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On Annular Bound For The Zeros Of A Polynomial*

Younseok Choo†

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Abstract

In this paper we present some results on the annular bound for the zeros of a polynomial based on the identities related to the generalized Fibonacci sequence with arbitrary initial condition. Several recently reported results in the same direction are special cases of our results.

1 Introduction

Several attempts have been made to obtain an explicit annular bound containing all the zeros of a polynomial based on the identities related to the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ ($F_0 = 0,\ F_1 = 1$, and $F_{n+1} = F_n + F_{n-1},\ n \geq 1$) or generalized Fibonacci sequence $\{F_n^{(a,b,c,d)}\}_{n=0}^{\infty}$ defined by

$$F_n^{(a,b,c,d)} = \begin{cases} aF_{n-1}^{(a,b,c,d)} + cF_{n-2}^{(a,b,c,d)}, & \text{if } n \text{ is even,} \\ bF_{n-1}^{(a,b,c,d)} + dF_{n-2}^{(a,b,c,d)}, & \text{if } n \text{ is odd,} \end{cases} \quad (n \geq 2)$$

where $F_0^{(a,b,c,d)} = 0$, $F_1^{(a,b,c,d)} = 1$, and $a, b, c, d > 0$.

Based on the identity

$$\sum_{k=1}^{n} 2^{n-k} 3^k F_k C(n,k) = F_{4n}, \tag{1}$$

where $C(n,k) = \frac{n!}{(n-k)!k!}$, Díaz-Barrero [1] proved the following theorem:

THEOREM A. A complex polynomial $P(z) = \sum_{k=0}^{n} d_k z^k$ ($d_k \neq 0$) has all its zeros in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \frac{3}{2} \min_{1 \leq k \leq n} \left\{ \frac{2^n F_k C(n,k)}{F_{4n}} \frac{|d_0|}{|d_k|} \right\}^{\frac{1}{k}} \quad \text{and} \quad r_2 = \frac{2}{3} \max_{1 \leq k \leq n} \left\{ \frac{F_{4n}}{2^n F_k C(n,k)} \frac{|d_{n-k}|}{|d_n|} \right\}^{\frac{1}{k}}.$$

Later Bidkham and Shashahani [3] derived the identity

$$\sum_{k=1}^{n} (a^2+1)^{n-k} (a^3+2a)^k F_k^{(a,a,1,1)} C(n,k) = F_{4n}^{(a,a,1,1)}, \tag{2}$$

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†Department of Electronic and Electrical Engineering, Hongik University, Sejong 339–701, Korea

and extended Theorem A as follows:

THEOREM B. All the zeros of a complex polynomial $P(z) = \sum_{k=0}^{n} d_k z^k$ $(d_k \neq 0)$ are contained in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{(a^2+1)^{n-k}(a^3+2a)^k F_k^{(a,a,1,1)} C(n,k)}{F_{4n}^{(a,a,1,1)}} \frac{|d_0|}{|d_k|} \right\}^{\frac{1}{k}},$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{F_{4n}^{(a,a,1,1)}}{(a^2+1)^{n-k}(a^3+2a)^k F_k^{(a,a,1,1)} C(n,k)} \frac{|d_{n-k}|}{|d_n|} \right\}^{\frac{1}{k}}.$$

Recently Rather and Mattoo [5] proved the identity

$$\sum_{k=1}^{n} (abc+c^2)^{n-k}(ab+2c)^k a^{\xi(k)} (ab)^{[\frac{k}{2}]} F_k^{(a,b,c,c)} C(n,k) = F_{4n}^{(a,b,c,c)}, \tag{3}$$

where $\xi(k) = k - 2[\frac{k}{2}]$, and then extended Theorem A and Theorem B as follows:

THEOREM C. All the zeros of a complex polynomial $P(z) = \sum_{k=0}^{n} d_k z^k$ $(d_k \neq 0)$ lie in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$, where

$$r_1 = \min_{1\leq k\leq n} \left\{ \frac{(abc+c^2)^{n-k}(ab+2c)^k a^{\xi(k)} (ab)^{[\frac{k}{2}]} F_k^{(a,b,c,c)} C(n,k)}{F_{4n}^{(a,b,c,c)}} \frac{|d_0|}{|d_k|} \right\}^{\frac{1}{k}},$$

and

$$r_2 = \max_{1\leq k\leq n} \left\{ \frac{F_{4n}^{(a,b,c,c)}}{(abc+c^2)^{n-k}(ab+2c)^k a^{\xi(k)} (ab)^{[\frac{k}{2}]} F_k^{(a,b,c,c)} C(n,k)} \frac{|d_{n-k}|}{|d_n|} \right\}^{\frac{1}{k}}.$$

In this paper we present further results in the same direction. Two theorems on the annular bound for the zeros of a polynomial are given respectively based on the identities related to the generalized Fibonacci sequences $\{F_n^{(a,b,c,d)}\}_{n=0}^{\infty}$ and $\{F_n^{(a,b,c,c)}\}_{n=0}^{\infty}$ with arbitrary initial conditions. The second one includes Theorem C as a special case.

2 Main Results

Before presenting our main results, we state some preliminary results.

LEMMA 1. Let r and s $(r \neq s)$ be nonzero roots of $x^2 - ax - b = 0$. Then the following three statements are equivalent:

(i) $B_j = b_0 \left(\frac{r^{j+1}-s^{j+1}}{r-s}\right) + (b_1 - ab_0)\left(\frac{r^j - s^j}{r-s}\right)$ for $j \geq 0$.

(ii) $B_j = aB_{j-1} + bB_{j-2}, \;\; j \geq 2$ with $B_0 = b_0$, $B_1 = b_1$.

(iii) $b_0 x^{j+1} + (b_1 - ab_0)x^j = xB_j + bB_{j-1}$ for $j \geq 1$ and $x = r, s$ where $B_0 = b_0$.

PROOF. (i)⇒(ii) follows from the fact that $B_0 = b_0$, $B_1 = b_1$ and, for $j \geq 0$

$$\begin{aligned} & r^{j+2} - s^{j+2} - a(r^{j+1} - s^{j+1}) - b(r^j - s^j) \\ = \;& (r^{j+2} - s^{j+2} - (r+s)(r^{j+1} - s^{j+1}) + rs(r^j - s^j) \\ = \;& 0. \end{aligned}$$

To prove (ii)⇒(iii), we proceed by induction as in [4]. If (ii) holds, (iii) is true for $j = 1$ since

$$\begin{aligned} b_0 x^2 + (b_1 - ab_0)x &= b_0(ax + b) + (b_1 - ab_0)x \\ &= xb_1 + bb_0 \\ &= xB_1 + bB_0. \end{aligned}$$

If (iii) holds for $j = m$, then, for $j = m + 1$

$$\begin{aligned} b_0 x^{m+2} + (b_1 - ab_0)x^{m+1} &= x[b_0 x^{m+1} + (b_1 - ab_0)x^m] \\ &= x^2 B_j + xbB_{j-1} \\ &= (ax + b)B_j + xbB_{j-1} \\ &= x(aB_j + bB_{j-1} + bB_j \\ &= xB_{j+1} + bB_j, \end{aligned}$$

hence (iii) follows. Now suppose (iii) holds. Then

$$b_0 r^{j+1} + (b_1 - ab_0)r^j = rB_j + bB_{j-1},$$

$$b_0 s^{j+1} + (b_1 - ab_0)s^j = sB_j + bB_{j-1},$$

and so

$$B_j = b_0 \left(\frac{r^{j+1} - s^{j+1}}{r - s} \right) + (b_1 - ab_0) \left(\frac{r^j - s^j}{r - s} \right) \text{ for } j \geq 1.$$

Since $B_0 = b_0$, (i) also holds, and the proof is completed.

REMARK 1. Although the closed-form expression for B_j in (i) satisfying the recurrence relation (ii) can also be computed by using the generating function, Lemma 1 provides another simple way to obtain the formula for B_j.

Lemma 2 and Lemma 3 below are slight generalizations of Theorem 1 and Theorem 2 in [2]. The proof of Lemma 3 is similar to that of [2, Theorem 2] and is omitted.

LEMMA 2. Let r and s $(r \neq s)$ be nonzero roots of $x^2 - ax - b = 0$. Define two sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ by $A_n = \gamma_n(cr^n + ds^n)$, where $\gamma_n, c, d \in \mathbb{R}$, and $B_n = \frac{r^n - s^n}{r - s}$. Then for $j \geq 2$ and $l \geq 0$

$$\sum_{k=1}^{n} C(n,k)(bB_{j-1})^{n-k}(B_j)^k \frac{\gamma_{jn+l}}{\gamma_{k+l}} A_{k+l} = A_{jn+l}. \tag{4}$$

PROOF. Using the equivalence (i) and (iii) in Lemma 1 for $b_0 = 0$ and $b_1 = 1$, we have

$$\begin{aligned}
\frac{A_{jn+l}}{\gamma_{jn+l}} &= cr^{jn+l} + ds^{jn+l} \\
&= cr^l(r^j)^n + ds^l(s^j)^n \\
&= cr^l(bB_{j-1} + rB_j)^n + ds^l(bB_{j-1} + sB_j)^n \\
&= \sum_{k=1}^{n} C(n,k)(bB_{j-1})^{n-k}(B_j)^k(cr^{k+l} + ds^{k+l}) \\
&= \sum_{k=1}^{n} C(n,k)(bB_{j-1})^{n-k}(B_j)^k \frac{A_{k+l}}{\gamma_{k+l}}.
\end{aligned}$$

LEMMA 3. With the same notation as in Lemma 2, all the zeros of a complex polynomial $P(z) = \sum_{k=0}^{n} d_k z^k$ $(d_k \neq 0)$ are contained in the annulus $C = \{z : r_1 \leq |z| \leq r_2\}$ where

$$r_1 = \min_{1 \leq k \leq n} \left\{ \frac{C(n,k)(bB_{j-1})^{n-k}(B_j)^k \gamma_{jn+l} A_{k+l}}{\gamma_{k+l} A_{jn+l}} \frac{|d_0|}{|d_k|} \right\}^{\frac{1}{k}},$$

and

$$r_2 = \max_{1 \leq k \leq n} \left\{ \frac{\gamma_{k+l} A_{jn+l}}{C(n,k)(bB_{j-1})^{n-k}(B_j)^k \gamma_{jn+l} A_{k+l}} \frac{|d_{n-k}|}{|d_n|} \right\}^{\frac{1}{k}}.$$

Now consider the generalized Fibonacci sequence $\{F_n^{(a,b,c,d)}\}_{n=0}^{\infty}$ defined in Section 1 with initial condition $F_0^{(a,b,c,d)} = f_0$, $F_1^{(a,b,c,d)} = f_1$. It is easily seen that

$$F_n^{(a,b,c,d)} = (ab + c + d)F_{n-2}^{(a,b,c,d)} - cdF_{n-4}^{(a,b,c,d)} \text{ for } n \geq 4.$$

Let $G_n^{(a,b,c,d)} = F_{2n}^{(a,b,c,d)}$, $n \geq 0$. Then $G_0^{(a,b,c,d)} = f_0$, $G_1^{(a,b,c,d)} = af_1 + cf_0$ and

$$G_n^{(a,b,c,d)} = (ab + c + d)G_{n-1}^{(a,b,c,d)} - cdG_{n-2}^{(a,b,c,d)} \text{ for } n \geq 2.$$

Hence, from Lemma 1, we have

$$\begin{aligned}
G_n^{(a,b,c,d)} &= G_0^{(a,b,c,d)} \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \right) \\
&\quad + \{G_1^{(a,b,c,d)} - (ab + c + d)G_0^{(a,b,c,d)}\} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \\
&= c_1\alpha_n + d_1\beta_n,
\end{aligned}$$

where α and β are roots of the equation $x^2-(ab+c+d)x+cd=0$, and

$$c_1=\frac{(\alpha-ab-d)f_0+af_1}{\alpha-\beta},\quad d_1=-\frac{(\beta-ab-d)f_0+af_1}{\alpha-\beta}.$$

On the other hand, let $\hat{G}_n^{(a,b,c,d)}=F_{2n+1}^{(a,b,c,d)}$ for $n\geq 0$. Then

$$\hat{G}_n^{(a,b,c,d)}=\frac{1}{a}(G_{n+1}^{(a,b,c,d)}-cG_n^{(a,b,c,d)})=c_2\alpha_n+d_2\beta_n,$$

where

$$c_2=\frac{c_1(\alpha-c)}{a}\text{ and }d_2=\frac{d_1(\beta-c)}{a}.$$

Hence, from Lemma 2 and Lemma 3, we obtain the following theorem.

THEOREM 1. Consider a complex polynomial $P(z)=\sum_{k=0}^{n}d_kz^k$ $(d_k\neq 0)$, and let $B_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}$, $n\geq 0$, where α and β are roots of $x^2-(ab+c+d)x+cd=0$. Then, for $j\geq 2$ and $l\geq 0$, all the zeros of $P(z)$ lie in the annulus $C=\{z:r_1\leq|z|\leq r_2\}$ or $\hat{C}=\{z:\hat{r}_1\leq|z|\leq\hat{r}_2\}$ where

$$r_1=\min_{1\leq k\leq n}\left\{\frac{C(n,k)(-cdB_{j-1})^{n-k}(B_j)^kF_{2(k+l)}^{(a,b,c,d)}}{F_{2(jn+l)}^{(a,b,c,d)}}\frac{|d_0|}{|d_k|}\right\}^{\frac{1}{k}},$$

$$r_2=\max_{1\leq k\leq n}\left\{\frac{F_{2(jn+l)}^{(a,b,c,d)}}{C(n,k)(-cdB_{j-1})^{n-k}(B_j)^kF_{2(k+l)}^{(a,b,c,d)}}\frac{|d_{n-k}|}{|d_n|}\right\}^{\frac{1}{k}},$$

$$\hat{r}_1=\min_{1\leq k\leq n}\left\{\frac{C(n,k)(-cdB_{j-1})^{n-k}(B_j)^kF_{2(k+l)+1}^{(a,b,c,d)}}{F_{2(jn+l)+1}^{(a,b,c,d)}}\frac{|d_0|}{|d_k|}\right\}^{\frac{1}{k}},$$

and

$$\hat{r}_2=\max_{1\leq k\leq n}\left\{\frac{F_{2(jn+l)+1}^{(a,b,c,d)}}{C(n,k)(-cdB_{j-1})^{n-k}(B_j)^kF_{2(k+l)+1}^{(a,b,c,d)}}\frac{|d_{n-k}|}{|d_n|}\right\}^{\frac{1}{k}}.$$

Next we consider the case where $c=d$. To this end we first find the formulae for $G_n^{(a,b,c,c)}=F_{2n}^{(a,b,c,c)}$ and $\hat{G}_n^{(a,b,c,c)}=F_{2n+1}^{(a,b,c,c)}$ in terms of $r=\sqrt{\alpha}$, $s=-\sqrt{\beta}$. It is easily seen that r and s are roots of the equation $x^2-\sqrt{ab}-c=0$. Now we have

$$\begin{aligned}G_n^{(a,b,c,c)} &= G_0^{(a,b,c,c)}\Big(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\Big)\\ &\quad+\{G_1^{(a,b,c,c)}-(ab+2c)G_0^{(a,b,c,c)}\}\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big)\\ &= \frac{1}{\sqrt{ab}}\Big[G_0^{(a,b,c,c)}\Big(\frac{r^{2n+2}-s^{2n+2}}{r-s}\Big)\\ &\quad+\{G_1^{(a,b,c,c)}-(ab+2c)G_0^{(a,b,c,c)}\}\Big(\frac{r^{2n}-s^{2n}}{r-s}\Big)\Big].\end{aligned}$$

On the other hand

$$\begin{aligned}\hat{G}_n^{(a,b,c,c)} &= \frac{1}{a}(G_{n+1}^{(a,b,c,c)} - cG_n^{(a,b,c,c)})\\ &= \frac{1}{a}\Big[G_0^{(a,b,c,c)}\Big\{\frac{(r^2-c)r^{2n+3}}{r(r^2-s^2)} - \frac{(s^2-c)s^{2n+3}}{s(r^2-s^2)}\Big\}\\ &\quad +\{G_1^{(a,b,c,c)} - (ab+2c)G_0^{(a,b,c,c)}\}\Big\{\frac{(r^2-c)r^{2n+1}}{r(r^2-s^2)} - \frac{(s^2-c)s^{2n+1}}{s(r^2-s^2)}\Big\}\Big]\\ &= \frac{1}{a}\Big[G_0^{(a,b,c,c)}\Big(\frac{r^{2n+3}-s^{2n+3}}{r-s}\Big)\\ &\quad +\{G_1^{(a,b,c,c)} - (ab+2c)G_0^{(a,b,c,c)}\}\Big(\frac{r^{2n+1}-r^{2n+1}}{r-s}\Big)\Big].\end{aligned}$$

Consequently $F_n(a,b,c,c)$ can be expressed as

$$\begin{aligned}F_n^{(a,b,c,c)} &= \frac{1}{a^{\xi(n)}(\sqrt{ab})^{1-\xi(n)}}\Big[G_0^{(a,b,c,c)}\Big(\frac{r^{n+2}-s^{n+2}}{r-s}\Big)\\ &\quad +\{G_1^{(a,b,c,c)} - (ab+2c)G_0^{(a,b,c,c)}\}\Big(\frac{r^n-s^n}{r-s}\Big)\Big]\\ &= \frac{1}{a^{\xi(n)}(\sqrt{ab})^{1-\xi(n)}}(c_3r^n + d_3s^n),\end{aligned}$$

where

$$c_3 = \frac{(r^2-ab-c)f_0 + af_1}{r-s},\quad d_3 = -\frac{(s^2-ab-c)f_0 + af_1}{r-s}.$$

From Lemma 2, we obtain the identity

$$\sum_{k=1}^{n} C(n,k)(cB_{j-1})^{n-k}(B_j)^k\frac{a^{\xi(k+l)}(\sqrt{ab})^{\xi(jn+l)}}{a^{\xi(jn+l)}(\sqrt{ab})^{\xi(k+l)}}F_{k+l}^{(a,b,c,c)} = F_{jn+l}^{(a,b,c,c)}, \tag{5}$$

where $B_n = \frac{r^n-s^n}{r-s}$. Finally, from Lemma 3, we obtain the following theorem.

THEOREM 2. Consider a complex polynomial $P(z) = \sum_{k=0}^{n} d_kz^k$ $(d_k \neq 0)$, and let $B_n = \frac{r^n-s^n}{r-s}$, $n \geq 0$, where r and s are roots of $x^2 - \sqrt{ab}x - c = 0$. Then, for $j \geq 2$ and $l \geq 0$, all the zeros of $P(z)$ lie in the annulus $C = \{z : r_3 \leq |z| \leq r_4\}$, where

$$r_3 = \min_{1\leq k\leq n}\left\{\frac{C(n,k)(cB_{j-1})^{n-k}(B_j)^k a^{\xi(k+l)}(\sqrt{ab})^{\xi(jn+l)}F_{k+l}^{(a,b,c,c)}}{a^{\xi(jn+l)}(\sqrt{ab})^{\xi(k+l)}F_{jn+l}^{(a,b,c,c)}}\frac{|d_0|}{|d_k|}\right\}^{\frac{1}{k}},$$

$$r_4 = \max_{1\leq k\leq n}\left\{\frac{a^{\xi(jn+l)}(\sqrt{ab})^{\xi(k+l)}F_{jn+l}^{(a,b,c,c)}}{C(n,k)(cB_{j-1})^{n-k}(B_j)^k a^{\xi(k+l)}(\sqrt{ab})^{\xi(jn+l)}F_{k+l}^{(a,b,c,c)}}\frac{|d_{n-k}|}{|d_n|}\right\}^{\frac{1}{k}}.$$

REMARK 2. Since $\{B_n\}_{n=0}^{\infty}$ satisfies the recurrence relation

$$B_{n+1} = \sqrt{ab}B_n + cB_{n-1} \text{ for } j \geq 1$$

with $B_0 = 0$, $B_1 = 1$, we have

$$B_3 = ab + c \text{ and } B_4 = \sqrt{ab}(ab + 2c).$$

Then, setting $j = 4$, $l = 0$, (5) reduces to (3), and so Theorem C is also a special case of Theorem 2.

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