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$\mathcal{I}$-Sequential Topological Spaces*

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Abstract

In this paper a new notion of topological spaces namely, I-sequential topological spaces is introduced and investigated. This new space is a strictly weaker notion than the first countable space. Also I-sequential topological space is a quotient of a metric space.

1 Introduction

The idea of convergence of real sequence have been extended to statistical convergence by [2, 14, 15] as follows: If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$ then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ stands for the cardinality of the set K_n. The natural density of the subset K is defined by

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

provided the limit exists.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric space (X, ρ) is said to be statistically convergent to l if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : \rho(x_k, l) \geq \varepsilon\}$ has natural density zero. A lot of investigation has been done on this convergence and its topological consequences after initial works by [5, 13].

It is easy to check that the family $I_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible ideal of $\mathbb{N}$ (recall that $I \subset 2^{\mathbb{N}}$ is called an ideal if (i) $A, B \in I$ implies $A \cup B \in I$ and (ii) $A \in I$, $B \subset A$ implies $B \in I$. I is called non-trivial if $I \neq \{\phi\}$ and $\mathbb{N} \notin I$. I is admissible if it contains all the singletons, cf. [8]). Thus one may consider an arbitrary ideal I of $\mathbb{N}$ and define I-convergence of a sequence by replacing a set of density zero in the definition of statistical convergence by a member of I.

In a topological space X, a set A is open if and only if every $a \in A$ has a neighborhood contained in A. A is sequentially open if and only if no sequence in $X \setminus A$ has a limit in A. In this paper using the idea of ideal convergence in topological spaces (cf. [9]), we define, I-sequentially open set and hence I-sequential topological space. Though the concept of these two sets, open and I-sequentially open are the same in case of metric spaces. We give an example of a topological space which is not I-sequential.

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Next we formulate an equivalent result for a topological space to be I-sequential and show that every I-sequential topological space is a quotient of a metric space. Finally we give an example of a topological space which is I-sequential but not first countable.

Throughout the paper we assume X to be a topological space and I be a non-trivial admissible ideal in $\mathbb{N}$.

2 Main Results

We first introduce the following definitions.

DEFINITION 2.1. A set $O \subset X$ is said to be open in X if and only if every $a \in O$ has a neighborhood contained in O.

DEFINITION 2.2. O is I-sequentially open if and only if no sequence in $X \setminus O$ has an I-limit in O. i.e. sequence can not I-converge out of a I-sequentially closed set.

DEFINITION 2.3. A topological space is I-sequential when any set O is open if and only if it is I-sequentally open.

We first show that the concept of these two sets are the same in case of metric spaces.

THEOREM 2.1. If X is a metric space, then the notion of open and I-sequentially open are equivalent.

PROOF. Let O be open and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $X \setminus O$. let $y \in O$. Then there is a neighborhood U of y which contained in O. Hence U can not contain any term of $\{x_n\}_{n\in\mathbb{N}}$. So y is not an I-limit of the sequence and O is I-sequentially open. Conversely, if O is not open then there is an $y \in O$ such that any neighborhood of y intersects $X \setminus O$. In particular we can pick an element $x_n \in (X \setminus O) \cap B(y, \frac{1}{n+1})$ for all $n \in \mathbb{N}$. Now the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X \setminus O$ converges and hence I-converges to $y \in O$, so O is not I-sequentially open.

The implication from open to I-sequentially open is true in any topological space.

THEOREM 2.2. In any topological space X, if O is open then O is I-sequentially open.

PROOF. The proof is similar to the first part of the Theorem 2.1.

Now we give an example of a topological space which is not I-sequential.

EXAMPLE 2.1. Consider $(\mathbb{R}, \tau_{cc})$, the countable complement topology on $\mathbb{R}$. Thus $A \subset \mathbb{R}$ is closed if and only if $A = \mathbb{R}$ or A is countable. Suppose that a sequence $\{x_n\}_{n\in\mathbb{N}}$ has an I-limit y. Then the neighborhood $(\mathbb{R} \setminus \{x_n : n \in \mathbb{N}\}) \cup \{y\}$ of y must contain x_n for infinitely many n. This is only possible when $x_n = y$ for n large enough. Consequently, a sequence in any set A can only I-converge to an element of A, so every subset of $\mathbb{R}$ is I-sequentially open. But as $\mathbb{R}$ is uncountable, not every subset is open. So $(\mathbb{R}, \tau_{cc})$ is not I-sequential.

The next theorem shows that if the space is first countable then it is I-sequential.

THEOREM 2.3. Every first countable space is I-sequential.

PROOF. Let $A \subset X$ is not open. Then there exists $y \in A$ such that every neighborhood of y intersects $X \setminus A$. Let $\{U_n : n \in \mathbb{N}\}$ be a countable basis at y. Now for every $n \in \mathbb{N}$ choose $x_n \in (X \setminus A) \cap (\bigcap_{i=1}^{n} U_i)$. Then for every neighborhood V of y there exists $n \in \mathbb{N}$ such that $U_n \subset V$ and hence $x_m \in V$ for every $m \geq n$. Clearly $\{x_n\}_{n\in\mathbb{N}}$ is I-convergent to y. Therefore A is not I-sequentially open.

In the following Lemma we give a necessary and sufficient condition for a set $A \subset X$ to be I-sequentially open.

LEMMA 2.1. Let X be a topological space. Then $A \subset X$ is I-sequentially open if and only if every sequence with I-limit in A has all but finitely many terms in A. Where the index set of the part in A of the sequence does not belong to I.

PROOF. If A is not I-sequentially open, then by definition there is a sequence with terms in $X \setminus A$ but I-limit in A. Conversely, suppose $\{x_n\}_{n\in\mathbb{N}}$ is a sequence with infinitely many terms in $X \setminus A$ such that I-converges to $y \in A$ and the index set of the part in A of the sequence does not belong to I. Then $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence in $X \setminus A$ that must still converges to $y \in A$, so A is not sequentially open.

THEOREM 2.4. The following are equivalent for any topological space X.

(i) X is I-sequential.

(ii) For any topological space Y and function $f : X \to Y$, f is continuous if and only if it preserves I-convergence.

PROOF. Suppose X is I-sequential. Any continuous function preserves I-convergence of sequences [1], so we only need to prove that if $f : X \to Y$ preserves I-convergence, then f is continuous. Suppose to the contrary that f is not continuous. Then there is an open set $U \subset Y$ such that $f^{-1}(U)$ is not open in X. As X is I-sequential, $f^{-1}(U)$ is also not I-sequentially open, so there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X \setminus f^{-1}(U)$ that I-converges to an $y \in f^{-1}(U)$. However $\{f(x_n)\}_{n\in\mathbb{N}}$ is then a sequence in the closed set $Y \setminus U$, so it can not have $f(y)$ as an I-limit. So f does not preserves I-convergence, as required. Thus assertions (ii) holds.

Suppose that the topological space (X, τ) is not I-sequential. Let (X, τ_{Iseq}) be the topological space where $A \subset X$ is open if and only if A is I-sequentially open in (X, τ).

Since X is not I-sequential, the topology τ_{Iseq} is strictly finer than τ. Hence the identity map from τ to τ_{Iseq} is not continuous. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is I-convergent to y in (X, τ). Then every open neighborhood A of y in (X, τ_{Iseq}) is I-sequentially open in (X, τ), so A contains all but finitely many terms of $\{x_n\}_{n\in\mathbb{N}}$. Therefore X is I-sequential.

We now give some result to prove the fact that all I-sequential topological spaces are the quotients of some metric spaces [3, 4].

First we recall the definition of a quotient space. Let X be a topological space and let $\sim$ be an equivalence relation on X. Consider the set of equivalence classes $X/\sim$ and the projection mapping $\Pi : X \to X/\sim$. Now we consider $X/\sim$ as a topological space by defining $A \subset X/\sim$ to be open if and only if $\Pi^{-1}(A)$ is open in X [10].

PROPOSITION 2.1. Any quotient space $X/\sim$ of an I-sequential topological space X is I-sequential.

PROOF. Suppose that $A \subset X/\sim$ is not open. By definition of quotient space $\Pi^{-1}(A)$ is not open in X. As X is I-sequential, there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X \setminus \Pi^{-1}(A)$ that I-converges to some $y \in \Pi^{-1}(A)$. As Π is continuous it preserve convergence. Hence $\{\Pi(x_n)\}_{n\in\mathbb{N}}$ is a sequence in $(X/\sim) \setminus A$ with I-limit $\Pi(y) \in A$. Thus A is not I-sequentially open. Hence $X/\sim$ is I-sequential.

PROPOSITION 2.2. Every I-sequential space X is a quotient of some metric space.

PROOF. Let M be the set of all sequences $\{x_n\}_{n\in\mathbb{N}}$ in X that I-converges to their first term, i.e. $x_n \xrightarrow{I} x_0$. Consider the subspace $Y = \{0\} \cup \{\frac{1}{n+1}, n \in \mathbb{N}\}$ of $\mathbb{R}$ with the standard metric. Thus $A \subset Y$ is open if and only if $0 \notin A$ or A contains all but finitely many elements of Y. Now consider the disjoint sum

$$S = \bigoplus_{\{x_n\}_{n\in\mathbb{N}} \in M} \{x_n\}_{n\in\mathbb{N}} \times Y.$$

$A \subset S$ is open if and only if for every $\{x_n\}_{n\in\mathbb{N}} \in M$ the set $\{y \in Y : (\{x_n\}_{n\in\mathbb{N}}, y) \in A\}$ is open in Y. Consider the map $f : S \to X$ by $(\{x_n\}_{n\in\mathbb{N}}, 0) \to x_0$ and $(\{x_n\}_{n\in\mathbb{N}}, \frac{1}{i+1}) \to x_i$ for all $i \in \mathbb{N}$. Here f is clearly surjective as for all $x \in X$ the constant sequence I-converges to x, so $x = f(\{x\}, 0)$.

Suppose that $A \subset X$ is open. As X is I-sequential, every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X that I-converging to some $a \in A$ must have all but finitely many terms in A where, the index set of the part in A of the sequence does not belong to I by Definition 2.2. Hence if $(\{x_n\}_{n\in\mathbb{N}}, 0) \in f^{-1}(A)$ we have $f^{-1}(A)$ contains all but finitely many elements of $\{x_n\}_{n\in\mathbb{N}} \times Y$. So for each $\{x_n\}_{n\in\mathbb{N}} \in M$, the set $\{y \in Y : (\{x_n\}_{n\in\mathbb{N}}, y) \in f^{-1}(A)\}$ is open in Y. Hence $f^{-1}(A)$ is open in S. Conversely, if A is not open in X then there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $X \setminus A$ that I-converges to some $a \in A$. But then $\{y \in Y : (\{x_n\}_{n\in\mathbb{N}}, y) \in f^{-1}(A)\} = \{0\}$ is not open in Y, so $f^{-1}(A)$ is not open in S.

We can now easily prove that I-sequential topological space is a strictly weaker notion than first countable topological space: There exists an I-sequential space X which is not first countable.

EXAMPLE 2.2. Consider $\mathbb{R}$ with standard topology and the quotient relation $\sim$ on $\mathbb{R}$, the equivalence classes are $\mathbb{N}$ and $\{x\}$ for every $x \in \mathbb{R} \setminus \mathbb{N}$. The quotient space $\mathbb{R}/\sim$ is I-sequential as a quotient of a metric space. Suppose that $\{U_n : n \in \mathbb{N}\}$

is any countable collection of neighborhood of $\mathbb{N}$. Then for all $n \in \mathbb{N}, \Pi^{-1}(U_n)$ is a neighborhood of n in $\mathbb{R}$ with the standard topology, so there is a $\varepsilon_n > 0$ such that $B(n, \varepsilon_n) \subset \Pi^{-1}(U_n)$. Now consider $\Pi(\bigcup_{n\in\mathbb{N}} B(n, \frac{\varepsilon_n}{2}))$, this is a neighborhood of $\mathbb{N}$ in $\mathbb{R}/\sim$, but it does not contain U_n for any $n \in \mathbb{N}$. So $\{U_n : n \in \mathbb{N}\}$ is not a countable basis at $\mathbb{N}$.

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