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Further Solutions Of The General Abel Equation Of The Second Kind: Use Of Julia's Condition*

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Abstract

In this paper, we propose a direct method to obtain an implicit solution of the Abel equation of the second kind

$$[g_0(x) + g_1(x)u]\, u' = f_0(x) + f_1(x)u + f_2(x)u^2.$$

We first reduce it into an equivalent equation, and assume that the coefficient functions $f_i(x),\ i = 0, 1, 2$ and $g_i(x),\ i = 0, 1$ satisfy the well-known Julia's condition. Therefore the given Abel equation can be transformed into a first-order linear differential equation, which can be easily solved, and then the implicit solutions of this equation are obtained.

1 Introduction

The Abel equation of the second kind has the general form

$$[g_0(x) + g_1(x)u]\, u' = f_0(x) + f_1(x)u + f_2(x)u^2. \tag{1}$$

This equation was derived in 1829 in the context of the studies of N.H. Abel [1] on the theory of elliptic functions. The first important well-known result in the analysis of the Abel equation is that: If $g_0,\ g_1\ \in C^1(a, b)$, $g_1(x) \neq 0$ and $g_0(x)+g_1(x) \neq 0$, then Abel's differential equation of the second kind can be reduced to Abel's differential equation of the first kind by substituting $g_0(x) + g_1(x)u = \frac{1}{z}$. The second important result is that: Eq.(1) can be reduced to the canonical form [2], by using various admissible functional transformations,

$$uu'_x - u = \Phi(x),$$

where the function $\Phi(x)$ is defined parametrically. It is often very difficult, if not impossible, to find explicit solutions of such nonlinear differential equations. But a number of solutions of the Abel equation of the second kind can be obtained by assuming that the coefficients $f_i(x),\ i = 0, 1, 2$ and $g_i(x),\ i = 0, 1$ satisfy some particular constraints. In 1933, the French mathematician Gaston Julia [3] proved that the equation

$$du + \frac{Au^2 + Bu + C}{Du + E} dx = 0,$$

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for $A,\ B,\ C,\ D$ and E functions of x, has an implicit solution if the condition

$$E(2A-D')=D(B-E'),\ D\neq 0$$

is satisfied. Then the solution is implicitly given by

$$D\frac{u^2}{2}e^{\int\frac{2A-D'}{D}dx}+Eue^{\int\frac{2A-D'}{D}dx}+\int ce^{\int\frac{2A-D'}{D}dx}dx=\lambda,$$

where λ is any constant. The Julia's result can be summarized by the following theorem:

THEOREM 1. For the general form of the Abel equation of the second kind. If the coefficients of Eq.(1) satisfy the functional relation

$$g_0(x)(2f_2(x)+g_1'(x))=g_1(x)(f_1(x)+g_0'(x)), \tag{2}$$

where $g_1(x)\neq 0$, then the implicit solutions of Eq.(1) are given by

$$\frac{2g_0(x)u+g_1(x)u^2}{2g_1(x)J(x)}=\int\frac{f_0(x)}{g_1(x)J(x)}dx+c, \tag{3}$$

where c is an integration constant and

$$J(x)=\exp(\int\frac{2f_2(x)}{g_1(x)}dx).$$

A new functional relation between the variable coefficients that can lead to the general solutions of Eq.(1) is presented in [4] as follows:

THEOREM 2. For the general form of the Abel equation of the second kind Eq.(1). If there exists a constant λ such that

$$2B_1(x)g_0(x)=\lambda B_2(x)g_1(x),\ g_i(x)\neq 0,\ i=0,1,$$

then Eq.(1) admits the general solution

$$B_1(x)u^2+\lambda B_2(x)u=2\int\frac{f_0(x)}{g_1(x)}B_1(x)dx+c,$$

where $B_1(x)=\exp(-2\int\frac{f_2(x)}{g_1(x)}dx)$ and $B_2(x)=\exp(-\int\frac{f_1(x)}{g_0(x)}dx)$.

In this note, a new technique is analyzed to establish new different solutions of the general Abel equation of the second kind, we first reduce it into an equivalent equation, and then we formulate the relations between the coefficient functions $f_i(x),\ i=0,1,2$ and $g_i(x),\ i=0,1$ to obtain the well-known Julia's condition. This leads to a first-order linear differential equation, which can be solved in a closed form. Therefore the given Abel equation can be solved implicitly.

2 Main Result

Here, we prove the following result

THEOREM 3. For the general form of the Abel equation of the second kind. If the coefficients of Eq.(1) satisfy the Julia's condition (2), where $g_0,\ g_1\ \in C^1(a,b)$, $g_0(x) \neq 0$ and $f_i(x) \in C(a,b),\ i = 0,1,2$. Then the the general solution of Eq.(1) can be exactly obtained by

$$\frac{2g_0(x)u + g_1(x)u^2}{2g_0(x)L(x)} = \int \frac{f_0(x)}{g_0(x)L(x)}dx + c, \tag{4}$$

where c is a constant and

$$L(x) = e^{\int \frac{f_1(x)}{g_0(x)}dx}.$$

REMARK. Clearly, this solution is completely different from the one obtained by Julia [3], which can be regarded as a new implicit form of the Abel equation.

PROOF. First of all, we begin our approach by writing Eq.(1) in an equivalent form as

$$u' + \frac{g_1}{g_0}uu' = \frac{f_0}{g_0} + \frac{f_1}{g_0}u + \frac{f_2}{g_0}u^2 \tag{5}$$

in view of

$$\left(\frac{g_1}{g_0}u^2\right)' = 2\frac{g_1}{g_0}uu' + \left(\frac{g_1}{g_0}\right)'^2.$$

Thus Eq.(5) can be written as

$$u' + \left(\frac{g_1}{2g_0}u^2\right)' = \frac{f_0}{g_0} + \frac{f_1}{g_0}u + \left[\frac{f_2}{g_0} + \left(\frac{g_1}{2g_0}\right)'\right]u^2.$$

It follows

$$\left(u + \frac{g_1}{2g_0}u^2\right)' = \frac{f_0}{g_0} + \frac{f_1}{g_0}\left[u + \frac{\frac{f_2}{g_0} + \left(\frac{g_1}{2g_0}\right)'}{f_1/g_0}u^2\right]. \tag{6}$$

We assume now that

$$\frac{\frac{f_2}{g_0} + \left(\frac{g_1}{2g_0}\right)'}{f_1/g_0} = \frac{g_1}{2g_0},$$

or

$$\frac{f_2}{f_1} + \frac{g_1'}{2f_1} - \frac{g_1g_0'}{2f_1g_0} = \frac{g_1}{2g_0}.$$

A direct calculation produces the following equation

$$g_0\left(2f_2 + g_1'(x)\right) = g_1\left(f_1 + g_0'\right),$$

which is indeed the Julia's condition (2). Eq.(6) is then easily integrated to give a solution of Eq.(1). The substitution of Eq. (2) into Eq. (6) leads to the following equation

$$\left(u + \frac{g_1}{2g_0}u^2\right)' = \frac{f_0}{g_0} + \frac{f_1}{g_0}\left[u + \frac{g_1}{2g_0}u^2\right]. \tag{7}$$

Let $\psi = u + \frac{g_1}{2g_0}u^2$. Thus Eq.(7) becomes

$$\psi' = \frac{f_0(x)}{g_0(x)} + \frac{f_1(x)}{g_0(x)}\psi, \tag{8}$$

which is a first-order linear differential equation, and we can obtain its explicit solution form

$$\psi(x) = \frac{\int\left[\frac{f_0(x)}{g_0(x)}e^{-\int\frac{f_1(x)}{g_0(x)}dx}dx\right] + c}{e^{-\int\frac{f_1(x)}{g_0(x)}dx}}.$$

Hence

$$u + \frac{g_1(x)}{2g_0(x)}u^2 = \frac{\int\left[\frac{f_0(x)}{g_0(x)}e^{-\int\frac{f_1(x)}{g_0(x)}dx}dx\right] + c}{e^{-\int\frac{f_1(x)}{g_0(x)}dx}}.$$

This completes the proof of the theorem.

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