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Some Inequalities For Complete Elliptic Integrals*

Li Yin[†], Feng Qi[‡]

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Abstract

In this paper, by using the Lupaş integral inequality, the authors find some new inequalities for the complete elliptic integrals of the first and second kinds. These results improve some known inequalities.

1 Introduction

Legendre's complete elliptic integrals of the first and second kind are defined for real numbers $0 < r < 1$ by

$$\kappa(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2 t}} \,\mathrm{d}\, t = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-r^2t^2)}} \,\mathrm{d}\, t \tag{1}$$

and

$$\varepsilon(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2 t} \,\mathrm{d}\, t = \int_0^1 \sqrt{\frac{1-r^2t^2}{1-t^2}} \,\mathrm{d}\, t \tag{2}$$

respectively. They can also defined by

$$\kappa(r,s) = \int_0^{\pi/2} \frac{1}{\sqrt{r^2 \cos^2 t + s^2 \sin^2 t}} \,\mathrm{d}\, t \tag{3}$$

and

$$\varepsilon(r,s) = \int_0^{\pi/2} \sqrt{r^2 \cos^2 t + s^2 \sin^2 t} \,\mathrm{d}\, t. \tag{4}$$

Let $r' = \sqrt{1-r^2}$. We often denote

$$\kappa'(r) = \kappa(r') \quad \text{and} \quad \varepsilon'(r) = \varepsilon(r').$$

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[†]Department of Mathematics, Binzhou University, Binzhou City, Shandong Province, 256603, China

[‡]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region, 028043, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China; Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, 454010, Henan Province, 454010, China

These integrals are special cases of the Gauss hypergeometric function

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!},$$

where $(a,n)=\prod_{k=0}^{n-1}(a+k)$. Indeed, we have

$$\kappa(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)\quad\text{and}\quad\varepsilon(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right).$$

Recently, some bounds for $\varepsilon(r)$ and $\kappa(r)$ were discovered in the paper [6]. For example, Theorem 1 in [6] states that, for $0<r<1$,

$$\frac{\pi}{2}-\frac{1}{2}\ln\frac{(1+r)^{1-r}}{(1-r)^{1+r}}<\varepsilon(r)<\frac{\pi-1}{2}+\frac{1-r^2}{4r}\ln\frac{1+r}{1-r}. \tag{5}$$

For more information on inequalities of complete elliptic integrals, please refer to [1, 2, 3, 7, 8] and a short survey in [9, pp. 40–46].

Motivated by the double inequality (5), some estimates for $\varepsilon(r)$ in terms of rational functions of the arithmetic, geometric, and roots square means were obtained in [5, 10, 11].

The aim of this paper is to establish some new inequalities for the complete elliptic integrals.

2 A Lemma

In order to prove our main results, the following lemma is necessary.

LEMMA 2.1. If $f',g'\in L_2[a,b]$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)g(t)\,\mathrm{d}\,t-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}\,t\frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}\,t\right|\le\frac{b-a}{\pi^2}\|f'\|_2\,\|g'\|_2, \tag{6}$$

where

$$\|f'\|_2=\left(\int_a^b\left|f'^2\right|\mathrm{d}\,t\right)^{1/2}\quad\text{and}\quad\|g'\|_2=\left(\int_a^b\left|g'^2\right|\mathrm{d}\,t\right)^{1/2}.$$

The inequality (6) is called the Lupaş integral inequality, see [4, p. 57].

3 Main Results

Now we are in a position to find some inequalities for complete elliptic integrals.

THEOREM 3.1. For $r\in(0,1)$, we have

$$\frac{\pi\sqrt{6+2\sqrt{1-r^2}-3r^2}}{4\sqrt{2}}\le\varepsilon(r)\le\frac{\pi\sqrt{10-2\sqrt{1-r^2}-5r^2}}{4\sqrt{2}}. \tag{7}$$

PROOF. Taking

$$f(t)=g(t)=\sqrt{1-r^2\sin^2 t}$$

and letting $a=0$ and $b=\frac{\pi}{2}$ in the inequality (6) yield

$$\begin{aligned}
&\left|\frac{2}{\pi}\int_0^{\pi/2}\bigl(1-r^2\sin^2 t\bigr)\,\mathrm{d}\,t-\frac{4}{\pi^2}\varepsilon^2(r)\right|\\
=&\left|\frac{4}{\pi^2}\varepsilon^2(r)-\frac{2-r^2}{2}\right|\le\frac{1}{2\pi}\int_0^{\pi/2}\frac{r^4\sin^2 t\cos^2 t}{1-r^2\sin^2 t}\,\mathrm{d}\,t\\
=&\frac{1}{2\pi}\int_0^{\pi/2}r^4\sin^2 t\cos^2 t\sum_{n=0}^{\infty}r^{2n}\sin^{2n} t\,\mathrm{d}\,t\\
=&\frac{1}{2\pi}\sum_{n=0}^{\infty}\int_0^{\pi/2}r^{2n+4}\left(\sin^{2n+2} t-\sin^{2n+4} t\right)\mathrm{d}\,t=\frac{1}{4}h(r),
\end{aligned}\tag{8}$$

where we use

$$\int_0^{\pi/2}\sin^{2i} t\,\mathrm{d}\,t=\frac{\pi}{2}\frac{(2i-1)!!}{(2i)!!}\tag{9}$$

for $i\in\mathbb{N}$, and

$$h(r)=\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n+2)!!}\frac{r^{2n+4}}{2n+4}.$$

A direct calculation yields

$$h'(r)=\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n+2)!!}r^{2n+3}=r\sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n+2)!!}r^{2n+2}=r\left(\frac{1}{\sqrt{1-r^2}}-1\right),$$

where we use

$$\frac{1}{\sqrt{1-t^2}}=\sum_{n=0}^{\infty}\frac{(2i-1)!!}{(2i)!!}t^{2n},\quad |t|<1.\tag{10}$$

Hence, we have

$$h(r)=h(0)+\int_0^r h'(r)\,\mathrm{d}\,r=1-\sqrt{1-r^2}-\frac{r^2}{2}.\tag{11}$$

Substituting this equality into (8) gives

$$\left|\frac{4}{\pi^2}\varepsilon^2(r)-\frac{2-r^2}{2}\right|\le\frac{1}{4}-\frac{\sqrt{1-r^2}}{4}-\frac{r^2}{8}.$$

This means the double inequality (7).

REMARK 3.1. By the well-known software MATHEMATICA, we can show that

(i) the left-hand side inequality in (7) refines the corresponding one in (5);

(ii) the right-hand side inequalities in (7) and (5) are not contained in each other;

(iii) when $r \in [\frac{1}{4}, \frac{3}{4}]$, the right-hand side inequality in (7) is better than the corresponding one in (5).

THEOREM 3.2. We have that

$$\kappa(r) \le \frac{\pi\sqrt{32(1-r^2)+r^4}}{8\sqrt{2}\sqrt[4]{(1-r^2)^3}} \text{ for } r \in (0,1) \tag{12}$$

and

$$\kappa(r) \ge \frac{\pi\sqrt{32(1-r^2)-r^4}}{8\sqrt{2}\sqrt[4]{(1-r^2)^3}} \text{ for } x \in \left(0, 2\sqrt{3\sqrt{2}-4}\right).$$

PROOF. Taking

$$f(t) = g(t) = \frac{1}{\sqrt{1-r^2\sin^2 t}}$$

and letting $a = 0$ and $b = \frac{\pi}{2}$ in the inequality (6) lead to

$$\begin{aligned}
&\left|\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{1-r^2\sin^2 t}\,\mathrm{d}\,t - \frac{4}{\pi^2}\kappa^2(r)\right| \\
= &\left|\frac{4}{\pi^2}\kappa^2(r) - \frac{2}{\pi}\int_0^{\pi/2}\sum_{n=0}^{\infty} r^{2n}\sin^{2n} t\,\mathrm{d}\,t\right| \\
= &\left|\frac{4}{\pi^2}\kappa^2(r) - \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}r^{2n}\right| = \left|\frac{4}{\pi^2}\kappa^2(r) - \frac{1}{\sqrt{1-r^2}}\right| \\
\le &\ \frac{1}{2\pi}\int_0^{\pi/2}\frac{r^4\sin^2 t\cos^2 t}{\left(1-r^2\sin^2 t\right)^3}\,\mathrm{d}\,t \\
= &\ \frac{1}{4\pi}\int_0^{\pi/2} r^4\sin^2 t\cos^2 t\sum_{n=0}^{\infty}(n+2)(n+1)r^{2n}\sin^{2n} t\,\mathrm{d}\,t \\
= &\ \frac{1}{8}\sum_{n=0}^{\infty}(n+2)(n+1)r^{2n+4}\frac{(2n+1)!!}{(2n+2)!!}\frac{1}{2n+4} = \frac{r^3}{32}p(r),
\end{aligned}$$

where

$$p(r) = \sum_{n=0}^{\infty}\frac{(2n+2)(2n+1)!!}{(2n+2)!!}r^{2n+1}$$

satisfies $p(0) = 0$,

$$\int_0^r p(r)\,\mathrm{d}\,r = \sum_{n=0}^{\infty}\frac{(2n+1)!!}{(2n+2)!!}r^{2n+2} = \frac{1}{\sqrt{1-r^2}} - 1,$$

and so

$$p(r) = \frac{r}{\sqrt{(1-r^2)^3}}. \tag{13}$$

Consequently, we find

$$\left|\frac{4}{\pi^2}\kappa^2(r)-\frac{1}{\sqrt{1-r^2}}\right| \le \frac{r^4}{32\sqrt{(1-r^2)^3}}. \tag{14}$$

The double inequality (12) follows.

THEOREM 3.3. For $r>0$ and $s>0$, we have

$$\frac{\pi}{8}\sqrt{\frac{8rs(r^2+s^2)-(s^2-r^2)^2}{rs}} \le \varepsilon(r,s) \le \frac{\pi}{8}\sqrt{\frac{8rs(r^2+s^2)+(s^2-r^2)^2}{rs}}. \tag{15}$$

PROOF. Taking

$$f(t)=g(t)=\sqrt{r^2\cos^2 t+s^2\sin^2 t}$$

and letting $a=0$ and $b=\frac{\pi}{2}$ in the inequality (6) reveal

$$\begin{aligned}
&\left|\frac{2}{\pi}\int_0^{\pi/2}\left(r^2\cos^2 t+s^2\sin^2 t\right)\mathrm{d}\,t-\frac{4}{\pi^2}\varepsilon^2(r,s)\right| \\
={}& \left|\frac{4}{\pi^2}\varepsilon^2(r,s)-\frac{r^2+s^2}{2}\right| \le \frac{1}{2\pi}\int_0^{\pi/2}\frac{(s^2-r^2)^2\sin^2 t\cos^2 t}{r^2\cos^2 t+s^2\sin^2 t}\mathrm{d}\,t \\
\le{}& \left[\frac{(s^2-r^2)^2}{2\pi}\right]\frac{1}{4}\int_0^{\pi/2}\frac{1}{r^2\cos^2 t+s^2\sin^2 t}\mathrm{d}\,t \\
={}& \left[\frac{(s^2-r^2)^2}{8\pi}\right]\frac{1}{rs}\arctan\left(\frac{s}{r}\tan t\Big|_0^{\pi/2}\right)=\frac{(s^2-r^2)^2}{16rs}.
\end{aligned}$$

This means the double inequality (15).

THEOREM 3.4. For $s>r>0$, we have

$$\left|\frac{4}{\pi^2}\kappa^2(r,s)-\frac{1}{\pi rs}\right| \le \frac{s^2-r^2}{32rs}\left(\frac{1}{s^2}+\frac{1}{r^2}\right). \tag{16}$$

PROOF. Taking

$$f(t)=g(t)=\frac{1}{\sqrt{r^2\cos^2 t+s^2\sin^2 t}}$$

and letting $a = 0$ and $b = \frac{\pi}{2}$ in the inequality (6), we acquire

$$\begin{aligned}
&\left|\frac{2}{\pi}\int_0^{\pi/2}\frac{1}{r^2\cos^2 t + s^2\sin^2 t}\,\mathrm{d}\,t - \frac{4}{\pi^2}\kappa^2(r,s)\right| \\
= \quad &\left|\frac{4}{\pi^2}\kappa^2(r,s) - \frac{1}{\pi rs}\right| \le \frac{1}{2\pi}\int_0^{\pi/2}\frac{\left(s^2-r^2\right)^2\sin^2 t\cos^2 t}{\left(r^2\cos^2 t + s^2\sin^2 t\right)^3}\,\mathrm{d}\,t \\
= \quad &\frac{r^2-s^2}{8\pi}\int_0^{\pi/2}\sin t\cos t\,\mathrm{d}\,\frac{1}{\left(r^2\cos^2 t + s^2\sin^2 t\right)^2} \\
= \quad &\frac{r^2-s^2}{8\pi}\left[\int_0^{\pi/2}\frac{\sin^2 t}{\left(r^2\cos^2 t + s^2\sin^2 t\right)^2}\,\mathrm{d}\,t - \int_0^{\pi/2}\frac{\cos^2 t}{\left(r^2\cos^2 t + s^2\sin^2 t\right)^2}\,\mathrm{d}\,t\right] \\
= \quad &\frac{r^2-s^2}{8\pi}\left[\int_0^{\pi/2}\frac{\csc^2 t}{\left(r^2\cot^2 t + s^2\right)^2}\,\mathrm{d}\,t - \int_0^{\pi/2}\frac{\sec^2 t}{\left(r^2 + s^2\tan^2 t\right)^2}\,\mathrm{d}\,t\right] \\
= \quad &\frac{r^2-s^2}{8\pi}\left[\int_0^{\infty}\frac{1}{(r^2u^2+s^2)^2}\,\mathrm{d}\,u - \int_0^{\pi/2}\frac{1}{(r^2+s^2\lambda^2)^2}\,\mathrm{d}\,\lambda\right] \\
= \quad &\frac{s^2-r^2}{32rs}\left(\frac{1}{s^2}+\frac{1}{r^2}\right).
\end{aligned}$$

The proof is complete.

REMARK 3.2. From (16), we easily obtain

$$\frac{\pi}{2}\sqrt{\frac{32r^2s^2-\pi(s^4-r^4)}{32\pi r^3s^3}} \le \kappa(r,s) \le \frac{\pi}{2}\sqrt{\frac{32r^2s^2+\pi(s^4-r^4)}{32\pi r^3s^3}}. \tag{17}$$

By the software MATHEMATICA, we can show that the double inequality in (17) and the inequality in [6, Theorem 2] are not contained in each other.

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