$.

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A natural question was raised whether assertion (ii) of THEOREM 1 could be extended to the convex sequences of order $k \geq 3$. A correct and elegant answer of this question was given in [3] as below.

THEOREM 2. Let $(a_n)_{n=1}^{\infty}$ be a positive sequence. Then the k-th order convexity of the sequence $(a_n)_{n=1}^{\infty}$, implies the k-th order convexity the sequence $(A_n)_{n=1}^{\infty}$, where A_n is defined by (1).

Various generalizations of convexity were studied by many authors. In [4] a sequence $(a_n)_{n=1}^{\infty}$ is said to be p-convex for a positive real number p if $L_p(a_n) \geq 0$ for all $n = 1, 2, \ldots$, where the difference operator L_p is defined by

$$L_p(a_n) = a_{n+2} - (1 + p)a_{n+1} + pa_n.$$

Another generalization uses the operator

$$L_{p,q}(a_n) = a_n - (p + q)a_{n+1} + pqa_{n+2},$$

where

$$L_p(a_n) = a_n - pa_{n+1} \quad \text{and} \quad L_{p,q}(a_n) = L_p(a_n) - qL_p(a_{n+1})$$

with $p, q \in \mathbb{R}$, $0 < p < 1$, $0 < q < 1$, see [5]. In the same paper are given the following definitions:

DEFINITION 2. A sequence $(a_n)_{n=1}^{\infty}$ is called $p-$monotone if $L_p(a_n) \geq 0$ for every n. A sequence $(a_n)_{n=1}^{\infty}$ is called $(p, q)-$convex sequence if $L_{p,q}(a_n) \geq 0$ for every n.

The application of $(p, q)-$convex sequences has led to the proving a generalized statement which in particular case when $p \to 1$ and $q \to 1$ implied a well-known inequality having an important application in Fourier analysis (see REMARK 7 at the end of the paper).

THEOREM 3 ([5]). Let $0 < p < 1$, $0 < q < 1$, $p \neq q$ and $(a_n)_{n=1}^{\infty}$ be a bounded $(p, q)-$convex sequence. Then $(a_n)_{n=1}^{\infty}$ is p-monotone. In addition, if $L_{p,q}(a_n) \geq 0$ and $0 \leq a_n \leq 1$, then we have

$$0 \leq L_{p,q}(a_n) \leq n \left(\sum_{k=1}^{n} \frac{p^k - q^k}{p - q} \right)^{-1}.$$

Let $(a_n)_{n=1}^{\infty}$ be an arbitrary real sequence. For a natural number r we define the difference operators $L_{p;r}$ with

$$L_{p;r}(a_n) = a_n - p^r a_{n+r}, \quad (n = 1, 2, \ldots),$$

and

$$L_{p,q;r}(a_n) = L_{p;r}(a_n) - q^r L_{p;r}(a_{n+r}), \quad (n = 1, 2, \ldots),$$

where $p, q \in \mathbb{R}$.

It is easy to verify the following properties of the operator $L_{p,q;r}$:

$$L_{p,q;r}(Ca_n) = CL_{p,q;r}(a_n), \;\; C-\text{is a constant}$$

and

$$L_{p,q;r}(a_n+b_n) = L_{p,q;r}(a_n) + L_{p,q;r}(b_n).$$

Now we introduce the concepts of $(p;r)-$monotonicity and $(p,q;r)-$convexity of an arbitrary real sequence.

DEFINITION 3. A sequence $(a_n)_{n=1}^{\infty}$ is called $(p;r)-$monotone if $L_{p;r}(a_n) \geq 0$ for every n and r. A sequence $(a_n)_{n=1}^{\infty}$ is called $(p,q;r)-$convex sequence if $L_{p,q;r}(a_n) \geq 0$ for every n and r.

Note that in particular case, the class of $(1,1;r)$-convex sequences is a wider class than the class of ordinary convex sequences as shows next example.

EXAMPLE 1. Let $(a_n)_{n=1}^{\infty}$ be an real sequence given by

$$a_n = \begin{cases} (-1)^n, & \text{for } n \text{ odd} \\ 0, & \text{for } n \text{ even.} \end{cases}$$

Then $\triangle^2 a_n = 4\cdot(-1)^n$ which means that for all n the sequence $(a_n)_{n=1}^{\infty}$ is not convex. On the other hand $L_{1,1;r}(a_n) = 2\cdot(-1)^n[1-(-1)^r]$, from which we conclude that the sequence $(a_n)_{n=1}^{\infty}$ is $(1,1;r)$-convex for all numbers n and for an arbitrary even number r.

We shall generalize THEOREM 2 using $(1,1;2)-$convexity instead of the ordinary second order convexity and THEOREM 3 using $(p,q;r)-$convexity, in general form, instead of $(p,q)-$convexity which are the main aims of this paper.

2 Main Results

Firstly we prove the following:

THEOREM 4. Let $(a_n)_{n=1}^{\infty}$ be a real sequence and let the sequences $(A_n)_{n=1}^{\infty}$ be defined by

$$A_n = \frac{1}{n}\sum_{k=1}^{n} a_k, \quad (n=1,2,\dots).$$

If the sequence $(a_n)_{n=1}^{\infty}$ is $(1,1;2)$-convex, then the sequence $(A_n)_{n=1}^{\infty}$ is $(1,1;2)$-convex as well.

PROOF. Let $(a_n)_{n=1}^{\infty}$ be a $(1,1;2)$-convex sequence. Then we have to prove that

$$\frac{a_1+a_2+\cdots+a_n+a_{n+2}+a_{n+3}+a_{n+4}}{n+4} \\ -2\cdot\frac{a_1+a_2+\cdots+a_n+a_{n+1}+a_{n+2}}{n+2} + \frac{a_1+a_2+\cdots+a_n}{n} \geq 0,$$

holds for all $n = 1, 2, \ldots$.

Multiplying the above inequalities by $n(n+2)(n+4)$ we obtain

$$\begin{aligned} n(n+2)(a_1 + a_2 + \cdots + a_n + a_{n+2} + a_{n+3} + a_{n+4}) \\ -2n(n+4)(a_1 + a_2 + \cdots + a_n + a_{n+1} + a_{n+2}) \\ +(n+2)(n+4)(a_1 + a_2 + \cdots + a_n) \geq 0, \end{aligned}$$

for all $n = 1, 2, \ldots$.

Now canceling similar terms, in the above inequalities, we obtain the equivalent inequalities

$$8(a_1 + a_2 + \cdots + a_n) - n(n+6)(a_{n+1} + a_{n+2}) + n(n+2)(a_{n+3} + a_{n+4}) \geq 0 \quad (2)$$

for all $n = 1, 2, \ldots$.

Subsequently, it is enough to prove (2). Let the sequence $(a_n)_{n=1}^{\infty}$ be $(1, 1; 2)$-convex and $n = 2k - 1$, $k \in \mathbb{N}$. Then adding the inequalities

$$\begin{aligned} &2 \cdot 4 \cdot (a_1 - 2a_3 + a_5) \geq 0, \\ &4 \cdot 6 \cdot (a_3 - 2a_5 + a_7) \geq 0, \\ &6 \cdot 8 \cdot (a_5 - 2a_7 + a_9) \geq 0, \\ &\qquad\vdots \\ &(2k-4)(2k-2)(a_{2k-5} - 2a_{2k-3} + a_{2k-1}) \geq 0, \\ &(2k-4)2k(a_{2k-3} - 2a_{2k-1} + a_{2k+1}) \geq 0, \\ &2k(2k+2)(a_{2k-1} - 2a_{2k+1} + a_{2k+3}) \geq 0, \end{aligned}$$

we obtain

$$8(a_1 + a_3 + \cdots + a_{2k-1}) - (4k^2 + 12k)a_{2k+1} + (4k^2 + 4k)a_{2k+3} \geq 0. \quad (3)$$

Now let $n = 2k$, $k \in \mathbb{N}$, be an even number. Similarly, adding the inequalities

$$\begin{aligned} &2 \cdot 4 \cdot (a_2 - 2a_4 + a_6) \geq 0, \\ &4 \cdot 6 \cdot (a_4 - 2a_6 + a_8) \geq 0, \\ &6 \cdot 8 \cdot (a_6 - 2a_8 + a_{10}) \geq 0, \\ &\qquad\vdots \\ &(2k-4)(2k-2)(a_{2k-4} - 2a_{2k-2} + a_{2k}) \geq 0, \\ &(2k-4)2k(a_{2k-2} - 2a_{2k} + a_{2k+2}) \geq 0, \\ &2k(2k+2)(a_{2k} - 2a_{2k+2} + a_{2k+4}) \geq 0, \end{aligned}$$

we obtain

$$8(a_2 + a_4 + \cdots + a_{2k}) - (4k^2 + 12k)a_{2k+2} + (4k^2 + 4k)a_{2k+4} \geq 0. \quad (4)$$

Finally, adding inequalities (3) and (4) we immediately obtain (2). The proof is completed.

THEOREM 5. Let $0<p<1$, $0<q<1$, $p\neq q$, $r\in\mathbb{N}$, and let $(a_n)_{n=1}^{\infty}$ be a bounded $(p,q;r)-$convex sequence. Then $(a_n)_{n=1}^{\infty}$ is $(p;r)$-monotone. In addition, if $L_{p,q;r}(a_n)\geq 0$ and $0\leq a_n\leq 1$, then we have

$$0\leq L_{p,q;r}(a_{nr})\leq n\left(\sum_{j=1}^{n}\frac{p^{jr}-q^{jr}}{p^r-q^r}\right)^{-1}.$$

PROOF. Assume that $L_{p,q;r}(a_n)\geq 0$ for all n. Then from $L_{p;r}(a_n)\geq q^r L_{p;r}(a_{n+r})$ and every $k>n$, $k=n+r,n+2r,n+3r,\dots,n+mr,\dots$, $(m=1,2,\dots)$, we get

$$L_{p;r}(a_k)\leq q^{n-k}L_{p;r}(a_n).$$

Hence, for $m=1,2,\dots,$ and assumptions of the theorem we obtain

$$\begin{aligned}\frac{1-(p/q)^{mr}}{1-(p/q)^r}L_{p;r}(a_n) &= \sum_{i=n,n+r,\dots,n+(m-1)r}(p/q)^{i-n}\,L_{p;r}(a_n)\\ &\geq \sum_{i=n,n+r,\dots,n+(m-1)r}(p/q)^{i-n}\,q^{i-n}L_{p;r}(a_i)\\ &= a_n-p^{mr}a_{n+mr}\geq -p^{mr},\end{aligned}$$

i.e.

$$\frac{1-(p/q)^{mr}}{1-(p/q)^r}L_{p;r}(a_n)+p^{mr}\geq 0.$$

Since last inequality holds true for all $mr>0$, then it clearly implies $L_{p;r}(a_n)\geq 0$ for all $n\in\mathbb{N}$.

Now the boundedness of the sequence $(a_n)_{n=1}^{\infty}$ implies

$$\begin{aligned}n &= \sum_{j=1}^{n}1\geq\sum_{j=1}^{n}a_{jr}\\ &= \sum_{j=1}^{n}\frac{1-p^{jr}}{1-p^r}L_{p;r}(a_{jr})+p^r a_{(n+1)r}\frac{1-p^{nr}}{1-p^r}\\ &\geq L_{p;r}(a_{nr})\sum_{j=1}^{n}\frac{1-p^{jr}}{1-p^r}q^{(n-j)r}\\ &= L_{p;r}(a_{nr})\big[q^{(n-1)r}+(1+p^r)q^{(n-2)r}+(1+p^r+p^{2r})q^{(n-3)r}\\ &\quad +\cdots+(1+p^r+p^{2r}+\cdots+p^{(n-1)r})\big]\\ &= L_{p;r}(a_{nr})\big\{\big[q^{(n-1)r}+p^rq^{(n-2)r}+p^{2r}q^{(n-3)r}+\cdots+p^{(n-1)r}\big]\\ &\quad +\big[q^{(n-2)r}+(1+p^r)q^{(n-3)r}+\cdots+(1+p^r+p^{2r}+\cdots+p^{(n-2)r})\big]\big\}\\ &\geq L_{p;r}(a_{nr})\big[q^{(n-1)r}+p^rq^{(n-2)r}+p^{2r}q^{(n-3)r}+\cdots+p^{(n-1)r}\big]\\ &= L_{p;r}(a_{nr})\sum_{j=1}^{n}\frac{p^{jr}-q^{jr}}{p^r-q^r}.\end{aligned}$$

The proof is completed.

REMARK 6. If we take $r = 1$ in THEOREM 5, then THEOREM 3 is an immediate results of it.

REMARK 7. If we take $r = 1$ and let $p \to 1$, $q \to 1$ in THEOREM 5, then we obtain

$$0 \leq \triangle(a_n) \leq \frac{2}{n+1}, \quad (n = 1, 2, \dots).$$

In fact, this is a well-known result that was appeared in [7]: If $(a_n)_{n=1}^{\infty}$ is bounded, say $0 \leq a_n \leq 1$ and convex, then $\triangle(a_n)_{n=1}^{\infty}$ is also bounded. This result, as we have mentioned in the first section of this paper, has an important application in Fourier analysis and its extends (as we have done) surly would be important in applications to this field.

In [6], any real sequence $(a_n)_{n=1}^{\infty}$ has been defined the following differences

$$\overline{L}_{pq}(a_n) = a_{n+2} - (p+q)a_{n+1} + pqa_n, \quad n = 1, 2, \dots,$$

and it is said that the sequence $(a_n)_{n=1}^{\infty}$ is p, q-convex if

$$\overline{L}_{pq}(a_n) \geq 0, \quad n = 1, 2, \dots,$$

and this definition differs from the definition given in [5]. Here we generalize the class of p, q-convex sequences in the following way: For any real sequence $(a_n)_{n=1}^{\infty}$ we define the following differences

$$\overline{L}_{p,q;r}(a_n) = a_{n+2r} - (p^r + q^r)a_{n+r} + p^r q^r a_n, \quad n, r = 1, 2, \dots.$$

We say that the sequence $(a_n)_{n=1}^{\infty}$ is $p, q; r$-convex if $\overline{L}_{p,q;r}(a_n) \geq 0, \quad n, r = 1, 2, \dots.$

Of what we said so far for $p, q; r$-convex sequences we are in able to prove the following result which plays an important role for the future investigations.

LEMMA 8. Let $r \in \{1, 2, \dots\}$. Then the sequence

$$w_n = \begin{cases} \frac{p^n - q^n}{p - q} & \text{if } p \neq q \\ np^{n-1} & \text{if } p = q \end{cases}$$

satisfies the relation $\overline{L}_{p,q;r}(w_n) = 0, \quad n \in \{1, 2, \dots\}$.

PROOF. The proof follows by direct calculations.

Taking the value $r = 1$ to the LEMMA 8, we obtain Lemma 1 proved previously by others, see [6] page 2.

THEOREM 9. Let $(a_n)_{n=1}^{\infty}$ be a $p, q; r$-convex sequence of real numbers, the integer $m \geq 2$, and $r \in \{1, 2, \dots\}$. If for the terms a_r and a_{2r} of the sequence $(a_n)_{n=1}^{\infty}$ holds

$$\frac{a_{2r}}{w_{m+r}} \geq \frac{a_r}{w_m}, \tag{5}$$

then the sequence $\left(\frac{a_{n+2r}}{w_{m+n+r}}\right)_{n=1}^{\infty}$ is monotone non-decreasing for all $n \in \{1,2,\dots\}$.

PROOF. Let $(a_n)_{n=1}^{\infty}$ be a $p,q;r$-convex sequence of real numbers, the integer $m \geq 2$, and $r \in \{1,2,\dots\}$. Then by the assumptions we have

$$\begin{aligned}
&[a_{n+2r} - (p^r + q^r)a_{n+r} + (pq)^r a_n]\,\frac{w_{m+n}}{(pq)^{m+n}} \geq 0,\\
&[a_{n+r} - (p^r + q^r)a_n + (pq)^r a_{n-r}]\,\frac{w_{m+n-r}}{(pq)^{m+n-r}} \geq 0,\\
&[a_n - (p^r + q^r)a_{n-r} + (pq)^r a_{n-2r}]\,\frac{w_{m+n-2r}}{(pq)^{m+n-2r}} \geq 0,\\
&\qquad\qquad\vdots\\
&[a_{5r} - (p^r + q^r)a_{4r} + (pq)^r a_{3r}]\,\frac{w_{m+3r}}{(pq)^{m+3r}} \geq 0,\\
&[a_{4r} - (p^r + q^r)a_{3r} + (pq)^r a_{2r}]\,\frac{w_{m+2r}}{(pq)^{m+2r}} \geq 0,\\
&[a_{3r} - (p^r + q^r)a_{2r} + (pq)^r a_r]\,\frac{w_{m+r}}{(pq)^{m+r}} \geq 0.
\end{aligned}$$

Adding the above inequalities we obtain

$$\frac{a_{n+2r}w_{m+n} - a_{n+r}w_{m+n+r}}{(pq)^{m+n}} + \frac{a_r w_{m+r} - a_{2r}w_m}{(pq)^m} \geq 0.$$

Thus, from this inequality and (5) we clearly obtain the assertion of the theorem.

OPEN PROBLEM 10. If the sequence $(a_n)_{n=1}^{\infty}$ is $(p,q;r)$-convex, then the sequence $(A_n)_{n=1}^{\infty}$ is $(p,q;r)$-convex as well for $r \in \{3,4,\dots\}$.

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