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Some Remarks On Global Total Domination In Graphs*

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Abstract

A subset S of vertices in a graph G is a global total dominating set, or just GTDS, if S is a total dominating set of both G and $\overline{G}$. The global total domination number $\gamma_{gt}(G)$ of G is the minimum cardinality of a GTDS of G. In this paper, we show that the decision problem for $\gamma_{gt}(G)$ is NP-complete, and then characterize graphs G of order n with $\gamma_{gt}(G) = n - 1$.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n. We denote the open neighborhood of a vertex v of G by $N_G(v)$, or just $N(v)$, and its closed neighborhood by $N_G[v]$ or $N[v]$. For a vertex set $S \subseteq V(G)$, we denote $N(S) = \cup_{v\in S}N(v)$ and $N[S] = \cup_{v\in S}N[v]$. The degree of a vertex x, $\deg(x)$ (or $\deg_G(x)$ to refer G) in a graph G denotes the number of neighbors of x in G. We refer $\Delta(G)$ and $\delta(G)$ as the maximum degree and the minimum degree of the vertices of G, respectively. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A leaf in a graph is a vertex of degree one, and a support vertex is one that is adjacent to a leaf. The distance between two vertices x and y, denoted by $d(x, y)$ (or $d_G(x, y)$ to refer to G) is the length of a shortest path from x to y. The diameter, $\mathrm{diam}(G)$, of a graph G is the maximum distance over all pairs of vertices of G. A set of vertices S in G is a dominating set, if $N[S] = V(G)$. The domination number, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. A set of vertices S in an isolate-free graph G is a total dominating set, or just TDS, if $N(S) = V(G)$. The total domination number, $\gamma_t(G)$, of G is the minimum cardinality of a total dominating set of G. For references and also terminology on domination in graphs see for example [6, 7].

The concept of global domination in graphs is introduced by Sampathkumar in [9], and further studied by Brigham et al. [2, 3], Dutton et al. [4, 5] and Arumugam et al.

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[1]. A subset S of vertices of a graph G is a global dominating set if S is a dominating set of both G and $\overline{G}$. The global domination number of a graph G, $\gamma_g(G)$, is the minimum cardinality of a global dominating set of G. Kulli et al. in [8] initiated the study of global total domination in graphs. A subset S of vertices in a graph G is a global total dominating set, or just DTDS, if S is a TDS of both G and $\overline{G}$. The global total domination number of G, $\gamma_{gt}(G)$, is the minimum cardinality of a GTDS of G. If a graph G of order n has a GTDS, then $\delta(G) \geq 1$ and $\Delta(G) \leq n-2$. That is neither G nor $\overline{G}$ have an isolated vertex. We define $\gamma_{gt}(G) = 0$ if G or $\overline{G}$ has an isolated vertex. For all graphs G which we study in this paper we assume that $\gamma_{gt}(G) > 0$.

In this paper, we study global total domination and obtain some new results and characterizations on the global total domination number of a graph G. In section 2 we present some preliminary results. In section 3, we show that the decision problem for $\gamma_{gt}(G)$ is NP-complete. In section 4, we characterize graphs G of order n with $\gamma_{gt}(G) = n-1$.

2 Preliminary Results

We start this section with the following observation that can be easily obtained from the definition.

OBSERVATION 1.

(1) For any graph G, $\gamma_{gt}(G) \geq \max\{\gamma_t(G), \gamma_t(\overline{G})\}$.

(2) If G is a disconnected graph, then $\gamma_{gt}(G) = \gamma_t(G)$.

(3) If S is a GTDS in a graph G, then for any vertex $x \in S$, $1 \leq \deg_{G[S]}(x) \leq |S|-2$.

LEMMA 2. The following statements hold.

(1) If $\gamma_t(G) > \Delta(G)+1$, then $\gamma_{gt}(G) = \gamma_t(G)$.

(2) If $\gamma_t(G) < \gamma_{gt}(G)$, then $|V(G)| \leq \Delta(G)(\Delta(G)+1)$.

PROOF. (1) Let S be a $\gamma_t(G)$-set with $|S| > \Delta(G)+1$ and $x \in V(G)$. Since $\deg(x) \leq \Delta(G)$, we have $S \nsubseteq N(x)$. So S is a GTDS for G. (2) Let $\gamma_t(G) < \gamma_{gt}(G)$. By part (1) $\gamma_t(G) \leq \Delta(G)+1$. Let $S = \{x_1, x_2, ..., x_k\}$ be a TDS of cardinality $k \leq \Delta(G)+1$. Since $G[S]$ has no isolated vertex, we obtain

$$\left|N(x_i) \cap \overline{S}\right| \leq \Delta(G) - 1$$

for any $i = 1, 2, ..., k$. On the other hand $\cup_i (N(x_i) \cap \overline{S}) = \overline{S}$. This implies that $|\overline{S}| \leq k(\Delta(G)-1)$ and

$$|V(G)| \leq k + k(\Delta(G)-1) \leq \Delta(G)(\Delta(G)+1).$$

Kulli et al. in [8] obtained the following results.

PROPOSITION 3 ([8]). Let G be a graph. Then the following statements hold.

(1) If $\mathrm{diam}(G) = 3$, then $\gamma_{gt}(G) \leq \gamma_t(G) + 2$,

(2) If $\mathrm{diam}(G) = 4$, then $\gamma_{gt}(G) \leq \gamma_t(G) + 1$,

(3) If $\mathrm{diam}(G) \geq 5$, then $\gamma_{gt}(G) = \gamma_t(G)$.

It is also noted in [8] that if $\mathrm{diam}(G) = 2$, then the difference between $\gamma_{gt}(G)$ and $\gamma_t(G)$ may be very large. However the graph G in the example posed in [8] satisfies Proposition 3 since $\mathrm{diam}(\overline{G}) > 2$. Furthermore, in that graph G, $\gamma_{gt}(\overline{G}) \leq \gamma_t(\overline{G}) + 2$. So the case $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$ remained open. We show that for any $k \geq 4$ there is a graph G such that $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$ and $\gamma_{gt}(G) = k$.

PROPOSITION 4. For any $k \geq 4$ there is a graph G such that $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$ and $\gamma_{gt}(G) = k$.

PROOF. Let
$$X = V(\overline{K_l}) = \{x_1, x_2, ..., x_l\}$$
for some integer $l \geq 6$. For any pair $i, j \in \{1, 2, ..., l\}$ we add a new vertex $x_{i,j}$ and join $x_{i,j}$ to x_i, x_j to obtain a graph H_1. Let $Y = \{x_{i,j} : 1 \leq i, j \leq l\}$. Let H_2 be a graph obtain from H_1 by joining any pair $x_{i,j}, x_{k,s}$ where $|\{i, j\} \cap \{k, s\}| = 1$. Let G be a graph obtained from H_2 by adding two new vertices x, y and joining them to every vertex in Y. It is a routine matter to see that $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. Let S be a GTDS for G. Any vertex of X is dominated by some vertex in Y. Since any vertex of Y dominates two vertices of X, we obtain $|S \cap Y| \geq \lceil \frac{l}{2} \rceil$. Since x (as a vertex of $\overline{G}$) is not dominated by $S \cap Y$, we have $S \cap X \neq \emptyset$. This implies that $\gamma_{gt}(G) \geq \lceil \frac{l}{2} \rceil + 1$. On the other hand $\{x_{2i+1}, x_{2i+2} : 0 \leq i < \lceil \frac{l}{2} \rceil\} \cup \{1\}$ is a GTDS for G. We conclude that $\gamma_{gt}(G) = \lceil \frac{l}{2} \rceil + 1$. Now the proof will be completed if we put $l = 2k - 2$ for a given $k \geq 4$.

We next determine the global total domination number of a tree.

THEOREM 5. For a tree T,
$$\gamma_{gt}(T) = \begin{cases} 0 & \text{if } \mathrm{diam}(T) \leq 2, \\ 4 & \text{if } \mathrm{diam}(T) = 3, \\ \gamma_t(T) + 1 & \text{if } \mathrm{diam}(T) = 4, \\ \gamma_t(T) & \text{if } \mathrm{diam}(T) \geq 5. \end{cases} \tag{1}$$

PROOF. Let T be a tree. If $\mathrm{diam}(T) \leq 2$, then $\overline{T}$ has an isolated vertex and so $\gamma_{gt}(T) = 0$. If $\mathrm{diam}(T) = 3$, then T is a double star and it is not hard to see that $\gamma_{gt}(T) = 4$. Assume that $\mathrm{diam}(T) = 4$. Let x be the central vertex of a diametrical path of T. If a minimum TDS S does not contain x, then it contains two support vertices a, b. Let $a_1 \in N(a) \cap S$ and $b_1 \in N(b) \cap S$. It is obvious that a_1 and b_1 are two leaves, since $\mathrm{diam}(G) = 4$. Then $(S \setminus \{a_1, b_1\}) \cup \{x\}$ is a TDS of G, a contradiction. So x belongs to every minimum TDS of T. Further, for any minimum TDS such S of T, $T[S]$ is a star with center x. This implies that $\gamma_{gt}(T) > \gamma_t(T)$. By Proposition 3 part (2) we obtain $\gamma_{gt}(T) = \gamma_t(T) + 1$. Assume now that $\mathrm{diam}(T) \geq 5$. By Proposition 3 part (3) $\gamma_{gt}(T) = \gamma_t(T)$.

3 Complexity

In this section we investigate the complexity of the following Problem:

GLOBAL TOTAL DOMINATING SET

INSTANCE: A graph $G = (V, E)$ and a positive integer k.

QUESTION: Does G has a GTDS of cardinality at most k?

To show that GTDS problem is NP-Complete for arbitrary graphs, we use the well-known NP-Completeness result for total domination which is defined by:

TOTAL DOMINATING SET

INSTANCE: A graph $G = (V, E)$ and a positive integer k.

QUESTION: Does G has a TDS of cardinality at most k?

THEOREM 6. GTDS is NP-complete for general graphs.

PROOF. To show that GTDS is an NP-Complete problem, we will establish a polynomial transformation from TDS to GTDS. Let G be an arbitrary instance of TDS. Let H be the graph $G \cup \overline{G}$. We show that obtaining the minimum TDS of G is equal to finding the minimum GTDS of H. Let S and S' be minimum TDS of G and $\overline{G}$, respectively. We prove that $S \cup S'$ is a minimum GTDS of H. It is obvious that $S \cup S'$ is a GTDS for H. We next show that $S \cup S'$ is a minimum GTDS for H. Suppose that L is a GTDS for H with $|L| < |S \cup S'|$. Let $L_1 = L \cap V(G)$ and $L_2 = L \cap V(\overline{G})$. Then $|L_1| < |S|$ or $|L_2| < |S'|$. Without loss of generality assume that $|L_1| < |S|$. Since any vertex of G is adjacent to some vertex in L, we deduce that L_1 is a TDS for G. This is a contradiction, since $|L_1| < \gamma_t(G)$. So $S \cup S'$ is a minimum GTDS for H. On the other hand let T be a minimum GTDS for H. Let $T_1 = T \cap V(G)$ and $T_2 = T \cap V(\overline{G})$. Then T_1 and T_2 are TDS for G and $\overline{G}$, respectively. We show that T_1 and T_2 are minimum TDS for G and $\overline{G}$, respectively. Without loss of generality assume that T_1 is not a minimum TDS of G. So there is a minimum TDS such D in G with $|D| < |T_1|$. Similar to the above discussions $D \cup T_2$ is a GTDS for H with $|D \cup T_2| < |T| = \gamma_{gt}(H)$. This is a contradiction. Since the construction of the graph H from G can be perform in time that is polynomial in n (the number of vertices of G), so the above reduction is polynomial and so GTDS is NP-Complete.

4 Graphs with Small and Large Global Total Domination Number

In [8] graphs G of order n with $\gamma_{gt}(G) = n$ have been characterized. In this section we characterize graphs G of order n with $\gamma_{gt}(G) = n-1$. For this purpose we first characterize graphs G with $\gamma_{gt}(G) = 4$.

THEOREM 7. [8] For a graph G, $\gamma_{gt}(G) = n$ if and only if $G = P_4$, mK_2 or $\overline{mK_2}$ for some $m \geq 2$.

By Observation 1, part (3), we obtain the following.

LEMMA 8. For any graph G, $\gamma_{gt}(G) \geq 4$.

Let $\mathcal{H}$ be the class of all graphs G such that one of the following hold:

(1) G contains a path P_4 as an induced subgraph, and any vertex in G outside P_4 is adjacent to at least one and at most three vertices of P_4,

(2) G contains a cycle C_4 (or $\overline{C_4}$) as an induced subgraph, and any vertex in G outside C_4 (or $\overline{C_4}$) is adjacent to at least one and at most three vertices of C_4 (or $\overline{C_4}$).

LEMMA 9. For a graph G, $\gamma_{gt}(G) = 4$ if and only if $G \in \mathcal{H}$.

PROOF. It is obvious that if $G \in \mathcal{H}$ then the vertices of P_4, C_4 or $\overline{C_4}$ form a GTDS for G. Let $\gamma_{gt}(G) = 4$ and let S be a $\gamma_{gt}(G)$-set. If $G[S]$ is connected, then $G[S] = P_4$ or C_4. If $G[S]$ is disconnected, then $G[S] = \overline{C_4}$. Now since S is a TDS in both G and $\overline{G}$, every vertex in $G - S$ has at least one and at most three neighbors in S.

So henceforth in this section we consider graphs G with $\gamma_{gt}(G) \geq 5$. Let $\mathcal{E}$ be the class of all graphs G such that G satisfies one of the following:

(1) G is obtained from a graph $K \in \{P_4, C_4, \overline{C_4}\}$ by adding a vertex x and joining x to at least one and at most three vertices of K.

(2) $G = mK_2 + P_3$ or $mK_2 + C_3$ for some integer $m \geq 1$.

LEMMA 10. Let $G \in \mathcal{E}$ be a graph of order n. If H is obtained from G be adding a vertex y and joining y to at most $n-1$ vertices of G, then $\gamma_{gt}(H) < n$.

PROOF. Let $G \in \mathcal{E}$. Assume that G is obtained from a graph $K \in \{P_4, C_4, \overline{C_4}\}$ by adding a vertex x and joining x to at least one and at most three vertices of K. If y is not adjacent to all vertices of K, then $V(K)$ is a GTDS for H. So assume that y is adjacent to all vertices in K. Since $\overline{G}$ has no isolated vertex, y is not adjacent to x. It is straightforward to check all possibilities for G to see that there is a GTDS for H of cardinality 4. In all cases $\gamma_{gt}(H) < n$. Next assume that $G = mK_2 + P_3$ or $mK_2 + C_3$ for some integer $m \geq 1$. Let $V(P_3)$ or $V(C_3)$ be $\{v_1v_2v_3\}$, where v_2 is adjacent to both v_1 and v_3. If y is not adjacent to all of the vertices of $G - v_1$, then $V(G - v_1)$ is a GTDS for H. So assume that y is adjacent to all of the vertices of $G - v_1$. Then $V(G) \setminus \{v_3\}$ is a GTDS for H. Thus $\gamma_{gt}(H) < n$.

THEOREM 11. For a graph G, $\gamma_{gt}(G) = n - 1$ if and only if G or $\overline{G}$ belongs to $\mathcal{E}$.

PROOF. It is straightforward to see that for any graph G or $\overline{G}$ in $\mathcal{E}$, $\gamma_{gt}(G) = |V(G)| - 1$. Let G be a graph of order n and $\gamma_{gt}(G) = n - 1$. We employee induction on n to prove that $G \in \mathcal{E}$. For the basis step of induction $\gamma_{gt}(G) = 4$. By Lemma 9 , G or $\overline{G}$ is in $\mathcal{E}$. Assume that for any graph G' of order $n' < n$ and global total domination number $n' - 1$, G' or $\overline{G'}$ belongs to $\mathcal{E}$. Let S be a $\gamma_{gt}(G)$-set, and $x \notin S$. Let T be a $\gamma_{gt}(G - x)$-set. We show that $|T| = n - 1$. If $|T| < n - 2$, then there are at least two vertices $a, b \in V(G) \setminus (T \cup \{x\})$. If a is not adjacent to x, then $T \cup \{a\}$ is a GTDS for G of cardinality less than $n - 1$, a contradiction. So a is adjacent to x. Similarly b is adjacent to x. But $\overline{G}$ has no isolated vertex. So x is not adjacent to all of the vertices of T. Now $T \cup \{x\}$ is a GTDS for G of cardinality less than $n - 1$. This contradiction implies that $|T| \geq n - 2$. If $|T| = n - 2$, then by the inductive hypothesis $G - x \in \mathcal{E}$, and by Lemma 10 , $\gamma_{gt}(G) < n - 1$, a contradiction. We deduce that $|T| = n - 1$. By Theorem 7 , $G - x = P_4$, mK_2 or $\overline{mK_2}$ for some $m \geq 2$. We consider the following Cases 1–3.

Case 1. $G - x = P_4$. Then G is obtained from a path P_4 by adding a vertex x and joining x to at least one and at most three vertices of P_4, and so $G \in \mathcal{E}$.

Case 2. $G - x = mK_2$ for some $m \geq 2$. If $m = 2$, then G is obtained from the graph $2K_2 = \overline{C_4}$ by adding a vertex x and joining x to at least one and at most three vertices of $\overline{C_4}$, and so $G \in \mathcal{E}$. So we assume that $m \geq 3$. Then $|T| = 2m$. Since $\overline{G}$ has no isolated vertex, $V(G) \setminus N[x] \neq \emptyset$. Let $w \in V(G) \setminus N[x]$. If x has a neighbor in all components of $G - x$, then $\{x, v_1, v_2, ..., v_m, w\}$ is a GTDS of G, where v_i is a vertex in the i'th component of $G - x$. This is a contradiction. So x does not have a neighbor in at least one component of $G - x$. As a result G is disconnected. If x has a neighbor in at least two components of $G - x$, then $V(G) \setminus \{v_i, v_j\}$ is a GTDS for G, where v_i, v_j are two vertices from two different components of $G - x$ which are adjacent to x, a contradiction. Thus x is adjacent to some vertex in exactly one component K_2 of $G - x$. Consequently, $G = (m - 1)K_2 + P_3$ or $(m - 1)K_2 + C_3$.

Case 3. $G - x = \overline{mK_2}$. Then $\overline{G}$ satisfies Case 2.

CONJECTURE 12. Every graph G with $\gamma_{gt}(G) = k$ can be obtained from a graph H with $\gamma_{gt}(H) = k - 1$ by adding a new vertex and joining it to at least one and at most $|V(H)| - 1$ vertices of H.

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