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Exact Controllability Of Wave Equation With Multiplicative Controls*

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Abstract

We consider the problem of exact controllability of the wave equation with Dirichlet boundary conditions and multiplicative controls. A multiplicative control is a coefficient like uy. Exact controllability result is stated and proved for some particular targets.

1 Introduction

Let $D \subset \mathbb{R}^n$, $n \in \mathbb{N}$ be a bounded domain with smooth boundary ∂D. We use the notations $Q_T = D \times (0,T)$ and $\sum_T = \partial D \times (0,T)$. Consider the following control problem with multiplicative control:

$$\begin{array}{rll} y_{tt} - \Delta y & = uy_t + vy, & \text{in } Q_T, \\ y & = g, & \text{on } \sum_T, \\ (y(x,0),\ y_t(x,0)) & = (y_0(x),\ y_1(x)) & \text{in } D, \end{array} \tag{1}$$

where $u,\ v \in L^\infty(Q_T)$ are controls, $g \in\ C(\overline{\sum_T})$, and $(y_0, y_1) \in H_0^1(D) \times L^2(D)$.

A problem like this arises, in the context of so-called “smart materials", whose properties can be altered by applying various external factors such as temperature, electrical current or magnetic field [1]. Russell [2] developed controllability and stability theory for wave equation. Ball, Marsden and Slemrod [3] consider the problem of global approximate controllability of the rod equation $u_{tt} + u_{xxx} + k(t)u_{xx} = 0$ with hinged ends and the wave equation $u_{tt} - u_{xx} + k(t)u = 0$ with Dirichlet boundary conditions, where k is control; which appears to be the first work on this subject in the framework of partial differential equations(pdes). In [1, Chapter 6], Khapalov proved, in a constructive way, that the set of equilibrium states like $(y_d, 0)$ of a vibrating string that can approximately be reached in $H_0^1(0,1) \times L^2(0,1)$ by varying its axial load and the gain of damping is dense in the subspace $H_0^1(0,1) \times \{0\}$ of this space; where as in Chapter 8, Khapalov talks about the extension of this problem to the semilinear case. It seems that the result for exact controllability of (1) obtained in this work is new. To prove our result we use the method of connecting the multiplicative controllability

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with the additive distributed controllability by means of some substitution used for parabolic equation by Fernandez [4].

REMARK 1. The zero state $(y_0, y_1) = (0, 0)$ is the fixed point for the solution mappings of system (1), regardless of the choice of controls u. Hence, it cannot be steered anywhere from this state. Hence, everywhere below we consider only non-zero initial states (y_0, y_1).

Some preliminary results used to prove our main Theorem are given in section 2. In section 3, we prove the main theorem.

2 Preliminaries

The exact controllability of system (1) is defined as follows:

DEFINITION 1. System (1) is said to be exact controllable in time $T_1 > 0$ if, for every initial data $(y_0, y_1) \in H_0^1(D) \times L^2(D)$ and desired profile $y_d \in H_0^1(D)$, there exist controls $u, v \in L^\infty(Q_T)$ such that $(y(x,T), y_t(x,T)) = (y_d, 0)$ in D, for all $T \geq T_1$.

We will need help of the following well known result from [5], while proving our main theorem.

LEMMA 1. Let $a \in L^\infty(Q_T)$, $v \in L^2(Q_T)$ and $(q_0, q_1) \in H_0^1(D) \times L^2(D)$ be given. Then the solution q of the linear problem

$$\begin{aligned} q_{tt}(x,t) - \Delta q(x,t) + a(x,t)q(x,t) &= v(x,t), && \text{in } Q_T, \\ q &= 0, && \text{on } \textstyle\sum_T, \\ (q(x,0), q_t(x,0)) &= (q_0(x), q_1(x)) && \text{in } D, \end{aligned}$$

satisfies $q \in C([0,T]; H_0^1(D))$, $q_t \in C([0,T]; L^2(D))$ and

$$\begin{aligned} & \|q(\cdot,t)\|^2_{C([0,T];H_0^1(D))} + \|q_t(\cdot,t)\|^2_{C([0,T];L^2(D))} \\ \leq \quad & e^{C_3 kT} \left(\|q(\cdot,T_2)\|^2_{H_0^1(D)} + \|q_t(\cdot,T_2)\|^2_{L^2(D)} + \|v\|^2_{L^2(Q_T)} \right), \end{aligned}$$

where $k = 1 + \|a\|_{L^\infty(Q_T)}$.

Also, we will use the following known result about the additive controllability from [6,7].

LEMMA 2. For every $(q_0,\ q_1) \in H_0^1(D) \times L^2(D)$ and $a \in L^\infty(Q_T)$, there exists a control function $v \in L^2(Q_T)$ such that the weak solution q to problem

$$\begin{aligned} q_{tt}(x,t) - \Delta q(x,t) + a(x,t)q(x,t) &= v(x,t), && \text{in } Q_T, \\ q &= 0, && \text{on } \textstyle\sum_T, \\ (q(x,0), q_t(x,0)) &= (q_0(x), q_1(x)) && \text{in } D, \end{aligned}$$

satisfies $(q(x,T), q_t(x,T)) = (0, 0)$. Moreover,

$$\|v\|^2_{L^2(Q_T)} \leq C(\|a\|_{L^\infty(Q_T)}) \left(\|q_0\|^2_{H_0^1(D)} + \|q_1\|^2_{L^2(D)} \right).$$

Now it is time to state and prove the main result.

3 The Main Result

Our main theorem is stated as follows:

THEOREM 1. Let θ be a function defined on $\overline{Q}_T$. If $\theta_t, \theta \in W^{2,\infty}(D)$, $0 < \theta$, $0 < \theta_t$ in $\overline{D}$, and $0 \leq \Delta\theta$, $\theta_t = \theta$ a.e. in D, $g \in C(\overline{\sum_T})$, $g = \theta$ on $\sum_T$, then there exists a $T(\theta) > 0$ such that for any non-zero initial state $(y_0,\ y_1) \in H_0^1(D) \times L^2(D)$, there exist multiplicative controls $u,\ v \in L^\infty(Q_T)$ such that the corresponding solution to (1) in $C([0,T]; H_0^1(D)) \cap C^1([0,T]; L^2(D))$ satisfies

$$(y(x,T), y_t(x,T)) = (\theta(x,T), \theta_t(x,T)),$$

in D, for all $T \geq T(\theta)$.

PROOF. Let $z = y - \theta$. Then $z_0 = z(x,0) = y(x,0) - \theta(x,0) = y_0 - \theta_0$, $z_1 = z_t(x,0) = y_t(x,0) - \theta_t(x,0) = y_1 - \theta_1$, and hence $(z_0, z_1) = (y_0 - \theta_0,\ y_1 - \theta_1)$. Thus from (1), z satisfies

$$\begin{aligned} z_{tt} - \Delta z &= u(z_t + \theta_t) + v(z + \theta) + \Delta\theta - \theta, && \text{in } Q_T, \\ z &= 0, && \text{on } \textstyle\sum_T, \\ (z(x,0), z_t(x,0)) &= (z_0, z_1), && \text{in } D. \end{aligned} \tag{2}$$

It is well known that given $(z_0,\ z_1) \in H_0^1(D) \times L^2(D)$, $u, v \in L^\infty(Q_T)$, θ as given in Theorem 1, the problem (2) admits a unique solution z in the space $C^1([0,T]; L^2(D)) \cap C([0,T]; H_0^1(D))$.

In order to prove Theorem 1, it is sufficient to prove that system (2) is exactly null controllable. We prove this in the following few steps:

STEP 1. In this step, we prove that given $T_1 > 0$, there exists $M > 0$, such that the corresponding solution to (4) satisfies

$$\|z(\cdot, T_1)\|^2_{H_0^1(D)} + \|z_t(\cdot, T_1)\|^2_{L^2(D)} \leq M.$$

Multiplying the system (4) by z_t and integrating over $Q_t = D \times (0,\ t)$, we obtain

$$\begin{aligned} 0 &= \int_{Q_t} \left[z_{tt} z_t - \Delta z z_t - u z_t^2 - u z_t \theta_t - z_t \Delta\theta - v z z_t - v z_t \theta + \theta z_t \right] dx d\tau \\ &= \int_{Q_t} \left[\frac{1}{2} \frac{d}{dt} (z_t^2 + \|\nabla z\|^2) - u z_t^2 - u z_t \theta_t - z_t \Delta\theta - v z z_t - v z_t \theta + \theta z_t \right] dx d\tau. \end{aligned}$$

Suppose $u,\ v$ are constants and $v = 0$. Then, for $t \in (0,\ T)$ we have

$$\begin{aligned} &\int_D \left[z_t^2(x,t) + |\nabla z(x,t)|^2 \right] dx \\ = &\int_D \left[z_1^2 + |\nabla z_0|^2 \right] dx + 2 \int_{Q_t} \left[u z_t^2 \right] dx d\tau + 2 \int_{Q_t} \left[u z_t \theta_t + z_t \Delta\theta - \theta z_t \right] dx d\tau \\ \leq &\ \|z_1\|^2_{L^2(D)} + \|\nabla z_0\|^2_{L^2(D)} + 2u \|z_t\|^2_{L^2(Q_t)} + |u| \|z_t\|^2_{L^2(Q_t)} + |u| \|\theta_t\|^2_{L^2(Q_t)} \\ &+ 2\|z_t\|^2_{L^2(Q_t)} + \|\Delta\theta\|^2_{L^2(Q_t)} + \|\theta\|^2_{L^2(Q_t)}. \end{aligned} \tag{3}$$

By Poincaré's inequality we have

$$\begin{aligned}\int_D \left[z_t^2(x,t)+z^2(x,t)\right]dx &\leq C_1(u+2)\int_{D_t}\left[z_t^2+z^2\right]dxd\tau + C_1\left(\|z_1\|^2_{L^2(D)}+\|\nabla z_0\|^2_{L^2(D)}\right)\\ &\quad +C_1\left(|u|\|\theta_t\|^2_{L^2(Q_t)}+\|\Delta\theta\|^2_{L^2(Q_t)}+\|\theta\|^2_{L^2(Q_t)}\right),\end{aligned}\tag{4}$$

where constant C_1 is independent of z. Hence, by Gronwall's lemma

$$\begin{aligned}&\|z(\cdot,t)\|^2_{H_0^1(D)}+\|z_t(\cdot,t)\|^2_{L^2(D)}\\ \leq\ & e^{(u+2)C_1t}\left(\|z_1\|^2_{L^2(D)}+\|\nabla z_0\|^2_{L^2(D)}\right)+C_1\int_0^t e^{C_1(u+2)(t-\tau)}\|z_1\|^2_{L^2(D)}d\tau\\ &+C_1\int_0^t e^{C_1(u+2)(t-\tau)}\|\nabla z_0\|^2_{L^2(D)}d\tau+|u|C_1\int_0^t e^{C_1(u+2)(t-\tau)}\|\theta_t\|^2_{L^2(Q_t)}d\tau\\ &+C_1\int_0^t e^{C_1(u+2)(t-\tau)}\|\Delta\theta\|^2_{L^2(Q_t)}d\tau+C_1\int_0^t e^{C_1(u+2)(t-\tau)}\|\theta\|^2_{L^2(Q_t)}d\tau\\ =\ & e^{(u+2)C_1t}\left(\|z_1\|^2_{L^2(D)}+\|\nabla z_0\|^2_{L^2(D)}\right)+\frac{1-e^{C_1(u+2)t}}{(u+2)}\|z_1\|^2_{L^2(D)}\\ &+\frac{1-e^{C_1(u+2)t}}{(u+2)}\|\nabla z_0\|^2_{L^2(D)}+|u|\frac{1-e^{C_1(u+2)t}}{(u+2)}\|\theta_t\|^2_{L^2(Q_t)}\\ &+\frac{1-e^{C_1(u+2)t}}{(u+2)}\|\Delta\theta\|^2_{L^2(Q_t)}+\frac{1-e^{C_1(u+2)t}}{(u+2)}\|\theta\|^2_{L^2(Q_t)}.\end{aligned}$$

Hence, for given $T_1>0$, we can select $u=u_1<-2$ depending on $(z_0,\ z_1)$ and $|u|$ is sufficiently large such that there exists a constant $M>0$, such that the corresponding solution of (2) satisfies

$$\|z(\cdot,T_1)\|^2_{H_0^1(D)}+\|z_t(\cdot,T_1)\|^2_{L^2(D)}\leq M.$$

STEP 2. In this step, we further prove that for any $\epsilon_0>0$, we can find controls u and v and $T_2(\theta)>0$ sufficiently large such that the corresponding solution to (2) satisfies

$$\|z(\cdot,T_2)\|^2_{H_0^1(D)}+\|z_t(\cdot,T_2)\|^2_{L^2(D)}\leq \epsilon_0.$$

As $\theta_t\in W^{2,\infty}(D)$, we have $\theta_t\in C(\overline{D})$ by Sobolev embedding theorem. Also, $\theta_t>0$ in $\overline{D}$, there exists a positive constant $\nu>0$ such that $\theta_t\geq\nu>0$ in $\overline{D}$, hence $0\leq\frac{\Delta\theta}{\theta_t}\in L^\infty(D)$. Let $z_{T_1}=z(x,T_1)$ and $z'_{T_1}=z_t(x,T_1)$. Select $u_2=-\frac{\Delta\theta}{\theta_t}+1$ and $v=0$ in $(T_1,\ T_2)$. Then, we have

$$\begin{aligned} z_{tt}-\Delta z &= -(\tfrac{\Delta\theta}{\theta_t}-1)z_t, && \text{in } D\times(T_1,\ T_2),\\ z &= 0, && \text{on } \partial D\times(T_1,\ T_2),\\ (z(x,T_1),z_t(x,T_1)) &= (z_{T_1},z'_{T_1}), && \text{in } D.\end{aligned}\tag{5}$$

Let $\lambda>1$ be the eigenvalue of $-\Delta$ in $H_0^1(D)$. Let $t\in(T_1,T_2)$ and $Q_t=D\times(T_1,t)$. Multiplying (5) by z_t and integrating in D, we have

$$\int_D\left[z_t^2(x,t)+|\nabla z(x,t)|^2\right]dx=\int_D\left[z_1^2+|\nabla z_0|^2\right]dx+2\int_{Q_t}\left(-(\lambda-1)\right)z_t^2dxd\tau.$$

As in step 1, using Poincaré's inequality and Gronwall's lemma we have

$$\|z(\cdot,T_2)\|^2_{H^1_0(D)} + \|z_t(\cdot,T_2)\|^2_{L^2(D)} \le C\left(\|z(\cdot,T_1)\|^2_{H^1_0(D)} + \|z_t(\cdot,T_1)\|^2_{L^2(D)}\right),$$

where $C = e^{-(\lambda-1)(T_2-T_1)}$. Thus by step 1,

$$\|z(\cdot,T_2)\|^2_{H^1_0(D)} + \|z_t(\cdot,T_2)\|^2_{L^2(D)} \le Me^{-(\lambda-1)(T_2-T_1)}.$$

Hence, for any $\epsilon_0 > 0$, there exists a $T_2(\theta) > 0$ sufficiently large such that

$$\|z(\cdot,T_2)\|^2_{H^1_0(D)} + \|z_t(\cdot,T_2)\|^2_{L^2(D)} \le \epsilon_0. \tag{6}$$

STEP 3. In this step, we achieve the result by means of the controllability result with the traditional additive distributive control. Let $z_{T_2} = z(x,T_2)$ and $z'_{T_2} = z_t(x,T_1)$. Consider the following system

$$\begin{aligned} z_{tt} - \Delta z &= u(z_t+\theta_t) + v(z+\theta) + \Delta\theta - \theta && \text{in } D\times(T_2,\ T_2+1),\\ z &= 0, && \text{on } \partial D\times(T_2,\ T_2+1),\\ (z(x,T_2), z_t(x,T_2)) &= (z_{T_2}, z'_{T_2}), && \text{in } D. \end{aligned} \tag{7}$$

As $\theta \in W^{2,\infty}(D)$, we have $\theta \in C(\overline{D})$ by Sobolev embedding theorem. Also, $\theta > 0$ in $\overline{D}$, there exists a positive constant $\mu > 0$ such that $\theta \ge \mu > 0$ in $\overline{D}$, hence $0 \le \frac{\Delta\theta}{\theta} \in L^\infty(D)$. Let $u = 0$ and $v = -\frac{\Delta\theta}{\theta} + 1 + v_3$. Then (7) becomes

$$\begin{aligned} z_{tt} - \Delta z + (\tfrac{\Delta\theta}{\theta} - 1)z &= v_3(z+\theta) && \text{in } D\times(T_2,\ T_2+1),\\ z &= 0, && \text{on } \partial D\times(T_2,\ T_2+1),\\ (z(x,T_2), z_t(x,T_2)) &= (z_{T_2}, z'_{T_2}), && \text{in } D. \end{aligned} \tag{8}$$

From (6) we have

$$\|z(\cdot,T_2)\|^2_{H^1_0(D)} + \|z_t(\cdot,T_2)\|^2_{L^2(D)} \le \epsilon_0, \tag{9}$$

where ϵ_0 will be fixed later. In place of (8), we consider the following system

$$\begin{aligned} z_{tt} - \Delta z + (\tfrac{\Delta\theta}{\theta} - 1)z &= w(x,t) && \text{in } D\times(T_2,\ T_2+1),\\ z &= 0, && \text{on } \partial D\times(T_2,\ T_2+1),\\ (z(x,T_2), z_t(x,T_2)) &= (z_{T_2}, z'_{T_2}), && \text{in } D. \end{aligned} \tag{10}$$

By Lemma 2, there exists a control $w \in L^2(D\times(T_2,\ T_2+1))$ such that the corresponding solution to (10) satisfies

$$(z(\cdot,T_2+1), z_t(\cdot,T_2+1)) = (0,0). \tag{11}$$

Moreover,

$$\|w\|^2_{L^2(D\times(T_2,\ T_2+1))} \le C_2(d)\left(\|z(x,T_2)\|^2_{H^1_0(D)} + \|z_t(x,T_2)\|^2_{L^2(D)}\right), \tag{12}$$

where $d = \|\Delta\theta/\theta - 1\|_{L^\infty(Q_{T_2})}$. Also, using Lemma 1, we have

$$\begin{aligned} &\|z(\cdot,t)\|^2_{C([T_2,T_2+1];H^1_0(D))} + \|z_t(\cdot,t)\|^2_{(C([T_2,T_2+1];L^2(D))}\\ &\le e^{C_3(1+d)}\left(\|z(\cdot,T_2)\|^2_{H^1_0(D)} + \|z_t(\cdot,T_2)\|^2_{L^2(D)} + \|w\|^2_{L^2(Q_{T_2})}\right), \end{aligned} \tag{13}$$

where $Q_{T_2} = D \times (T_2,\ T_2+1)$, and constant C_3 depends only on D. From (12) and (13), we have

$$\begin{aligned}&\|z(\cdot,t)\|^2_{C([T_2,T_2+1];H_0^1(D))} + \|z_t(\cdot,t)\|^2_{(C([T_2,T_2+1];L^2(D))}\\ &\le e^{C_3(1+d)}(1+C_2)\left(\|z(\cdot,T_2)\|^2_{H_0^1(D)} + \|z_t(\cdot,T_2)\|^2_{L^2(D)}\right),\end{aligned} \tag{14}$$

We now select

$$\epsilon_0 < \frac{1}{e^{C_3(1+d)}(1+C_2)} < \frac{\mu}{e^{C_3(1+d)}(1+C_2)}.$$

Here we may assume that $\mu > 1$, and by (6), select T_2 sufficiently large such that

$$(\|z(\cdot,T_2)\|^2_{H_0^1(D)} + \|z_t(\cdot,T_2)\|^2_{L^2(D)}) \le \epsilon_0.$$

Hence, we have

$$\|z(\cdot,t)\|^2_{C([T_2,T_2+1];H_0^1(D))} + \|z_t(\cdot,t)\|^2_{(C([T_2,T_2+1];L^2(D))} < \mu. \tag{15}$$

Thus, we can select the multiplicative control for (8)

$$v_3 = \frac{\mu}{z+\theta} \text{ a.e. in } D \times (T_2,\ T_2+1), \tag{16}$$

where z is the solution of (8). In view of $\theta \ge \mu > 0$, and (15), we have $v_3 \in L^\infty(D \times (T_2,\ T_2+1))$.

Hence, in the time interval $(T_2,\ T_2+1)$, in view of (16), the solution of (8) with the control v_3, i.e., the solution of (7) with the controls $u = 0$ and $v = -\frac{\Delta\theta}{\theta} + 1 + v_3$ and the solution of (10) with the control w become identical. Hence from (11), we have

$$(z(\cdot,T_2+1), z_t(\cdot,T_2+1)) = (0,0),$$

where z is the corresponding solution to (7) with $u = 0$ and $v = -\frac{\Delta\theta}{\theta} + 1 + v_3$.

By steps 1, 2 and 3, we have for any $(z_0,\ z_1) \in H_0^1(D) \times L^2(D)$, we can select $T_2(\theta) > 0$ sufficiently large such that the corresponding solution z to (2) with controls

$$u = \begin{cases} u_1 & \text{in } (0,T_1),\\ -\frac{\Delta\theta}{\theta_t} & \text{in } (T_1,T_2),\\ 0 & \text{in } (T_2,T_2+1),\end{cases} \quad \text{and } v = \begin{cases} 0 & \text{in } (0,T_1),\\ 0 & \text{in } (T_1,T_2),\\ -\frac{\Delta\theta}{\theta}+1+v_3 & \text{in } (T_2,T_2+1),\end{cases}$$

satisfies

$$(z(\cdot,T_2+1), z_t(\cdot,T_2+1)) = (0,0),$$

where $T(\theta) = T_2 + 1$ depends on θ only. This completes the proof of Theorem 1.

REMARK 2. In step 2, we can take $\epsilon_0 = \inf_{x\in D}\theta$. So, $\|z(\cdot,T_2)\|_{H_0^1(D)} \le \epsilon_0$. Now in step 3, due to null controllability z approaches 0. Thus in (16) we have $z+\theta > 0$ in $D \times (T_2, T_2+1)$.

From the proof of Theorem 1 we have the following theorem.

THEOREM 2. Let θ be a function defined on $\overline{Q}_T$. If $\theta, \theta_t \in W^{2,\infty}(D)$, $0 > \theta$, $0 > \theta_t$ in $\overline{D}$, and $0 \geq \Delta\theta$, $\theta_t = \theta$ a.e. in D, $g \in C(\overline{\sum_T})$, $g = \theta$ on $\sum_T$, then there exists a $T(\theta) > 0$ such that for any non-zero initial state $(y_0,\ y_1) \in H_0^1(D) \times L^2(D)$, there exist multiplicative controls $u,\ v \in L^\infty(Q_T)$ such that the corresponding solution to (1) in $C([0,T]; H_0^1(D)) \cap C^1([0,T]; L^2(D))$ satisfies $(y(x,T), y_t(x,T)) = (\theta(x,T), \theta_t(x,T))$, in D, for all $T \geq T(\theta)$.

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