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Existence Of Positive Periodic Solutions For Two Types Of Third-Order Nonlinear Neutral Differential Equations With Variable Delay*

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Abstract

In this article we study the existence of positive periodic solutions for two types of third-order nonlinear neutral differential equation with variable delay. The main tool employed here is the Krasnoselskii's fixed point theorem dealing with a sum of two mappings, one is a contraction and the other is completely continuous. The results obtained here generalize the work of Ren, Siegmund and Chen [14].

1 Introduction

In recent years, there have been a few papers written on the existence of periodic solutions, nontrivial periodic solutions and positive periodic solutions for several classes of functional differential equations with delays, which arise from a number of mathematical ecological models, economical and control models, physiological and population models and other models, see [1–14], [16–18] and the references therein.

In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of delay differential equations. Motivated by the papers [2, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17] and the references therein, we concentrate on the existence of positive periodic solutions for the two types of third-order nonlinear neutral differential equation with variable delay

$$\frac{d^3}{dt^3}\left(x\left(t\right)-g\left(t,x\left(t-\tau\left(t\right)\right)\right)\right)=a\left(t\right)x\left(t\right)-f\left(t,x\left(t-\tau\left(t\right)\right)\right), \tag{1}$$

and

$$\frac{d^3}{dt^3}\left(x\left(t\right)-g\left(t,x\left(t-\tau\left(t\right)\right)\right)\right)=-a\left(t\right)x\left(t\right)+f\left(t,x\left(t-\tau\left(t\right)\right)\right), \tag{2}$$

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where $a, \tau \in C(\mathbb{R}, (0,\infty))$, $g \in C(\mathbb{R} \times [0,\infty), \mathbb{R})$, $f \in C(\mathbb{R} \times [0,\infty), [0,\infty))$, and a, τ, $g(t,x)$, $f(t,x)$ are T-periodic in t where T is a positive constant. To reach our desired end we have to transform (1) and (2) into integral equations and then use Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions. The obtained equation splits into a sum of two mappings, one is a contraction and the other is compact. In the special case $g(t,x) = cx$ with $|c| < 1$, Ren et al. in [14] show that (1) and (2) have a positive periodic solutions by using Krasnoselskii's fixed point theorem.

The organization of this paper is as follows. In Section 2, we introduce some notations and lemmas, and state some preliminary results needed in later sections, then we give the Green's function of (1) and (2), which plays an important role in this paper. Also, we present the inversions of (1) and (2), and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [15]. In Section 3 and Section 4, we present our main results on existence of positive periodic solutions of (1) and (2), respectively. The results presented in this paper generalize the main results in [14].

2 Preliminaries

For $T > 0$, let C_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(C_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

Define

$$C_T^+ = \{x \in C_T : x > 0\}, \ C_T^- = \{x \in C_T : x < 0\}.$$

Denote

$$M = \sup\{a(t) : t \in [0,T]\}, \ m = \inf\{a(t) : t \in [0,T]\}, \ \beta = \sqrt[3]{M},$$

and

$$F(t,x) = f(t, x(t-\tau(t))) - a(t)\, g(t, x(t-\tau(t))).$$

LEMMA 2.1 ([14]). The equation

$$\frac{d^3}{dt^3} y(t) - M y(t) = h(t), \ h \in C_T^-,$$

has a unique T-periodic solution

$$y(t) = \int_0^T G_1(t,s)(-h(s))\, ds,$$

where if $0 \leq s \leq t \leq T$,

$$
\begin{aligned}
G_1(t,s) &= \frac{2\exp\left(\frac{\beta(s-t)}{2}\right)}{3\beta^2\left[1+\exp(-\beta T)-2\exp\left(-\frac{\beta T}{2}\right)\cos\left(\frac{\sqrt{3}\beta T}{2}\right)\right]}\left[\sin\left(\frac{\sqrt{3}}{2}\beta(t-s)+\frac{\pi}{6}\right)\right. \\
&\quad \left. -\exp\left(-\frac{1}{2}\beta T\right)\sin\left(\frac{\sqrt{3}}{2}\beta(t-s-T)+\frac{\pi}{6}\right)\right]+\frac{\exp(\beta(t-s))}{3\beta^2(\exp(\beta T)-1)},
\end{aligned}
$$

and if $0 \leq t \leq s \leq T$,

$$
\begin{aligned}
G_1(t,s) &= \frac{2\exp\left(\frac{\beta(s-t-T)}{2}\right)}{3\beta^2\left[1+\exp(-\beta T)-2\exp\left(-\frac{\beta T}{2}\right)\cos\left(\frac{\sqrt{3}\beta T}{2}\right)\right]} \\
&\quad \times\left[\sin\left(\frac{\sqrt{3}}{2}\beta(t-s+T)+\frac{\pi}{6}\right)-\exp\left(-\frac{1}{2}\beta T\right)\sin\left(\frac{\sqrt{3}}{2}\beta(t-s)+\frac{\pi}{6}\right)\right] \\
&\quad +\frac{\exp(\beta(t+T-s))}{3\beta^2(\exp(\beta T)-1)}.
\end{aligned}
$$

LEMMA 2.2 ([14]). $\int_0^\omega G_1(t,s)ds = 1/M$ and if $\sqrt{3}\beta T < 4\pi/3$ holds, then $G_1(t,s) > 0$ for all $t \in [0,T]$ and $s \in [0,T]$.

LEMMA 2.3 ([14]). The equation

$$\frac{d^3}{dt^3}y(t) - a(t)\,y(t) = h(t)\,,\ \ h \in C_T^-,$$

has a unique positive T-periodic solution

$$(P_1 h)(t) = (I - T_1 B_1)^{-1}\, T_1 h(t)\,,$$

where

$$(T_1 h)(t) = \int_0^T G_1(t,s)(-h(s))\,ds \text{ and } (B_1 y)(t) = [-M + a(t)]\,y(t)\,.$$

LEMMA 2.4 ([14]). If $\sqrt{3}\beta T < 4\pi/3$ holds, then P_1 is completely continuous and

$$0 < (T_1 h)(t) \leq (P_1 h)(t) \leq \frac{M}{m}\|T_1 h\|\,,\ \ h \in C_T^-.$$

The following lemma is essential for our results on existence of positive periodic solution of (1). The proof is similar to that of Section 6 of [14] and hence, we omit it.

LEMMA 2.5. If $x \in C_T$ then x is a solution of equation (1) if and only if

$$x(t) = g(t, x(t-\tau(t))) + P_1(-f(t, x(t-\tau(t))) + a(t)\,g(t, x(t-\tau(t))))\,. \qquad (3)$$

LEMMA 2.6 ([14]). The equation

$$\frac{d^3}{dt^3}y(t)+My(t)=h(t),\ h\in C_T^+,$$

has a unique T-periodic solution

$$y(t)=\int_0^T G_2(t,s)\,h(s)\,ds,$$

where if $0\le s\le t\le T$,

$$\begin{aligned} G_2(t,s) &= \frac{2\exp\left(\frac{\beta(t-s)}{2}\right)}{3\beta^2\left[1+\exp(\beta T)-2\exp\left(\frac{\beta T}{2}\right)\cos\left(\frac{\sqrt{3}\beta T}{2}\right)\right]}\left[\sin\left(\frac{\sqrt{3}}{2}\beta(t-s)-\frac{\pi}{6}\right)\right.\\ &\quad \left.-\exp\left(\frac{1}{2}\beta T\right)\sin\left(\frac{\sqrt{3}}{2}\beta(t-s-T)-\frac{\pi}{6}\right)\right]+\frac{\exp(\beta(s-t))}{3\beta^2(1-\exp(-\beta T))}, \end{aligned}$$

and if $0\le t\le s\le T$,

$$\begin{aligned} G_2(t,s) &= \frac{2\exp\left(\frac{\beta(t+T-s)}{2}\right)}{3\beta^2\left[1+\exp(\beta T)-2\exp\left(\frac{\beta T}{2}\right)\cos\left(\frac{\sqrt{3}\beta T}{2}\right)\right]}\\ &\quad\times\left[\sin\left(\frac{\sqrt{3}}{2}\beta(t+T-s)-\frac{\pi}{6}\right)-\exp\left(\frac{1}{2}\beta T\right)\sin\left(\frac{\sqrt{3}}{2}\beta(t-s)-\frac{\pi}{6}\right)\right]\\ &\quad+\frac{\exp(\beta(s-t-T))}{3\beta^2(1-\exp(-\beta T))}. \end{aligned}$$

LEMMA 2.7 ([14]). $\int_0^T G_2(t,s)\,ds = 1/M$ and if $\sqrt{3}\beta T < 4\pi/3$ holds, then $G_2(t,s)>0$ for all $t\in[0,T]$ and $s\in[0,T]$.

LEMMA 2.8 ([14]). The equation

$$\frac{d^3}{dt^3}y(t)+a(t)\,y(t)=h(t),\ h\in C_T^+,$$

has a unique positive T-periodic solution

$$(P_2h)(t)=(I-T_2B_2)^{-1}\,T_2h(t),$$

where

$$(T_2h)(t)=\int_0^T G_2(t,s)\,h(s)\,ds,\ (B_2y)(t)=[M-a(t)]\,y(t).$$

LEMMA 2.9 ([14]). If $\sqrt{3}\beta T<4\pi/3$ holds, then P_2 is completely continuous and

$$0<(T_2h)(t)\le(P_2h)(t)\le\frac{M}{m}\|T_2h\|,\ h\in C_T^+.$$

The following lemma is essential for our results on existence of positive periodic solution of (2).

LEMMA 2.10. If $x \in C_T$ then x is a solution of equation (2) if and only if

$$x(t) = g(t, x(t-\tau(t))) + P_2 (f(t, x(t-\tau(t))) - a(t) g(t, x(t-\tau(t)))). \tag{4}$$

PROOF. Let $x \in P_T$ be a solution of (2). Rewrite (2) as

$$\begin{aligned}
&\frac{d^3}{dt^3}[x(t) - g(t, x(t-\tau(t)))] + M[x(t) - g(t, x(t-\tau(t)))] \\
&= [M - a(t)][x(t) - g(t, x(t-\tau(t)))] + f(t, x(t-\tau(t))) - a(t) g(t, x(t-\tau(t))) \\
&= B_2 [x(t) - g(t, x(t-\tau(t)))] + f(t, x(t-\tau(t))) - a(t) g(t, x(t-\tau(t))).
\end{aligned}$$

From Lemma 2.6, we have

$$\begin{aligned}
x(t) - g(t, x(t-\tau(t))) &= T_2 B_2 [x(t) - g(t, x(t-\tau(t)))] \\
&\quad + T_2 (f(t, x(t-\tau(t))) - a(t) g(t, x(t-\tau(t)))).
\end{aligned}$$

This yields

$$(I - T_2 B_2)(x(t) - g(t, x(t-\tau(t)))) = T_2 (f(t, x(t-\tau(t))) - a(t) g(t, x(t-\tau(t)))).$$

Therefore,

$$\begin{aligned}
x(t) - g(t, x(t-\tau(t))) &= (I - T_2 B_2)^{-1} T_2 (f(t, x(t-\tau(t))) - a(t) g(t, x(t-\tau(t)))) \\
&= P_2 (f(t, x(t-\tau(t))) - a(t) g(t, x(t-\tau(t)))).
\end{aligned}$$

Obviously,

$$x(t) = g(t, x(t-\tau(t))) + P_2 (f(t, x(t-\tau(t))) - a(t) g(t, x(t-\tau(t)))).$$

This completes the proof.

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1) and (2). For its proof we refer the reader to ([15], p. 31).

THEOREM 2.1 (Krasnoselskii). Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that

(i) $x, y \in \mathbb{D}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,

(ii) $\mathcal{A}$ is completely continuous,

(iii) $\mathcal{B}$ is a contraction mapping.

Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3 Positive Periodic Solutions for (1)

To apply Theorem 2.1, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is a completely continuous. So, we let $(\mathbb{B}, \|.\|) = (C_T, \|.\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : L \leq \varphi \leq K\}$, where L is non-negative constant and K is positive constant. We express equation (3) as

$$\varphi(t) = (\mathcal{B}_1\varphi)(t) + (\mathcal{A}_1\varphi)(t) := (H_1\varphi)(t),$$

where $\mathcal{A}_1, \mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_1\varphi)(t) = P_1(-f(t, \varphi(t-\tau(t))) + a(t)\, g(t, \varphi(t-\tau(t)))), \tag{5}$$

and

$$(\mathcal{B}_1\varphi)(t) = g(t, \varphi(t-\tau(t))). \tag{6}$$

In this section we obtain the existence of a positive periodic solution of (1) by considering the three cases; $g(t,x) > 0$, $g(t,x) = 0$ and $g(t,x) < 0$ for all $t \in \mathbb{R}$, $x \in \mathbb{D}$. We assume that function $g(t,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|g(t,x) - g(t,y)| \leq k\|x-y\|, \text{ for all } t \in [0,T], \; x, y \in \mathbb{D}. \tag{7}$$

In the case $g(t,x) > 0$, we assume that there exist positive constants k_1 and k_2 such that

$$k_1 x \leq g(t,x) \leq k_2 x, \text{ for all } t \in [0,T], \; x \in \mathbb{D}, \tag{8}$$

$$k_2 < 1, \tag{9}$$

and for all $t \in [0,T]$, $x \in \mathbb{D}$,

$$k_1 m \leq F(t,x) \leq M. \tag{10}$$

LEMMA 3.1. Suppose that (7) holds. If $\mathcal{B}_1$ is given by (6) with

$$k < 1, \tag{11}$$

then $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction.

PROOF. Let $\mathcal{B}_1$ be defined by (6). Obviously, $\mathcal{B}_1\varphi$ is continuous and it is easy to show that $(\mathcal{B}_1\varphi)(t+T) = (\mathcal{B}_1\varphi)(t)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$|(\mathcal{B}_1\varphi)(t) - (\mathcal{B}_1\psi)(t)| \leq |g(t, \varphi(t-\tau(t))) - g(t, \psi(t-\tau(t)))| \leq k\|\varphi - \psi\|.$$

Then $\|\mathcal{B}_1\varphi - \mathcal{B}_1\psi\| \leq k\|\varphi - \psi\|$. Thus $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction by (11).

Besides, by the complete continuity of P_1, it is easy to verify the following lemma.

LEMMA 3.2. Suppose that $\sqrt{3}\beta T < 4\pi/3$ and the conditions (8)-(10) hold. Then $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$ is completely continuous.

THEOREM 3.1. Suppose that $\sqrt{3}\beta T < 4\pi/3$ and the conditions (7)-(11) hold with $L = \dfrac{k_1 m}{(1-k_1)M}$ and $K = \dfrac{M}{(1-k_2)m}$. Then equation (1) has a positive T-periodic solution x in the subset

$$\mathbb{D} = \left\{ \varphi \in \mathbb{B} : \frac{k_1 m}{(1-k_1)M} \le \varphi \le \frac{M}{(1-k_2)m} \right\}.$$

PROOF. By Lemma 3.1, the operator $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction. Also, from Lemma 3.2, the operator $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$ is completely continuous. Moreover, we claim that $\mathcal{B}_1\psi + \mathcal{A}_1\varphi \in \mathbb{D}$ for all $\varphi, \psi \in \mathbb{D}$. Since $F(t,x) \ge k_1 m > 0$ which implies $-f(t,x) + a(t) g(t,x) < 0$, then for any $\varphi, \psi \in \mathbb{D}$, by Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&(\mathcal{B}_1\psi)(t) + (\mathcal{A}_1\varphi)(t) \\
&= g(t, \psi(t-\tau(t))) + P_1(-f(t, \varphi(t-\tau(t))) + a(t) g(t, \varphi(t-\tau(t)))) \\
&\le k_2 \psi(t-\tau(t)) + \frac{M}{m} \| T_1(-f(t, \varphi(t-\tau(t))) + a(t) g(t, \varphi(t-\tau(t)))) \| \\
&\le \tfrac{k_2 M}{(1-k_2)m} + \frac{M}{m} \max_{t\in[0,T]} \left| \int_0^T G_1(t,s)(f(s, \varphi(s-\tau(s))) - a(s) g(s, \varphi(s-\tau(s))))\,ds \right| \\
&\le \tfrac{k_2 M}{(1-k_2)m} + \frac{M}{m} \max_{t\in[0,T]} \int_0^T G_1(t,s)(f(s, \varphi(s-\tau(s))) - a(s) g(s, \varphi(s-\tau(s))))\,ds \\
&\le \frac{k_2 M}{(1-k_2)m} + \frac{M}{m} \int_0^T G_1(t,s) M ds \\
&= \frac{k_2 M}{(1-k_2)m} + \frac{M}{m} M \frac{1}{M} = \frac{M}{(1-k_2)m}.
\end{aligned}$$

On the other hand, by Lemma 2.2 and Lemma 2.4,

$$\begin{aligned}
&(\mathcal{B}_1\psi)(t) + (\mathcal{A}_1\varphi)(t) \\
&= g(t, \psi(t-\tau(t))) + P_1(-f(t, \varphi(t-\tau(t))) + a(t) g(t, \varphi(t-\tau(t)))) \\
&\ge k_1 \psi(t-\tau(t)) + \int_0^T G_1(t,s)(f(s, \varphi(s-\tau(s))) - a(s) g(s, \varphi(s-\tau(s))))\,ds \\
&\ge \frac{k_1^2 m}{(1-k_1)M} + \int_0^T G_1(t,s) k_1 m ds \\
&= \frac{k_1^2 m}{(1-k_1)M} + k_1 m \frac{1}{M} = \frac{k_1 m}{(1-k_1)M}.
\end{aligned}$$

Then $\mathcal{B}_1\psi + \mathcal{A}_1\varphi \in \mathbb{D}$ for all $\varphi, \psi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}_1 x + \mathcal{B}_1 x$. By Lemma 2.5 this fixed point is a solution of (1) and the proof is complete.

EXAMPLE 3.1. Consider the following third-order nonlinear neutral differential equation with variable delay

$$\frac{d^3}{dt^3}[x(t) - g(t, x(t-\tau(t)))] = a(t) x(t) - f(t, x(t-\tau(t))), \tag{12}$$

where $T = \pi$, $\tau(t) = \sin^2(t)$, $a(t) = \frac{1}{5}\sin^2(t) + 0.8$, $g(t,x) = 0.6\sin\left(\frac{x}{2}\right)$, and

$$f(t,x) = \frac{\sin^2(t)}{x^2 + 1.6} + 0.12\sin^2(t)\sin\left(\frac{x}{2}\right) + 0.48\sin^2\left(\frac{x}{2}\right) + 0.2.$$

Then Equation (12) has a positive π-periodic solution x satisfying $0.2 \leq x \leq 2.5$. To see this, a simple calculation yields

$$k = 0.3,\ m = 0.8,\ M = 1,\ k_1 = 0.2,\ k_2 = 0.5,\ L = 0.2,\ K = 2.5.$$

Define the set $\mathbb{D} = \{\varphi \in \mathbb{B} : 0.2 \leq \varphi \leq 2.5\}$. Then for $x \in [0.2, 2.5]$ we have

$$F(t,x) = \frac{\sin^2(t)}{x^2 + 1.6} + 0.2 \leq 0.81 < 1 = M.$$

On the other hand,

$$F(t,x) = \frac{\sin^2(t)}{x^2 + 1.6} + 0.2 \geq 0.2 > 0.16 = k_1 m.$$

By Theorem 3.1, Equation (12) has a positive π-periodic solution x such that $0.2 \leq x \leq 2.5$.

REMARK 3.1. When $g(t,x) = cx$, Theorem 3.1 reduces to Theorem 6.2 of [14].

In the case $g(t,x) = 0$, we have the following theorem.

THEOREM 3.2 ([14]). If $\sqrt{3}\beta T < 4\pi/3$ holds, $k_2 = 0$ and $0 < F(t,x) \leq M$, then equation (1) has a positive T-periodic solution x in the subset

$$\mathbb{D}_1 = \left\{\varphi \in \mathbb{B} : 0 < \varphi \leq \frac{M}{m}\right\}.$$

In the case $g(t,x) < 0$, we substitute conditions (8)-(10) with the following conditions respectively. We assume that there exist negative constants k_3 and k_4 such that

$$k_3 x \leq g(t,x) \leq k_4 x, \text{ for all } t \in [0,T],\ x \in \mathbb{D}, \tag{13}$$

$$-k_3 < \frac{m}{M}, \tag{14}$$

and for all $t \in [0,T]$, $x \in \mathbb{D}$

$$-k_3 M < F(t,x) \leq m. \tag{15}$$

THEOREM 3.3. Suppose that $\sqrt{3}\beta T < 4\pi/3$, (7) and (11)-(15) hold with $L = 0$ and $K = 1$. Then equation (1) has a positive T-periodic solution x in the subset $\mathbb{D}_2 = \{\varphi \in \mathbb{B} : 0 < \varphi \leq 1\}$.

PROOF. By Lemma 3.1, the operator $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction. Also, from Lemma 3.2, the operator $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$ is completely continuous. Moreover, we claim that $\mathcal{B}_1\psi + \mathcal{A}_1\varphi \in \mathbb{D}$ for all $\varphi, \psi \in \mathbb{D}$. In fact, for any $\varphi, \psi \in \mathbb{D}$, by Lemma 2.2 and Lemma 2.4, we have

$$\begin{aligned}
&(\mathcal{B}_1\psi)(t) + (\mathcal{A}_1\varphi)(t) \\
&= g(t, \psi(t-\tau(t))) + P_1(-f(t, \varphi(t-\tau(t))) + a(t)\, g(t, \varphi(t-\tau(t)))) \\
&\le k_4\psi(t-\tau(t)) + \frac{M}{m}\left\| T_1(-f(t, \varphi(t-\tau(t))) + a(t)\, g(t, \varphi(t-\tau(t))))\right\| \\
&\le \frac{M}{m}\max_{t\in[0,T]}\left|\int_0^T G_1(t,s)\,(f(s, \varphi(s-\tau(s))) - a(s)\, g(s, \varphi(s-\tau(s))))\,ds\right| \\
&\le \frac{M}{m}\max_{t\in[0,T]}\int_0^T G_1(t,s)\,(f(s, \varphi(s-\tau(s))) - a(s)\, g(s, \varphi(s-\tau(s))))\,ds \\
&\le \frac{M}{m}\int_0^T G_1(t,s)\, m ds = \frac{M}{m} m \frac{1}{M} = 1.
\end{aligned}$$

On the other hand, by Lemma 2.2 and Lemma 2.4,

$$\begin{aligned}
&(\mathcal{B}_1\psi)(t) + (\mathcal{A}_1\varphi)(t) \\
&= g(t, \psi(t-\tau(t))) + P_1(-f(t, \varphi(t-\tau(t))) + a(t)\, g(t, \varphi(t-\tau(t)))) \\
&\ge k_3\psi(t-\tau(t)) + \int_0^T G_1(t,s)\,(f(s, \varphi(s-\tau(s))) - a(s)\, g(s, \varphi(s-\tau(s))))\,ds \\
&\ge k_3 + \int_0^T G_1(t,s)\,(-k_3M)\,ds \\
&= k_3 + (-k_3M)\frac{1}{M} = 0.
\end{aligned}$$

Then $\mathcal{B}_1\psi + \mathcal{A}_1\varphi \in \mathbb{D}$ for all $\varphi, \psi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}_1x + \mathcal{B}_1x$. Since $F(t,x) > -k_3M$, it is clear that $x(t) > 0$, hence $x \in \mathbb{D}_2$. By Lemma 2.5 this fixed point is a solution of (1) and the proof is complete.

REMARK 3.2. When $g(t,x) = cx$, Theorem 3.3 reduces to Theorem 6.6 of [14].

4 Positive Periodic Solutions for (2)

We express equation (4) as

$$\varphi(t) = (\mathcal{B}_2\varphi)(t) + (\mathcal{A}_2\varphi)(t) := (H_2\varphi)(t),$$

where $\mathcal{A}_2, \mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_2\varphi)(t) = P_2(f(t, \varphi(t-\tau(t))) - a(t)\, g(t, \varphi(t-\tau(t)))), \tag{16}$$

and

$$\left(\mathcal{B}_2\varphi\right)(t) = g\left(t, \varphi\left(t - \tau\left(t\right)\right)\right). \tag{17}$$

Moreover, by the complete continuity of P_2, it is easy to verify

LEMMA 4.1. Suppose that $\sqrt{3}\beta T < 4\pi/3$ and the conditions (8)-(10) hold. Then $\mathcal{A}_2 : \mathbb{D} \to \mathbb{B}$ is completely continuous.

REMARK 4.1. Notice that $\mathcal{B}_2$ in this section is defined exactly the same as that in Section 3. Hence Lemma 3.1 still holds true.

Similar to the results in Section 3, we have

THEOREM 4.1. Assume that the hypotheses of Theorem 3.1 hold, then equation (2) has a positive T-periodic solution x in the subset

$$\mathbb{D} = \left\{\varphi \in \mathbb{B} : \frac{k_2}{M} \leq \varphi \leq \frac{1}{m}\right\}.$$

THEOREM 4.2. Assume that the hypotheses of Theorem 3.2 hold, then equation (2) has a positive T-periodic solution x in the subset

$$\mathbb{D}_1 = \left\{\varphi \in \mathbb{B} : 0 < \varphi \leq \frac{1}{m}\right\}.$$

THEOREM 4.3. Assume that the hypotheses of Theorem 3.3 hold, then equation (2) has a positive T-periodic solution x in the subset

$$\mathbb{D}_2 = \left\{\varphi \in \mathbb{B} : 0 < \varphi \leq 1\right\}.$$

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