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Large Solutions To Non-Monotone Quasilinear Elliptic Systems*

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Abstract

We present the existence of entire large positive radial solutions for the non-monotonic system

$$\begin{cases} \triangle_p u = m(|x|)g(v) & \text{on } \mathbf{R}^N, \\ \triangle_q v = n(|x|)f(u) & \text{on } \mathbf{R}^N, \end{cases}$$

where $N \geq 3$ and $p \geq q \geq 2$. The functions f and g satisfy a Keller-Osserman type condition while nonnegative functions m and n are required to satisfy some decay conditions. Furthermore, m and n are such that $\min\{m, n\}$ does not have compact support.

1 Introduction

In this paper, we consider the quasilinear elliptic system

$$\begin{cases} \triangle_p u = m(|x|)g(v) & \text{for } x \in \mathbf{R}^N, \\ \triangle_q v = n(|x|)f(u) & \text{for } x \in \mathbf{R}^N, \end{cases} \tag{1}$$

for $N \geq 3$, $p \geq q \geq 2$. We are concerned with existence of entire large solutions of (1), i.e. solutions such that $u(|x|) \longrightarrow \infty$ and $v(|x|) \longrightarrow \infty$ as $|x| \longrightarrow \infty$. Such problems arise in the theory of quasi-regular and quasi-conformal mappings as well as in the study of non-Newtonian fluids. In the latter case, the pair (p, q) is a characteristic of the medium. Media with $(p, q) > (2, 2)$ are called dilatant fluids and those with $(p, q) < (2, 2)$ are called pseudo-plastics. If $(p, q) = (2, 2)$, they are Newtonian fluids.

When $p = q = 2$, the system

$$\begin{cases} \triangle u = m(|x|)g(v) & \text{for } x \in \mathbf{R}^N, \\ \triangle v = n(|x|)f(u) & \text{for } x \in \mathbf{R}^N \end{cases} \tag{2}$$

have received much attention recently. We list here, for example, [3, 4, 8, 11–13, 15, 16] and the references therein.

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When $p = q = 2$, $m(|x|)g(v) = p(|x|)v^\alpha$, $n(|x|)f(u) = q(|x|)u^\beta$, system (2) becomes

$$\begin{cases} \triangle u = p(|x|)v^\alpha & \text{for } x \in \mathbf{R}^N, \\ \triangle v = q(|x|)u^\beta & \text{for } x \in \mathbf{R}^N, \end{cases} \tag{3}$$

for which existence results for entire positive solutions can be found in [17]. Lair and Wood established that all positive entire radial solutions of (3) are explosive provided that

$$\int_0^\infty tp(t)dt = \infty \text{ and } \int_0^\infty tq(t)dt = \infty.$$

If, on the other hand,

$$\int_0^\infty tp(t)dt < \infty \text{ and } \int_0^\infty tq(t)dt < \infty,$$

then all positive entire radial solutions of (3) are bounded. In addition, [17] shows the set of central values, $\{(u(0), v(0))\}$, such that the system has an entire solution is a closed, bounded, and convex subset of $R^+ \times R^+$.

Jesse D. Peterson and Aihua W. Wood [3], Cirstea and Radulescu [16] extended the above results to a large class of systems (2).

Carcia-Melian and Rossi [18] studied the existence and non-existence results for boundary blow-up solutions, which can be obtained to the elliptic system

$$\begin{cases} \triangle u = u^{m_1}v^{n_1} & \text{for } x \in \Omega \in \mathbf{R}^N, \\ \triangle v = u^{m_2}v^{n_2} & \text{for } x \in \Omega \in \mathbf{R}^N. \end{cases} \tag{4}$$

Wu and Yang [19] extended the above results to a class of systems

$$\begin{cases} \text{div}\left((|\nabla u|^{p-2}\nabla u)\right) = u^{m_1}v^{n_1} & \text{for } x \in \Omega, \\ \text{div}\left((|\nabla v|^{q-2}\nabla v)\right) = u^{m_2}v^{n_2} & \text{for } x \in \Omega. \end{cases}$$

In [20], author discussed the existence of positive entire solutions of the quasilinear elliptic system

$$\begin{cases} \text{div}(|\nabla u|^{p-2}\nabla u) = a(r)v^\alpha & \text{for } x \in \mathbf{R}^N, \\ \text{div}(|\nabla v|^{q-2}\nabla v) = b(r)u^\beta & \text{for } x \in \mathbf{R}^N, \end{cases} \tag{5}$$

where a and b satisfy $a(r) \le \frac{K_1}{r^\lambda}$ and $b(r) \le \frac{K_2}{r^\mu}$ for $r \ge r_0 > 0$, and K_1, K_2 are positive constants and

$$\lambda > \frac{\alpha}{q-1}(q-\mu) + p \text{ and } \mu > \frac{\beta}{p-1}(p-\lambda) + q,$$

then (5) has infinitely many spherically symmetric positive entire solutions.

In [6, 21], Yang showed the existence of entire explosive radial solutions for quasilinear elliptic system (1), where f and g are positive and non-decreasing functions on $(0, \infty)$ satisfying the Keller-Osserman condition. Other results can be seen in [5, 7, 10, 14].

A key feature common to all results known to us is that the functions f and g are required to be non-decreasing. This condition is necessary to construct monotonic sequences of functions converging to solutions of (1). The combination of the absence of a meaningful maximum principle for systems and a lack of monotonicity of the functions has been the major hurdle in understanding the solution structures for this type of system. This is reflected by the vacuum between results for systems and those for the corresponding single equations. Motivated by the results of the above cited papers, we further study the existence of large solutions for problem (1), the results of the semilinear problem are extended to the quasilinear ones. We can find the related results for $p = q = 2$ in [3]. This paper marks our first step toward filling this gap. In particular, we prove similar results for f and g being non-monotonic, rather "banded" between monotonic functions.

Given system (1), we define

$$G(t) := \min\left\{\inf_{t\le s}(f(s)), \inf_{t\le s}(g(s))\right\},$$

satisfying the following properties:

(G1) $G(0) = 0$,

(G2) $G(s) > 0$ for $s > 0$,

(G3) the Keller-Osserman condition

$$\int_1^\infty \left[\int_0^s G(t)dt\right]^{-\frac{1}{p}} ds < \infty.$$

Let

$$\psi(|x|) = \min\{m(|x|), n(|x|)\} \text{ and } \phi(|x|) = \max\{m(|x|), n(|x|)\}.$$

Notice $\psi(|x|) \le m(|x|) \le \phi(|x|)$ and $\psi(|x|) \le n(|x|) \le \phi(|x|)$. It is also clear that ψ and ϕ both satisfy

$$\int_0^\infty \left(t^{1-N}\int_0^t s^{N-1}\psi(s)ds\right)^{\frac{1}{p-1}} dt < \infty \text{ and } \int_0^\infty \left(t^{1-N}\int_0^t s^{N-1}\phi(s)ds\right)^{\frac{1}{q-1}} dt < \infty$$

when m and n satisfy

$$\int_0^\infty \left(t^{1-N}\int_0^t s^{N-1}m(s)ds\right)^{\frac{1}{p-1}} dt < \infty \text{ and } \int_0^\infty \left(t^{1-N}\int_0^t s^{N-1}n(s)ds\right)^{\frac{1}{q-1}} dt < \infty.$$

Given system (1), we define the following function

$$H(t) = \max\left\{\max_{0\le s\le t}(f(s)), \max_{0\le s\le t}(g(s))\right\}.$$

We can see that H and G are nonnegative, non-decreasing, and continuous functions satisfying $G(0) = H(0) = 0$ and $H(t) > 0$ when $t > 0$. Furthermore,

$$G(t) \le f(t) \le H(t) \text{ and } G(t) \le g(t) \le H(t).$$

Notice the function H necessarily satisfies the Keller-Osserman condition while G satisfies this requirement by hypothesis.

Inspired by [2, 3, 6, 24], we establish the following main theorems.

THEOREM 1. Suppose $m,\, n \in C(R^N)$ are nonnegative functions such that

$$\int_0^\infty \left[t^{1-N}\int_0^t s^{N-1}m(s)ds\right]^{\frac{1}{p-1}} dt < \infty,\ \int_0^\infty \left[t^{1-N}\int_0^t s^{N-1}n(s)ds\right]^{\frac{1}{q-1}} dt < \infty, \quad (6)$$

and $\min\{m, n\}$ does not have compact support. Suppose $f,\, g \in C[0, \infty)$ are locally Lipschitz continuous on $(0, \infty)$ such that $G(t)$ satisfies (G1), (G2), and the Keller-Osserman condition (G3). Then there are infinitely many entire positive solutions of system (1).

REMARK 1. If $N \geq 3$ and $N > p$, then condition (6) can be replaced by

$$0 < \int_1^\infty r^{\frac{1}{p-1}} m(r)^{\frac{1}{p-1}} dr < \infty \text{ if } 1 < p \leq 2, \tag{A}$$

$$0 < \int_1^\infty r^{\frac{(p-2)N+1}{p-1}} m(r) dr < \infty \text{ if } p \geq 2, \tag{B}$$

$$0 < \int_1^\infty r^{\frac{1}{q-1}} n(r)^{\frac{1}{q-1}} dr < \infty \text{ if } 1 < q \leq 2, \tag{C}$$

$$0 < \int_1^\infty r^{\frac{(q-2)N+1}{q-1}} n(r) dr < \infty \text{ if } q \geq 2. \tag{D}$$

We let

$$J(r) = \int_0^r \left[t^{1-N}\int_0^t s^{N-1}\psi(s)ds\right]^{\frac{1}{p-1}} dt.$$

In fact, if $1 < p \leq 2$, then

$$J(r) \leq C_1 + \int_1^r t^{\frac{1-N}{p-1}} \left[\int_0^t s^{N-1}\psi(s)ds\right]^{1/(p-1)} dt.$$

Using the assumption $N \geq 3$ in the computation of the first integral above and Jensen's inequality to estimate the last one, we obtain

$$J(r) \leq C_2 + C_3 \int_1^r t^{\frac{3-N-p}{p-1}} \int_1^t s^{\frac{N-1}{p-1}} \psi(s)^{\frac{1}{p-1}} ds dt.$$

Computing the above integral, we obtain

$$J(r) \leq C_2 + C_4 \int_1^r t^{\frac{1}{p-1}} \psi(t)^{\frac{1}{p-1}} dt.$$

Applying (A) in the integral above, we infer that $H_\infty = \lim_{r\to\infty} J(r) < \infty$. On the other hand, if $p \geq 2$, set

$$H(t) = \int_0^t s^{N-1}\psi(s)ds$$

and note that either, $H(t) \le 1$ for $t > 0$ or $H(t_0) = 1$ for some $t_0 > 0$. In the first case, $H^{\frac{1}{p-1}} \le 1$, and hence,

$$J(r) = \int_0^r t^{\frac{1-N}{p-1}} H(t)^{\frac{1}{p-1}} dt \le C_5 + \int_1^r t^{\frac{1-N}{p-1}} dt$$

so that $J(r)$ has a finite limit because $p < N$. In the second case, $H(s)^{\frac{1}{p-1}} \le H(s)$ for $s \ge s_0$ and hence,

$$J(r) \le C_6 + \int_1^r t^{\frac{1-N}{p-1}} \int_0^t s^{N-1}\psi(s) ds dt.$$

Estimating and integrating by parts, we obtain

$$\begin{aligned} J(r) &\le C_6 + \frac{p-1}{N-p}\int_0^1 t^{N-1}\psi(t)dt \\ &\quad + \frac{p-1}{N-p}\left[\int_1^r t^{\frac{(p-2)N+1}{p-1}}\psi(t)dt - r^{\frac{p-N}{p-1}}\int_0^r t^{N-1}\psi(t)dt\right] \\ &\le C_7 + C_8\int_1^r t^{\frac{(p-2)N+1}{p-1}}\psi(t)dt. \end{aligned}$$

By (B), we obtain that $H_\infty = \lim_{r\to\infty} J(r) < \infty$.

Using the notation $R^+ \equiv [0,\infty)$ and defining the set of central values

$$S = \{(a,b) \in R^+ \times R^+ : u(0) = a, v(0) = b\}$$

where (u,v) is an entire solution of (1), we have the following theorem.

THEOREM 2. Given the hypotheses of Theorem 1, the set S is a closed bounded subset of $R^+ \times R^+$.

Finally, defining the edge set

$$E = \{(a,b) \in \partial S : a > 0, b > 0\},$$

we have an existence theorem for entire large solutions of (1).

THEOREM 3. Given the hypotheses of Theorem 1, any entire positive radial solution of system (1) with central value $(u(0), v(0)) \in E$ is large.

LEMMA 1. Under the hypotheses of Theorem 1, for any central values $(c,d) \in R^+ \times R^+$ where $c > 0$, $d > 0$, system (1) has a solution in a finite ball of radius ρ centered at 0 (denoted by $B(0,\rho)) \subset R^N$.

PROOF. Note that radial solutions of (1) are solutions of the system

$$\begin{cases} (|u'^{p-2}u')' + \frac{N-1}{r}|u'^{p-2}u' = m(r)g(v), \\ (|v'^{q-2}v')' + \frac{N-1}{r}|v'^{q-2}v' = n(r)f(u). \end{cases} \tag{7}$$

Thus, solutions of (1) are simply fixed points of the operator $T : C[0,\infty)\times C[0,\infty) \longrightarrow C[0,\infty)\times C[0,\infty)$ given by

$$T(u,v) = (\widehat{u},\widehat{v}) = \begin{cases} c + \int_0^r (t^{1-N}\int_0^t s^{N-1}m(s)g(v)ds)^{\frac{1}{p-1}}dt & \text{for } 0\le r<\rho, \\ d + \int_0^r (t^{1-N}\int_0^t s^{N-1}n(s)f(u)ds)^{\frac{1}{q-1}}dt & \text{for } 0\le r<\rho. \end{cases}$$

To prove such a fixed point exists, we will apply a version of Schauder's Fixed Point Theorem. We consider the Banach space $C[0,\rho]\times C[0,\rho]$, where $\rho>0$ with norm

$$\|(u,v)\| = \max\{\|u\|, \|v\|\}.$$

Define the subset

$$X = \{(u,v)\in C[0,\rho]\times C[0,\rho] : \|(u,v)-(c,d)\|_\infty \le \min\{c,d\}\}.$$

Clearly X is closed, bounded, and convex in $C[0,\rho]\times C[0,\rho]$. Further, T is a compact operator (refer to [1]). If we can show T maps X into X, then Schauder's Fixed Point Theorem will guarantee we have a solution. Let $(u,v)\in X$ be arbitrary and $\|(u,v)\|_\infty \le Q$. We have the estimate

$$\begin{aligned} & \int_0^r \left[t^{1-N}\int_0^t s^{N-1}m(s)g(v(s))ds\right]^{\frac{1}{p-1}} dt \\ \le\ & \int_0^r \left[t^{1-N}\int_0^t s^{N-1}\phi(s)H(v(s))ds\right]^{\frac{1}{p-1}} dt \\ \le\ & H^{\frac{1}{p-1}}(v(r))\int_0^r \left[t^{1-N}\int_0^t s^{N-1}\phi(s)ds\right]^{\frac{1}{p-1}} dt \\ \le\ & H^{\frac{1}{p-1}}(Q)\rho\int_0^\rho \phi^{\frac{1}{p-1}}(s)ds. \end{aligned}$$

Similarly, we may show

$$\int_0^r \left[t^{1-N}\int_0^t s^{N-1}n(s)f(u(s))ds\right]^{\frac{1}{q-1}} dt \le H^{\frac{1}{q-1}}(Q)\rho\int_0^\rho \phi^{\frac{1}{q-1}}(s)ds.$$

Then choose $\rho>0$ small enough so that

$$\max\left\{H^{\frac{1}{p-1}}(Q)\rho\int_0^\rho \phi^{\frac{1}{p-1}}(s)ds, H^{\frac{1}{q-1}}(Q)\rho\int_0^\rho \phi^{\frac{1}{q-1}}(s)ds\right\} < \min\{c,d\}.$$

Doing so, we then have $T(u(r),v(r)) = (\widehat{u}(r),\widehat{v}(r))$, where

$$c \le \widehat{u}(r) = c + \int_0^r \left[t^{1-N}\int_0^t s^{N-1}m(s)g(v(s))ds\right]^{\frac{1}{p-1}} dt \le c + \min\{c,d\}$$

and similarly, $d \le \widehat{v}(r) \le d + \min\{c,d\}$ for all $0\le r\le\rho$. Thus $(\widehat{u}(r),\widehat{v}(r))\in X$. Therefore a fixed point of T, a solution to (1) in the ball $B(0,\rho)$, exists.

From [9, 22, 23], we give the following lemma

LEMMA 2. (Weak comparison principle) Let Ω be a bounded domain in $R^N (N \geq 2)$ with smooth boundary $\partial\Omega$ and $\theta : (0, \infty) \to (0, \infty)$ is continuous and nondecreasing. Let $u_1,\ u_2 \in W^{1,p}(\Omega)$ satisfy

$$\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \nabla \psi dx + \int_\Omega \theta(u_1)\psi dx \leq \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \nabla \psi dx + \int_\Omega \theta(u_2)\psi dx,$$

for all non-negative $\psi \in W_0^{1,p}(\Omega)$ satisfy $u_1 \leq u_2$ on $\partial\Omega$. Then $u_1 \leq u_2$ on Ω.

2 Proof of Theorem 1

Let $T : C[0, \infty) \times C[0, \infty) \longrightarrow C[0, \infty) \times C[0, \infty)$ be the same as in Lemma 1. We have shown that T has a fixed point in $C([0, \rho])$. Next, we prove that T has a fixed point in $C([0, \infty))$. Define

$$(\overline{u_k}, \overline{v_k}) = T(u_k, v_k) = \begin{cases} c + \int_0^r \left[t^{1-N} \int_0^t s^{N-1} m(s) g(v_k) ds\right]^{\frac{1}{p-1}} dt, \\ d + \int_0^r \left[t^{1-N} \int_0^t s^{N-1} n(s) f(u_k) ds\right]^{\frac{1}{q-1}} dt. \end{cases}$$

Just as in the proof of Lemma 1, we examine $(\overline{u_k},\ \overline{v_k})$ over $[0, 1] \times [0, 1]$ and then we have $\{\overline{u_k}\}$ and $\{\overline{v_k}\}$ are each uniformly bounded on $[0, 1]$. The nonnegative sequences of derivatives $\{\overline{u_k}'\}$ and $\{\overline{v_k}'\}$ are also bounded on $[0, 1]$ implying the sequence has a uniformly convergent subsequence. Call this subsequence $\{(w_k^1, z_k^1)\}$, and let $(w_k^1, z_k^1) \longrightarrow (\widehat{u_1}, \widehat{v_1})$ as $k \longrightarrow \infty$, uniformly on $[0, 1] \times [0, 1]$. Notice that $(\widehat{u_1}, \widehat{v_1})$ is a solution of (1) in $B(0, 1)$ with central value $(a_0, b_0) \in \partial S$. Likewise, the subsequences $\{w_k^1\}$ and $\{z_k^1\}$ are each uniformly bounded and equicontinuous on $[0, 2]$, so there exists a subsequence $\{(w_k^2, z_k^2)\}$ of $\{(w_k^1, z_k^1)\}$ such that $(w_k^2, z_k^2) \longrightarrow (\widehat{u_2}, \widehat{v_2})$ as $k \longrightarrow \infty$, uniformly on $[0, 2] \times [0, 2]$. Since $\{(w_k^2, z_k^2)\} \subseteq \{(w_k^1, z_k^1)\}$, we see $(\widehat{u_2}, \widehat{v_2}) = (\widehat{u_1}, \widehat{v_1})$ on $[0, 1] \times [0, 1]$. Continuing, we obtain a sequence $\{(\widehat{u_k}, \widehat{v_k})\}$, each a solution of (1) in $B(0, k)$ with central value $(a_0, b_0) \in \partial S$, and each solution satisfies

$$\begin{aligned} (\widehat{u_k}, \widehat{v_k}) &= (\widehat{u_1}, \widehat{v_1}) \quad \text{for } r \in [0, 1], \\ (\widehat{u_k}, \widehat{v_k}) &= (\widehat{u_2}, \widehat{v_2}) \quad \text{for } r \in [0, 2], \\ &\vdots \\ (\widehat{u_k}, \widehat{v_k}) &= (\widehat{u_{k-1}}, \widehat{v_{k-1}}) \quad \text{for } r \in [0, k-1]. \end{aligned}$$

Thus, $(\widehat{u_k}, \widehat{v_k})$ converges to (u, v) where

$$(u(r), v(r)) = (\widehat{u_k}, \widehat{v_k}) \quad \text{if } 0 \leq r \leq k.$$

This convergence is uniform on any bounded set. Thus $(u, v) \in C[0, \infty) \times C[0, \infty)$ and satisfies $(Tu, Tv) = (u, v)$. The proof of Theorem 1 is completed.

3 Proof of Theorem 2

To prove that S is bounded, we first assume to the contrary that S is not bounded. Therefore, since $[0,a]\times[0,b]\subset S$ whenever $(a,b)\in S$, we must have either $[0,\infty)\times\{0\}\subset S$ or $\{0\}\times[0,\infty)\subset S$. Let h be a positive radial solution of

$$\begin{cases} \triangle_p h = \psi(r)G(\frac{h}{2}) & 0\le r<1, \\ \lim_{r\longrightarrow 1^-} h(r)=\infty. \end{cases} \tag{8}$$

(see [2] for the proof of existence). Let (u,v) be any solution which exists by hypotheses, to the system

$$\begin{cases} u(r) = a+\int_0^r \left[t^{1-N}\int_0^t s^{N-1}m(s)g(v(s))ds\right]^{\frac{1}{p-1}} dt & \text{for } r>0, \\ v(r) = b+\int_0^r \left[t^{1-N}\int_0^t s^{N-1}n(s)f(u(s))ds\right]^{\frac{1}{q-1}} dt & \text{for } r>0 \end{cases} \tag{9}$$

with $a>h(0)$ and $b=0$. Without loss of generality, we assume that $a\ge 1$. We now show that $h\le u+v$ for all $r\ge 0$, which, if proven, will contradict with the fact that $u+v$ exists for all $r\ge 0$. Clearly, $h(0)<a\le u(0)+v(0)$. Thus, there exists $\epsilon>0$ such that $h(r)<u(r)+v(r)$ for all $r\in[0,\epsilon)$. Let

$$R_0=\sup\{\epsilon>0 | h(r)<u(r)+v(r) \text{ for } r\in[0,\epsilon)\}.$$

We claim that $R_0=1$. Indeed, suppose that $R_0<1$. Since $h(r)<u(r)+v(r)$ in $[0,R_0)$,

$$g(v(r))\ge G(v(r))\ge G(\frac{u(r)+v(r)}{2}) \text{ if } v(r)\ge u(r),$$

$$f(u(r))\ge G(u(r))\ge G(\frac{u(r)+v(r)}{2}) \text{ if } u(r)\ge v(r),$$

and elementary estimates, we observe that

$$\begin{aligned} & h(0)+\int_0^{R_0}\left[t^{1-N}\int_0^t s^{N-1}\psi(s)G(\frac{h(s)}{2})ds\right]^{\frac{1}{p-1}} dt \\ < \ & h(0)+\int_0^{R_0}\left[t^{1-N}\int_0^t s^{N-1}\psi(s)G(\frac{u(s)+v(s)}{2})ds\right]^{\frac{1}{p-1}} dt \\ < \ & h(0)+\int_0^{R_0}\left[t^{1-N}\int_0^t s^{N-1}\psi(s)(g(v)+f(u))ds\right]^{\frac{1}{p-1}} dt \\ < \ & h(0)+\int_0^{R_0}\left[t^{1-N}\int_0^t s^{N-1}\left(\psi(s)g(v)+\psi(s)f(u)\right)\right]^{\frac{1}{p-1}} dt \\ < \ & h(0)+\int_0^{R_0}\left[t^{1-N}\int_0^t s^{N-1}\left(m(s)g(v)+n(s)f(u)\right)\right]^{\frac{1}{p-1}} dt \\ < \ & h(0)+\int_0^{R_0}\left[t^{1-N}\int_0^t s^{N-1}m(s)g(v)\right]^{\frac{1}{p-1}} dt \end{aligned}$$

$$
\begin{aligned}
& + \int_0^{R_0} \left[t^{1-N} \int_0^t s^{N-1} n(s) f(u) \right]^{\frac{1}{q-1}} dt \\
< \; & u(0) + v(0) + \int_0^{R_0} \left[t^{1-N} \int_0^t s^{N-1} m(s) g(v) \right]^{\frac{1}{p-1}} dt \\
& + \int_0^{R_0} \left[t^{1-N} \int_0^t s^{N-1} n(s) f(u) \right]^{\frac{1}{q-1}} dt \\
< \; & u(R_0) + v(R_0).
\end{aligned}
$$

Thus, since $h(R_0) < u(R_0) + v(R_0)$, there exists $\delta > 0$ such that $h(r) < u(r) + v(r)$ in $[0, R_0 + \delta)$. This contradicts with the fact that R_0 is a superemum. Thus, $R_0 = 1$, establishing the boundness of the set S.

To show S is closed, we will show the set contains its boundary. Let $h_1(r)$ and $h_2(r)$ be positive solutions of

$$
\begin{cases} \triangle_p h_1 = \psi(r) G(h_1) & \text{for } 0 \le r < R, \\ \lim_{r \longrightarrow R^-} h_1(r) = \infty, \end{cases} \tag{10}
$$

and

$$
\begin{cases} \triangle_q h_2 = \psi(r) G(h_2) & \text{for } 0 \le r < R, \\ \lim_{r \longrightarrow R^-} h_2(r) = \infty. \end{cases} \tag{11}
$$

where R is an arbitrary positive real number.

It is easy to show that by Lemma 2, $u \le h_1$ and $v \le h_2$. Thus, $u + v \le h_1 + h_2$ in $[0, R]$. Let $h_1 + h_2 = \eta$. Then $u + v \le \eta$ in $[0, R]$. Now, let $(a_0, b_0) \in \partial S$ and consider a positive, increasing sequence $\{R_k\}$ such that $R_k \longrightarrow \infty$ as $k \longrightarrow \infty$, and $\psi(|x|) > 0$ for $|x| = R_k$. We can find a sequence, since ψ is radial and does not have compact support, and then we have $B((a_0, b_0), 1/R_k) \cap S \ne \varnothing$. For each $k \ge 1$, we denote the arbitrary point $(a_0^k, b_0^k) \in B((a_0, b_0), 1/R_k) \cap S$. Notice that $(a_0^k, b_0^k) \longrightarrow (a_0, b_0)$ as $k \longrightarrow \infty$. We define the sequence

$$
(u_k, v_k) = \begin{cases} a_0^k + \int_0^r \left[t^{1-N} \int_0^t s^{N-1} m(s) g(v_k(s)) ds \right]^{\frac{1}{p-1}} dt & \text{if } r \ge 0, \\ b_0^k + \int_0^r \left[t^{1-N} \int_0^t s^{N-1} n(s) f(u_k(s)) ds \right]^{\frac{1}{q-1}} dt & \text{if } r \ge 0, \end{cases}
$$

where each (u_k, v_k) is an entire solution of (1). These solutions exist since each $(a_0^k, b_0^k) \in S$ and each $u_k + v_k \le \eta$. Examining (u_k, v_k) over $[0,1] \times [0,1]$, we have $u_k + v_k \le \eta(1) < \infty$ for $0 \le r \le 1$. Thus, $\{u_k\}$ and $\{v_k\}$ are each uniformly bounded on $[0,1]$. The nonnegative sequences of derivatives $\{u_k'\}$ and $\{v_k'\}$ are also bounded on $[0,1]$ implying the sequence has a uniformly convergent subsequence. Call this subsequence $\{(w_k^1, z_k^1)\}$, and let $(w_k^1, z_k^1) \longrightarrow (\widehat{u_1}, \widehat{v_1})$ as $k \longrightarrow \infty$, uniformly on $[0,1] \times [0,1]$. Notice that $(\widehat{u_1}, \widehat{v_1})$ is a solution to (1) in $B(0,1)$ with central value $(a_0, b_0) \in \partial S$. Likewise, the subsequences $\{w_k^1\}$ and $\{z_k^1\}$ are each uniformly bounded and equicontinuous on $[0,2]$, so there exists a subsequence $\{(w_k^2, z_k^2)\}$ of $\{(w_k^1, z_k^1)\}$ such that $(w_k^2, z_k^2) \longrightarrow (\widehat{u_2}, \widehat{v_2})$ uniformly on $[0,2] \times [0,2]$ as $k \longrightarrow \infty$. Since $\{(w_k^2, z_k^2)\} \subseteq \{(w_k^1, z_k^1)\}$, we see $(\widehat{u_2}, \widehat{v_2}) = (\widehat{u_1}, \widehat{v_1})$ on $[0,1] \times [0,1]$. Continuing, we obtain a sequence $\{(\widehat{u_k}, \widehat{v_k})\}$, each a solution to (1) in $B(0,k)$ with central value $(a_0, b_0) \in \partial S$, and each solution satisfies

$$\begin{aligned}
(\widehat{u_k}, \widehat{v_k}) &= (\widehat{u_1}, \widehat{v_1}) \quad \text{for } r \in [0,1], \\
(\widehat{u_k}, \widehat{v_k}) &= (\widehat{u_2}, \widehat{v_2}) \quad \text{for } r \in [0,2], \\
&\vdots \\
(\widehat{u_k}, \widehat{v_k}) &= (\widehat{u_{k-1}}, \widehat{v_{k-1}}) \quad \text{for } r \in [0,k-1].
\end{aligned}$$

Thus, $(\widehat{u_k}, \widehat{v_k})$ converges to (u,v) where

$$(u(r), v(r)) = (\widehat{u_k}, \widehat{v_k}) \quad \text{for } 0 \leq r \leq k.$$

This convergence is uniform on bounded sets, and thus (u,v) is an entire solution with central value $(a_0, b_0) \in \partial S$. We consider that $\partial S \subset S$, implying S is closed.

4 Proof of Theorem 3

Let u_n and v_n be defined as positive solutions of

$$\begin{cases} u_n = u(0) + \frac{1}{n} + \int_0^r \left[t^{1-N} \int_0^t s^{N-1} m(s) g(v_n(s)) ds \right]^{\frac{1}{p-1}} dt, \\ v_n = v(0) + \frac{1}{n} + \int_0^r \left[t^{1-N} \int_0^t s^{N-1} n(s) f(u_n(s)) ds \right]^{\frac{1}{q-1}} dt \end{cases} \tag{12}$$

where $(u(0), v(0)) \in E$. We denote that $u_n'(r) \geq 0$ and $v_n'(r) \geq 0$. Also, since $(u(0) + \frac{1}{n}, v(0) + \frac{1}{n}) \notin S$, for each $n = 1, 2, 3, ...,$ there exists $R_n < \infty$ such that

$$\lim_{r \longrightarrow R_n^-} u_n(r) = \infty, \quad \lim_{r \longrightarrow R_n^-} v_n(r) = \infty, \text{ and } R_1 \leq R_2 \leq R_3 \leq \cdots.$$

From (12) and the fact that $v_n'(r) \geq 0$, we get

$$\begin{aligned}
v_n(r) &\leq v(0) + \frac{1}{n} + H^{\frac{1}{q-1}}(u_n(r)) \int_0^\infty \left[t^{1-N} \int_0^t s^{N-1} n(s) ds \right]^{\frac{1}{q-1}} dt \\
&\leq C_1 u_n(r) + C_2 H^{\frac{1}{q-1}}(u_n(r)),
\end{aligned}$$

where value C_1 is any upper bound on $\frac{v(0)+\frac{1}{n}}{u(0)+\frac{1}{n}}$ and

$$C_2 = \int_0^\infty \left[t^{1-N} \int_0^t s^{N-1} n(s) ds \right]^{\frac{1}{q-1}} dt < \infty.$$

Next, we define $h(t) = H(C_1 t + C_2 H^{\frac{1}{q-1}}(t))$. It is an easy matter to show that $h(0) = 0$, $h(s) > 0$ for $s > 0$, and h satisfies the Keller-Osserman condition

$$\int_1^\infty \left[\int_0^s h(t) dt \right]^{-\frac{1}{p}} ds < \infty.$$

Define

$$F(s) = \int_s^\infty \frac{dt}{h^{\frac{1}{p-1}}},$$

which is well defined for $s > 0$. Notice that

$$F'(s) = -\frac{1}{h^{\frac{1}{p-1}}} < 0 \text{ and } F''(s) = \frac{h'(s)}{(p-1)h^{\frac{p}{p-1}}} > 0.$$

We now have

$$\begin{aligned} \operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) &= m(r)g(v_n(r)) \le m(r)H(v_n(r)) \\ &\le m(r)H\left[C_1 u_n(r) + C_2 H^{\frac{1}{q-1}}(u_n(r))\right] \\ &= m(r)h(u_n(r)). \end{aligned}$$

Then, we may calculate

$$\begin{aligned} & \operatorname{div}(|\nabla F(u_n)|^{p-2}\nabla F(u_n)) \\ = \;& -|F'(u_n)|^{p-1}\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) + (p-1)|\nabla u_n|^p|F'(u_n)|^{p-2}F''(u_n) \\ \ge \;& -|F'(u_n)|^{p-1}\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) \ge -|F'(u_n)|^{p-1}m(r)h(u_n) \\ = \;& -\left(\frac{1}{h^{\frac{1}{p-1}}(u_n)}\right)^{p-1} m(r)h(u_n) = -m(r). \end{aligned}$$

Hence,

$$\operatorname{div}(|\nabla F(u_n)|^{p-2}\nabla F(u_n)) \ge -m(r).$$

Rewriting the Laplacian in radial form and multiplying by r^{N-1}, we obtain

$$(r^{N-1}|F'(u_n)|^{p-2}F'(u_n))'^{N-1}m(r).$$

Integrating over $[0, r]$, where $0 < r < R_n$, we get

$$\frac{d}{dr}F(u_n) \ge -\left(r^{1-N}\int_0^r s^{N-1}m(s)ds\right)^{\frac{1}{p-1}}.$$

Next, we integrate over $[r, R_n]$. Note that $F(u_n(r)) \longrightarrow 0$ as $r \longrightarrow R_n^-$. This integration yields,

$$-F(u_n(r)) \ge -\int_r^{R_n}\left[t^{1-N}\int_0^r s^{N-1}m(s)ds\right]^{\frac{1}{p-1}} dt.$$

That is,

$$F(u_n(r)) \le \int_r^{R_n}\left[t^{1-N}\int_0^r s^{N-1}m(s)ds\right]^{\frac{1}{p-1}} dt.$$

Since $F'(s) < 0$ for $s > 0$, we have

$$u_n(r) \ge F^{-1}\left(\int_r^{R_n}\left[t^{1-N}\int_0^r s^{N-1}m(s)ds\right]^{\frac{1}{p-1}} dt\right).$$

Now, we let $n \longrightarrow \infty$, so $R_n \longrightarrow R$, and $u_n \longrightarrow u$, we have

$$F^{-1}\left(\int_r^R \left[t^{1-N}\int_0^r s^{N-1}m(s)ds\right]^{\frac{1}{p-1}} dt\right) \leq u(r).$$

Finally, let $r \longrightarrow R$, we have

$$\lim_{r\longrightarrow R} F^{-1}\left(\int_r^R \left[t^{1-N}\int_0^r s^{N-1}m(s)ds\right]^{\frac{1}{p-1}} dt\right) = \infty \leq \lim_{r\longrightarrow R} u(r).$$

However, $u(|x|)$ and $v(|x|)$ have a central value $(u(0), v(0)) \in E$ and entire. Thus $R = \infty$, and our proof is complete.

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