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Haar Wavelets Approach For Solving Multidimensional Stochastic Itô-Volterra Integral Equations*

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Abstract

A new computational method based on Haar wavelets is proposed for solving multidimensional stochastic Itô-Volterra integral equations. The block pulse functions and their relations to Haar wavelets are employed to derive a general procedure for forming stochastic operational matrix of Haar wavelets. Then, Haar wavelets basis along with their stochastic operational matrix are used to approximate solution of multidimensional stochastic Itô-Volterra integral equations. Convergenc and error analysis of the proposed method are discussed. In order to show the effectiveness of the proposed method, it is applied to some problems.

1 Introduction

The stochastic integral equations arise in many different fields e.g. biology, chemistry, epidemiology, mechanics, microelectronics, economics, and finance. The behavior of dynamical systems in these fields are often dependent on a noise source and a Gaussian white noise, governed by certain probability laws, so that modeling such phenomena naturally requires the use of various stochastic differential equations or, in more complicated cases, stochastic Volterra integral equations and stochastic integro-differential equations [1–5].

As analytic solutions of stochastic integral equations are not available in many cases, numerical approximation becomes a practical way to face this difficulty. In previous works various numerical methods have been used for approximating the solution of stochastic integral and differential equations. Here we only mention Kloeden and Platen [1], Oksendal [2], Maleknejad et al. [3, 4], Cortes et al. [5, 6], Murge et al. [7], Khodabin et al. [8, 9], Zhang [10, 11], Jankovic [12] and Heydari et al. [13].

Recently, different orthogonal basis functions, such as block pulse functions, Walsh functions, Fourier series, orthogonal polynomials and wavelets, are used to estimate solutions of functional equations. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis, and many other areas. Wavelets permit

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the accurate representation of a variety of functions and operators [14, 15]. Haar wavelets have been widely applied in system analysis, system identification, optimal control and numerical solution of integral and differential equations [16, 17].

The multidimensional Itô-Volterra integral equations arise in many problems such as an exponential population growth model with several independent white noise sources [3]. In this paper we consider the following multidimensional stochastic Volterra integral equation

$$X(t) = f(t) + \int_0^t k_0(s,t)X(s)ds + \sum_{i=1}^{n} \int_0^t k_i(s,t)X(s)dB_i(s), \quad t \in [0,T), \tag{1}$$

where $X(t)$, $f(t)$ and $k_i(s,t)$, $i = 0, 2, ..., n$ are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also

$$B(t) = (B_1(s), B_2(s), ..., B_n(s))$$

is a multidimensional Brownian motion process and $\int_0^t k_i(s,t)X(s)dB_i(s),\ i = 1, 2, ..., n$ are the Itô integral [2, 18]. In order to solving this multidimensional stochastic Itô-Volterra integral equation we first derive the Haar wavelets stochastic integration operational matrix. Then the stochastic operational matrix for Haar wavelets along with Haar wavelets basis are used to derive a numerical solution.

This paper is organized as follows: In section 2, some basic definition and preliminaries are described. In section 3, some basic properties of the Haar wavelets are presented. In section 4, stochastic operational matrix for Haar wavelets and a general procedure for deriving this matrix are introduced. In section 5, a new computational method based on stochastic operational matrix for Haar wavelets are proposed for solving multidimensional stochastic Itô-Volterra integral equations. Section 6 is devoted to convergence and error analysis of the proposed method. Some numerical examples are presented in section 7. Finally, a conclusion is given in section 8.

2 Preliminaries

In this section we review some basic definition of the stochastic calculus and the block pulse functions (BPFs).

2.1 Stochastic Calculus

DEFINITION 1. (Brownian motion process) A real-valued stochastic process $B(t)$, $t \in [0,T]$ is called Brownian motion if it satisfies the following properties.

(i) The process has independent increments for $0 \le t_0 \le t_1 \le ... \le t_n \le T$.

(ii) For all $t \ge 0$, $B(t+h) - B(t)$ has Normal distribution with mean 0 and variance h.

(iii) The function $t \to B(t)$ is continuous functions of t.

DEFINITION 2. Let $\{\mathcal{N}_t\}_{t\geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t,\omega) : [0,\infty) \times \Omega \rightarrow \mathbb{R}^n$ is called $\mathcal{N}_t$-adapted if for each $t \geq 0$ the function $\omega \rightarrow g(t,\omega)$ is $\mathcal{N}_t$-measurable.

DEFINITION 3. Let $\mathcal{V} = \mathcal{V}(S,T)$ be the class of functions $f(t,\omega) : [0,\infty) \rightarrow \Omega \times R$ such that:

(i) The function $(t,\omega) \rightarrow f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel algebra on $[0,\infty)$ and $\mathcal{F}$ is the σ -algebra on Ω.

(ii) f is adapted to $\mathcal{F}_t$, where $\mathcal{F}_t$ is the σ -algebra generated by the random variables $B(s), s \leq t$.

(iii) $E\left(\int_S^T f^2(t,\omega)dt\right) < \infty$.

DEFINITION 4. (The Itô integral) Let $f \in \mathcal{V}(S,T)$, then the Itô integral of f is defined by

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\rightarrow\infty} \int_S^T \varphi_n(t,\omega)dB_t(\omega) \quad (\lim \text{in } L^2(P)),$$

where, φ_n is a sequence of elementary functions such that

$$E\left(\int_s^T \left(f(t,\omega) - \varphi_n(t,\omega)\right)^2 dt\right) \rightarrow 0, \quad \text{as} \quad n \rightarrow \infty.$$

For more details about stochastic calculus and integration please see [2].

2.2 Block Pulse Functions

BPFs have been studied by many authors and applied for solving different problems. In this section we recall definition and some properties of the block pulse functions [3, 4, 19]. The m-set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & \text{for } (i-1)h \leq t < ih, \\ 0 & \text{otherwise,} \end{cases}$$

in which $t \in [0,T)$, $i = 1,2,...,m$ and $h = \frac{T}{m}$. The set of BPFs are disjointed with each other in the interval $[0,T)$ and

$$b_i(t)b_j(t) = \delta_{ij}b_i(t), i,j = 1,2,...,m,$$

where δ_{ij} is the Kronecker delta. The set of BPFs defined in the interval $[0,T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t)b_j(t)dt = h\delta_{ij}, \quad i,j = 1,2,...,m.$$

If $m \to \infty$, the set of BPFs is a complete basis for $L^2[0,T)$. So an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0,T)$, can be expanded into a block pulse series as

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t), \tag{2}$$

where

$$f_i = \frac{1}{h}\int_0^T b_i(t) f(t) dt, \quad i = 1, 2, ..., m.$$

Rewriting Eq. (2) in the vector form we have

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t) = F^T \Phi(t) = \Phi^T(t) F,$$

in which

$$F = [f_1, f_2, ..., f_m]^T \text{ and } \Phi(t) = [b_1(t), b_2(t), ..., b_m(t)]^T. \tag{3}$$

Moreover, any two dimensional function $k(s,t) \in L^2([0,T_1] \times [0,T_2])$ can be expanded with respect to BPFs such as

$$k(s,t) = \Phi^T(t) K \Phi(t),$$

where $\Phi(t)$ is the m-dimensional BPFs vectors respectively, and K is the $m \times m$ BPFs coefficient matrix with (i,j)-th element

$$k_{ij} = \frac{1}{h_2 h_2}\int_0^{T_1}\int_0^{T_2} k(s,t) b_i(t) b_j(s) dt ds, \quad i,j = 1, 2, ..., m,$$

and $h_1 = \frac{T_1}{m}$ and $h_2 = \frac{T_2}{m}$. Let $\Phi(t)$ be the BPFs vector. Then we have

$$\Phi^T(t)\Phi(t) = 1 \text{ and } \Phi(t)\Phi^T(t) = \begin{pmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_m(t) \end{pmatrix}_{m \times m}.$$

For an m-vector F, we have

$$\Phi(t)\Phi^T(t) F = \tilde{F}\Phi(t), \tag{4}$$

where $\tilde{F}$ is an $m \times m$ matrix and $\tilde{F} = diag(F)$. Also, it is easy to show that

$$\Phi^T(t) A \Phi(t) = \tilde{A}^T \Phi(t) \text{ for an } m \times m \text{ matrix } A, \tag{5}$$

where $\tilde{A} = diag(A)$ is a m-vector.

3 Haar Wavelets

The orthogonal set of Haar wavelets $h_n(t)$ consists a set of square waves defined as follows [14, 16, 17]

$$h_n(t) = 2^{\frac{j}{2}}\psi\left(2^j t - k\right), \quad j \geq 0, \quad 0 \leq k < 2^j, \quad n = 2^j + k, \quad n, j, k \in \mathbb{Z},$$

where

$$h_0(t) = 1, \quad 0 \leq t < 1 \text{ and } \psi(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2}, \\ -1 & \frac{1}{2} \leq t < 1. \end{cases}$$

Each Haar wavelets $h_n(t)$ has the support $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$, so that it is zero elsewhere in the interval $[0,1)$. The Haar wavelets $h_n(t)$ are pairwise orthonormal in the interval $[0,1)$ and

$$\int_0^1 h_i(t)h_j(t)dt = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Any square integrable function $f(t)$ in the interval $[0,1)$ can be expanded in terms of Haar wavelets as

$$f(t) = c_0 h_0(t) + \sum_{i=1}^{\infty} c_i h_i(t), \quad i = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j, \quad j, k \in \mathbb{N}, \tag{6}$$

where c_i is given by

$$c_i = \int_0^1 f(t)h_i(t)dt, \quad i = 0, 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j, \quad j, k \in \mathbb{N}. \tag{7}$$

The infinite series in Eq. (6) can be truncated after $m = 2^J$ terms (J is level of wavelet resolution), that is

$$f(t) \simeq c_0 h_0(t) + \sum_{i=1}^{m-1} c_i h_i(t), \quad i = 2^j + k, \quad j = 0, 1, ..., J-1, \quad 0 \leq k < 2^j,$$

rewriting this equation in the vector form we have,

$$f(t) \simeq C^T H(t) = H(t)^T C,$$

in which C and $H(t)$ are Haar coefficients and wavelets vectors as

$$C = [c_0, c_1, ..., c_{m-1}]^T,$$
$$H(t) = [h_0(t), h_1(t), ..., h_{m-1}(t)]^T. \tag{8}$$

Any two dimensional function $k(s,t) \in L^2([0,1) \times [0,1))$ can be expanded with respect to Haar wavelets as

$$k(s,t) = H^T(t)KH(t), \tag{9}$$

where $H(t)$ is the Haar wavelets vector and K is the $m \times m$ Haar wavelets coefficients matrix with (i,l)-th element can be obtained as

$$k_{il} = \int_0^1 \int_0^1 k(s,t)H_i(t)H_l(s)dtds, \quad i, l = 1, 2, ..., m.$$

3.1 Haar Wavelets and BPFs

In this section we will derive the relation between the BPFs and Haar wavelets. It is worth mentioning that in this section we set $T = 1$ in definition of BPFs.

THEOREM 1. Let $H(x)$ and $\Phi(x)$ be the m-dimensional Haar wavelets and BPFs vector respectively, the vector $H(x)$ can be expanded by BPFs vector $\Phi(x)$ as

$$H(t) = Q\Phi(t), \quad m = 2^J, \tag{10}$$

where Q is an $m \times m$ matrix and

$$Q_{il} = 2^{\frac{l}{2}} h_{i-1}\left(\frac{2l-1}{2m}\right), \quad i,l = 1,2,...,m, \quad i-1 = 2^j + k, \quad 0 \leq k < 2^j.$$

PROOF. Let $H_i(t), i = 1,2,...,m$, be the i-th element of Haar wavelets vector. Expanding $H_i(t)$ into an m-term vector of BPFs, we have

$$H_i(t) = \sum_{l=1}^{m} Q_{il} b_l(t) = Q_i^T B(t), \quad i = 1,2,...,m,$$

where Q_i is the i-th row and Q_{il} is the (i,l)-th element of matrix Q. By using the orthogonality of BPFs, we have

$$Q_{il} = \frac{1}{h}\int_0^1 H_i(t) b_l(t) dt = \frac{1}{h}\int_{\frac{l-1}{m}}^{\frac{l}{m}} H_i(t) dt = 2^{\frac{j}{2}} m \int_{\frac{l-1}{m}}^{\frac{l}{m}} h_{i-1}(t) dt.$$

So by using mean value theorem for integrals in the last equation, we can write

$$Q_{ij} = 2^{\frac{j}{2}} m \left(\frac{l}{m} - \frac{l-1}{m}\right) h_{i-1}(\eta_l) = 2^{\frac{j}{2}} h_{i-1}(\eta_l), \quad \eta_l \in \left(\frac{l-1}{m}, \frac{l}{m}\right).$$

As $h_{i-1}(t)$ is constant on the interval $\left(\frac{l-1}{m}, \frac{l}{m}\right)$, we can choose $\eta_l = \frac{2l-1}{2m}$. So we have

$$Q_{il} = 2^{\frac{j}{2}} h_{i-1}\left(\frac{2l-1}{2m}\right), \quad i,l = 1,2,...m.$$

and this proves the desired result.

REMARK 1. According to the definition of matrix Q in (10), it is easy to see that

$$Q^{-1} = \frac{1}{m} Q^T.$$

The following Remarks are the consequence of relations (4), (5) and Theorem 1.

REMARK 2. For an m-vector F, we have

$$H(t)H^T(t)F = \tilde{F} H(t),$$

in which $\tilde{F}$ is an $m \times m$ matrix as $\tilde{F} = Q\bar{F}Q^{-1}$ where $\bar{F} = diag\left(Q^T F\right)$.

REMARK 3. Let A be an arbitrary $m \times m$ matrix. Then for the Haar wavelets vector $H(t)$, we have

$$H^T(t)AH(t) = \hat{A}^T H(t),$$

where $\hat{A}^T = UQ^{-1}$ and $U = diag(Q^T AQ)$ is a m-vector.

4 Stochastic Integration Operational Matrix of Haar Wavelets

In this section we obtain the stochastic integration operational matrix for Haar wavelets. For this purpose we recall some useful results for BPFs [3, 4].

LEMMA 1 ([3]). Let $\Phi(t)$ be the BPFs vector defined in (3). Then integration of this vector can be derived as

$$\int_0^t \Phi(s)ds \simeq P\Phi(t),$$

where $P_{m\times m}$ is called the operational matrix of integration for BPFs and is given by

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{m\times m}. \tag{11}$$

LEMMA 2 ([3]). Let $\Phi(t)$ be the BPFs vector defined in (3). The Itô integral of this vector can be derived as

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t),$$

where P_s is called the stochastic operational matrix of integration for BPFs and is given by

$$P_s = \begin{bmatrix} B\left(\frac{h}{2}\right) & B\left(h\right) & \cdots & B\left(h\right) \\ 0 & B\left(\frac{3h}{2}\right) - B\left(h\right) & \cdots & B\left(2h\right) - B\left(h\right) \\ 0 & 0 & \cdots & B\left(3h\right) - B\left(2h\right) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B\left(\frac{(2m-1)h}{2}\right) - B\left(\left(m-1\right)h\right) \end{bmatrix}_{m\times m}. \tag{12}$$

Now we are ready to derive a new operational matrix of stochastic integration for the Haar wavelets basis. To this end we use BPFs and the matrix Q introduced in (10).

THEOREM 2. Suppose $H(t)$ is the Haar wavelets vector defined in (8). The integral of this vector can be derived as

$$\int_0^t H(s)ds \simeq \frac{1}{m} QPQ^T H(t) = \Lambda H(t), \tag{13}$$

where Q is introduced in (10) and P is the operational matrix of integration for BPFs derived in (11).

PROOF. Let $H(t)$ be the Haar wavelets vector, by using Theorem 1 and Lemma 1 we have

$$\int_0^t H(s)ds \simeq \int_0^t Q\Phi(s)ds = Q \int_0^t \Phi(s)ds = QP\Phi(t),$$

now, Theorem 1 and Remark 1 give

$$\int_0^t H(s)ds \simeq QP\Phi(t) = \frac{1}{m} QPQ^T H(t) = \Lambda H(t),$$

and this complete the proof.

THEOREM 3. Suppose $H(t)$ is the Haar wavelets vector defined in (8). The Itô integral of this vector can be derived as

$$\int_0^t H(s)dB(s) \simeq \frac{1}{m} QP_sQ^T H(t) = \Lambda_s H(t), \tag{14}$$

where Λ_s is called stochastic operational matrix for Haar wavelets, Q is introduced in (10) and P_s is the stochastic operational matrix of integration for BPFs derived in (12).

PROOF. Let $H(t)$ be the Haar wavelets vector. By using Theorem 1 and Lemma 2, we have

$$\int_0^t H(s)dB(s) \simeq \int_0^t Q\Phi(s)dB(s) = Q \int_0^t \Phi(s)dB(s) = QP_s\Phi(t),$$

now, Theorem 1 and Remark 1 result

$$\int_0^t H(s)dB(s) \simeq QP_s\Phi(t) = \frac{1}{m} QP_sQ^T H(t) = \Lambda_s H(t),$$

and this complete the proof.

5 Solving Multidimensional Stochastic Itô-Volterra Integral Equations

In this section, we apply the stochastic operational matrix of Haar wavelets for solving multidimensional stochastic Itô-Volterra integral equation. Consider the following multidimensional stochastic Itô-Volterra integral equation

$$X(t) = f(t) + \int_0^t k_0(s,t)X(s)ds + \sum_{i=1}^{n} \int_0^t k_i(s,t)X(s)dB_i(s), \quad t \in [0,T), \tag{15}$$

where $X(t)$, $f(t)$ and $k_i(s,t)$, $i = 0,1,2,...,n$ are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also

$$B(t) = (B_1(s), B_2(s), ..., B_n(s))$$

is a multidimensional Brownian motion process and $\int_0^t k_i(s,t)X(s)dB_i(s)$, $i = 1,2,...,n$ are the Itô integral [18, 2]. For solving this problem by using the stochastic operational matrix of Haar wavelets, we approximate $X(t)$, $f(t)$ and $k_i(s,t), i = 0,1,2...n$ in terms of Haar wavelets as follows

$$\begin{aligned} f(t) &= F^T H(t) = F H^T(t), \\ X(t) &= X^T H(t) = X H^T(t), \\ k_i(s,t) &= H^T(s) K_i H(t) = H^T(t) K_i^T H(s), \quad i = 0,1,2,...,n, \end{aligned} \tag{16}$$

where X and F are Haar wavelets coefficients vector, and $K_i, i = 0,1,2,...,n$ are Haar wavelets coefficient matrices defined in Eqs. (8) and (9). Substituting above approximations in Eq. (15), we have

$$\begin{aligned} X^T H(t) &= F^T H(t) + H^T(t) K_0 \left(\int_0^t H(s) H^T(s) X ds \right) \\ &\quad + \sum_{i=1}^{n} H^T(t) K_i \left(\int_0^t H(s) H^T(s) X dB_i(s) \right). \end{aligned}$$

By Remark 2, we get

$$X^T H(t) = F^T H(t) + H^T(t) K_0 \left(\int_0^t \tilde{X} H(s) ds \right) + \sum_{i=1}^{n} H^T(t) K_i \left(\int_0^t \tilde{X} H(s) dB_i(s) \right),$$

where $\tilde{X}$ is a $m \times m$ matrix. Now by applying the operational matrices Λ and Λ_s for Haar wavelets derived in Eqs. (13) and (14), we have

$$X^T H(t) = F^T H(t) + H^T(t) K_0 \tilde{X} \Lambda H(t) + \sum_{i=1}^{n} H^T(t) K_i \tilde{X} \Lambda_s H(t),$$

by setting $Y_0 = K_0\tilde{X}\Lambda$ and $Y_i = K_i\tilde{X}\Lambda_s$, $i = 1, 2, ..., n$, and using Remark 2 we derive

$$X^T H(t) - \hat{Y_0}^T H(t) - \sum_{i=1}^{n} \hat{Y}_i^T H(t) = F^T H(t),$$

in which $\hat{Y}_i, i = 0, 1, 2, ..., n$ are linear functions of vectors $\tilde{X}$ and so are linear function of X. This equation holds for all $t \in [0, 1)$, so we can write

$$X^T - \hat{Y_0}^T - \sum_{i=1}^{n} \hat{Y}_i^T = F^T. \tag{17}$$

Since $\hat{Y}_i, i = 0, 1, 2, ..., n$ are linear functions of X, Eq. (17) is a linear system of equations for unknown vector X. By solving this linear system and determining X, we can approximate solution of m-dimensional stochastic Itô-Volterra integral equation (15) by substituting the obtained vector X in Eq. (16).

6 Error Analysis

In this section, we investigate the convergence and error analysis of the Haar wavelets method for solution of m-dimensional stochastic Itô-Volterra integral equation.

THEOREM 4. Suppose that $f(t) \in L^2[0, 1)$ with bounded first derivative, $|f'(t)| \leq M$, and $e_m(t) = f(t) - \sum\limits_{i=0}^{m-1} f_i h_i(t)$. Then

$$\|e_m(t)\| \leq \frac{M}{\sqrt{3}m},$$

that is the Haar wavelets series will be convergent.

PROOF. By definition of error $e_m(t)$, we have

$$\|e_m(t)\|^2 = \int_0^1 \left(\sum_{i=m}^{\infty} f_i h_i(t)\right)^2 dt = \sum_{i=m}^{\infty} f_i^2,$$

where $i = 2^j + k,\ m = 2^J,\ J > 0$ and

$$f_i = \int_0^1 h_i(t) f(t) dt = 2^{\frac{j}{2}} \left(\int_{k2^{-j}}^{\left(k+\frac{1}{2}\right)2^{-j}} f(t)dt - \int_{\left(k+\frac{1}{2}\right)2^{-j}}^{(k+1)2^{-j}} f(t)dt \right).$$

So by the mean value theorem for integrals, there are $\eta_{j1} \in \left[k2^{-j}, \left(k + \frac{1}{2}\right)2^{-j}\right]$ and $\eta_{j2} \in \left[\left(k + \frac{1}{2}\right)2^{-j}, (k+1)\, 2^{-j}\right]$ such that

$$\begin{aligned} f_i &= \int_0^1 h_i(t) f(t) dt = 2^{\frac{j}{2}} \left(f(\eta_{j1}) \int_{k2^{-j}}^{\left(k+\frac{1}{2}\right)2^{-j}} dt - f(\eta_{j2}) \int_{\left(k+\frac{1}{2}\right)2^{-j}}^{(k+1)2^{-j}} dt \right) \\ &= 2^{\frac{j}{2}} \left(f(\eta_{j1}) \left[\left(k + \frac{1}{2}\right) 2^{-j} - k2^{-j} \right] - f(\eta_{j2}) \left[(k+1)\, 2^{-j} - \left(k + \frac{1}{2}\right) 2^{-j} \right] \right) \\ &= 2^{-\frac{j}{2}-1} \left(f(\eta_{j1}) - f(\eta_{j2}) \right) = 2^{-\frac{j}{2}-1} \left(\eta_{j1} - \eta_{j2} \right) f^{'}(\eta_j), \qquad \eta_{j2} < \eta_j < \eta_{j2}. \end{aligned}$$

It follows that

$$\begin{aligned}\|e_m(t)\|^2 &= \sum_{i=m}^{\infty} f_i^2 = \sum_{i=m}^{\infty} 2^{-j-2}\left(\eta_{j1}-\eta_{j2}\right)^2\left(f^{'}(\eta_j)\right)^2 \leq \sum_{i=m}^{\infty} 2^{-j-2}2^{-2j}M^2 \\ &= \sum_{i=m}^{\infty} 2^{-3j-2}M^2 = \sum_{j=J}^{\infty}\sum_{k=0}^{2^j-1} 2^{-3j-2}M^2 = \frac{M^2}{3m^2}. \qquad (18)\end{aligned}$$

In other words,

$$\|e_m(t)\| \leq \frac{M}{\sqrt{3}m}.$$

The proof is complete

THEOREM 5. Suppose that $f(s,t) \in L^2([0,1)\times[0,1))$ with bounded partial derivatives derivative, $\left|\frac{\partial^2 f}{\partial s \partial t}\right| \leq M$, and

$$e_m(s,t) = f(s,t) - \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} f_{ij}h_i(s)h_j(t).$$

Then

$$\|e_m(s,t)\| \leq \frac{M}{3m^2}.$$

PROOF. By definition of error $e_m(s,t)$, we have

$$\|e_m(s,t)\|^2 = \int_0^1 \left(\sum_{i=m}^{\infty}\sum_{l=m}^{\infty} f_{il}h_i(s)h_l(t)\right)^2 dt = \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} f_{il}^2,$$

where $i = 2^j + k,\ l = 2^{j'} + k,\ m = 2^J, J > 0$ and

$$f_{ij} = \int_0^1\int_0^1 h_i(s)h_l(t)f(s,t)dsdt.$$

By Theorem 4, there are $\eta_j,\ \eta_{j1},\ \eta_{j2},\ \eta_{j'},\ \eta_{j_1'}$ and $\eta_{j_2'}$ such that

$$f_{ij} = \int_0^1 h_i(s)\left(\int_0^1 h_l(t)f(s,t)dt\right)ds = \int_0^1 h_i(s)\left[2^{-\frac{j'}{2}-1}\left(\eta_{j_1'}-\eta_{j_2'}\right)\frac{\delta f(s,\eta_{j'})}{\delta t}\right]ds$$

and

$$2^{-\frac{j'}{2}-1}\left(\eta_{j_1'}-\eta_{j_2'}\right)\int_0^1 \frac{\delta f(s,\eta_{j'})}{\delta t}h_i(s) = 2^{-\frac{j}{2}-\frac{j'}{2}-2}\left(\eta_{j_1'}-\eta_{j_2'}\right)\left(\eta_{j_1}-\eta_{j_2}\right)\frac{\partial^2 f(\eta_j,\eta_{j'})}{\partial t\partial s}.$$

So we obtain that

$$\begin{aligned}\|e_m(s,t)\|^2 &= \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} f_{il}^2 = \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} 2^{-j-j'-4}\left(\eta_{j_1'}-\eta_{j_2'}\right)^2\left(\eta_{j_1}-\eta_{j_2}\right)^2\left|\frac{\partial^2 f(\eta_j,\eta_{j'})}{\partial t\partial s}\right|^2 \\ &\leq \sum_{i=m}^{\infty}\sum_{l=m}^{\infty} M^2 2^{-3j-3j'-4}.\end{aligned}$$

By using Eq.(18), we can derive

$$\|e_m(s,t)\|^2 \leq M^2 \sum_{i=m}^{\infty} 2^{-3j-2} \sum_{l=m}^{\infty} 2^{-3j'-2} = \frac{M^2}{(3m^2)^2}.$$

In other words

$$\|e_m(s,t)\| \leq \frac{M}{3m^2}.$$

The proof is complete.

THEOREM 6. Suppose $X(t)$ is the exact solution of (1) and $X_m(t)$ is its Haar wavelets approximate solution such that their coefficients are obtained by (7). Assume that the following conditions (a)–(c) hold:

(a) $\|X(t)\| \leq \rho$ for $t \in [0,1]$.

(b) $\|k_i(s,t)\| \leq M_i$ for $s,t \in [0,1] \times [0,1]$ and $i = 0,1,2,...n$.

(c) $(M_0 + \Gamma_{0m}) + \sum\limits_{i=1}^{n} \|B_i(t)\| (M_i + \Gamma_{im}) < 1$.

Then

$$\|X(t) - X_m(t)\| \leq \frac{\Upsilon_m + \rho\Gamma_{0m} + \rho \sum\limits_{i=1}^{n} \|B_i(t)\| \Gamma_{im}}{\left(1 - \left[(M_0 + \Gamma_{0m}) + \sum\limits_{i=1}^{n} \|B_i(t)\| (M_i + \Gamma_{im})\right]\right)},$$

where

$$\Upsilon_m = \sup_{t\in[0,1]} \frac{\left|f'(t)\right|}{\sqrt{3}\,m}, \quad \Gamma_{im} = \frac{1}{3m^2} \sup_{s,t\in[0,1]} \left|\frac{\partial^2 k_i(s,t)}{\partial s \partial t}\right|, \ i = 0,1,2,...,n.$$

PROOF. From (1) we have

$$\begin{aligned} X(t) - X_m(t) &= f(t) - f_m(t) + \int_0^t (k_0(s,t)X(s) - k_{0m}(s,t)X_m(s))\, ds \\ &\quad + \sum_{i=1}^{n} \int_0^t (k_i(s,t)X(s) - k_{im}(s,t)X_m(s))\, dB_i(s). \end{aligned}$$

So by the mean value theorem, we can write

$$\begin{aligned} \|X(t) - X_m(t)\| &\leq \|f(t) - f_m(t)\| + t\,\|(k_0(s,t)X(s) - k_{0m}(s,t)X_m(s))\| \\ &\quad + \sum_{i=1}^{n} B_i(t)\, \|(k_i(s,t)X(s) - k_{im}(s,t)X_m(s))\|. \end{aligned} \tag{19}$$

Now by using Theorems 4 and 5, we have

$$\begin{aligned} & \|(k_i(s,t)X(s) - k_{im}(s,t)X_m(s))\| \\ \leq\ & \|k_i(s,t)\| \, \|X(t) - X_m(t)\| + \|(k_i(s,t) - k_{im}(s,t))\| \, \|X(t)\| \\ & + \|(k_i(s,t) - k_{im}(s,t))\| \, \|X(t) - X_m(t)\| \\ \leq\ & (M_i + \Gamma_{im}) \, \|X(t) - X_m(t)\| + \rho\Gamma_{im} \end{aligned} \qquad (20)$$

for $i = 0, 1, 2, ..., n$. Substituting (20) in (19), we get

$$\begin{aligned} \|X(t) - X_m(t)\| \ \leq\ & \Upsilon_m + t\left[(M_0 + \Gamma_{0m}) \, \|X(t) - X_m(t)\| + \rho\Gamma_{0m}\right] \\ & + \sum_{i=1}^{n} B_i(t) \left[(M_i + \Gamma_{im}) \, \|X(t) - X_m(t)\| + \rho\Gamma_{im}\right], \end{aligned}$$

as assumption (c) is hold we get the inequality

$$\|X(t) - X_m(t)\| \leq \frac{\Upsilon_m + \rho\Gamma_{0m} + \rho \sum_{i=1}^{n} \|B_i(t)\| \, \Gamma_{im}}{\left(1 - \left[(M_0 + \Gamma_{0m}) + \sum_{i=1}^{n} \|B_i(t)\| \, (M_i + \Gamma_{im})\right]\right)},$$

and this proves the desired result.

7 Numerical Examples

In this section, we consider some numerical examples to illustrate the efficiency and reliability of the Haar wavelets operational matrices in solving multidimensional stochastic Itô-Volterra integral equation.

EXAMPLE 1. Let us consider the following three-dimensional stochastic Itô-Volterra integral equation [4]

$$X(t) = \frac{1}{12} + \int_0^t r(s)X(s)ds + \sum_{i=1}^{4} \int_0^t \alpha_i(s)X(s)dB_i(s) \qquad s, t \in [0, 1],$$

in which $r(s) = s^2$, $\alpha_1(s) = \sin(s)$, $\alpha_2(s) = \cos(s)$, and $\alpha_3(s) = s$. The exact solution of this three-dimensional stochastic Itô-Volterra integral equation is

$$X(t) = \frac{1}{12} \exp\left(\int_0^t \left(r(s) - \frac{1}{2}\sum_{i=1}^{3} \alpha_i^2(s)\right) ds + \sum_{i=1}^{3} \int_0^t \alpha_i(s)dB_i(s)\right),$$

where $X(t)$ is an unknown three-dimensional stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, and $B(t) = (B_1(t), B_2(t), B_3(t))$ is a three-dimensional Brownian motion process. This stochastic Itô-Volterra integral equation is solved by using the Haar wavelets stochastic operational matrix and the proposed method in section 5 for different values of $m = 2^J$. Figure 1 presents the approximate solution computed by the proposed method and exact solution are shown for $m = 2^8$. The absolute error of the numerical results for different values of m are shown in the Table 1. As the results show the proposed method is efficient for solving this multidimensional stochastic Itô-Volterra integral equations.

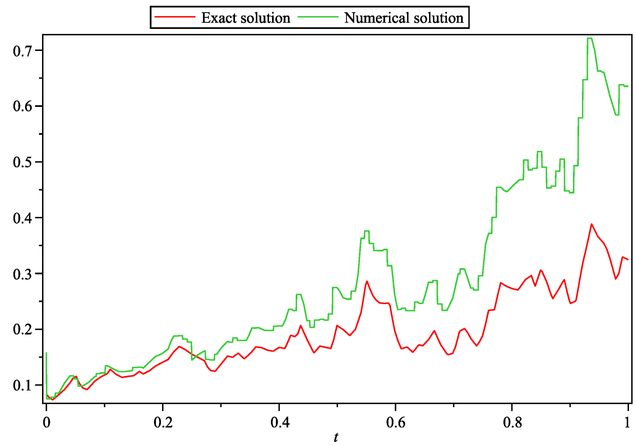

Figure 1: The approximate solution and exact solution for $m = 2^8$.

t	$m = 2^4$	$m = 2^5$	$m = 2^6$	$m = 2^7$
0.1	0.00116673	0.00905469	0.01317201	0.00582587
0.3	0.01503544	0.02546997	0.01830332	0.01603512
0.5	0.03381872	0.04105323	0.05022002	0.04959923
0.7	0.06688455	0.05083629	0.04467774	0.04956795
0.9	0.03255049	0.03664656	0.03311518	0.02844815

Table 1. The absolute error of the numerical results for different values of m.

EXAMPLE 2. We consider the following four-dimensional stochastic Itô-Volterra integral equation [4]

$$X(t) = \frac{1}{200} + \frac{1}{20}\int_0^t X(s)ds + \sum_{i=1}^{4}\int_0^t \alpha_i X(s)dB_i(s) \qquad s,t \in [0,1],$$

with $\alpha_1 = \frac{1}{50}$, $\alpha_2 = \frac{2}{50}$, $\alpha_3 = \frac{4}{50}$ and $\alpha_4 = \frac{9}{50}$. The exact solution of this four-dimensional stochastic Itô-Volterra integral equation is

$$X(t) = \frac{1}{200}\exp\left(\left(\frac{1}{20} - \frac{1}{2}\sum_{i=1}^{4}\alpha_i^2\right)t + \sum_{i=1}^{4}\alpha_i B_i(t)\right),$$

where $X(t)$ is an unknown four-dimensional stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, and $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t))$ is a four-dimensional Brownian motion process. The Haar wavelets stochastic operational matrix and the proposed method in section 5 is used for approximate solution of this four-dimensional stochastic Itô-Volterra integral equation for different values of $m = 2^J$. Figure 2 represent the approximate solution computed by the presented method and an exact solution for $m = 2^7$. Table 2 shows the absolute error of the numerical results for different values of m. As numerical results in Figure 2 and Table 2 reveal the proposed method is accurate and efficient for approximate the solution of this multidimensional stochastic Itô-Volterra integral equation.

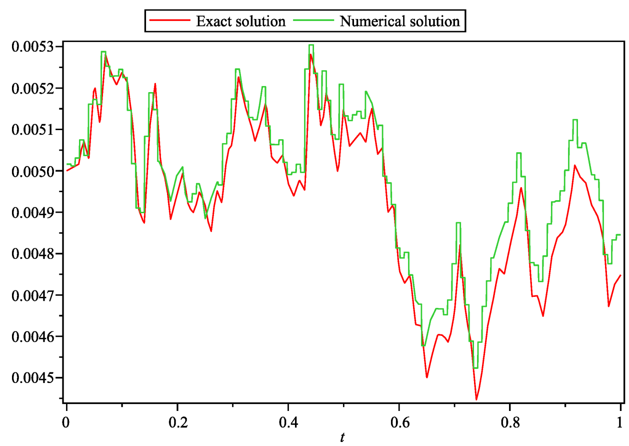

Figure 2: The approximate solution and exact solution for $m = 2^8$.

t	$m = 2^4$	$m = 2^5$	$m = 2^6$	$m = 2^7$
0.1	0.00016160	0.00005146	0.00001057	0.00001073
0.3	0.00014963	0.00013925	0.00009423	0.00009080
0.5	0.00003719	0.00001625	0.00003525	0.00006010
0.7	0.00049102	0.00035556	0.00036712	0.00035321
0.9	0.00035612	0.00035330	0.00034627	0.00034276

Table 2. The absolute error of the numerical results for different values of m.

8 Conclusion

The block pulse functions and their relations to Haar wavelets are employed to derive the stochastic operational matrix for Haar wavelets. By using this stochastic operational matrix a new computational method is proposed for solving multidimensional stochastic Itô-Volterra integral equations. The convergence and error analysis of the proposed method are investigated. Some numerical examples are included to demonstrate the efficiency and accuracy of the proposed method.

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