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Fixed Points Of (ψ, φ)-Almost Weakly Contractive Maps In G-Metric Spaces*

Gutti Venkata Ravindranadh Babu†, Dasari Ratna Babu‡, Kanuri Nageswara Rao§, Bendi Venkata Siva Kumar†

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Abstract

In this paper, we introduce (ψ, φ)-almost weakly contractive maps in G-metric spaces and prove the existence of fixed points. Our Theorem 4 generalizes the result of Aage and Salunke (Theorem 2, [1]). We also extend it to a pair of weakly compatible maps and prove the existence of common fixed points. We provide examples in support of our results.

1 Introduction and Preliminaries

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful results in fixed point theory. In the direction of generalization of contraction conditions, in 1997, Alber and Guerre-Delabriere [3] introduced weakly contractive maps which are extensions of contraction maps and obtained fixed point results in the setting of Hilbert spaces. Rhoades [16] extended this concept to metric spaces. In 2008, Dutta and Choudhury [12] introduced (ψ, φ)-weakly contractive maps and proved the existence of fixed points in complete metric spaces. In 2009, Doric [11] extended it to a pair of maps. For more literature in this direction, we refer to Choudhury, Konar and Rhoades [9], Babu, Nageswara Rao and Alemayehu [4], Sastry, Babu and Kidane [17], Babu and Sailaja [5] and Zhang and Song [19]. In continuation to the extensions of contraction maps, Berinde [7] introduced 'weak contractions' as a generalization of contraction maps. Berinde renamed 'weak contractions' as 'almost contractions' in his later work [8]. For more works on almost contractions and its generalizations, we refer to Babu, Sandhya and Kameswari [6], Abbas, Babu and Alemayehu [2] and the related references cited in these papers.

Throughout this paper, we denote $\mathbb{R}_+ = [0, \infty)$ and

$$\begin{aligned}\Psi \;=\; &\{\psi/\psi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is continuous on } \mathbb{R}_+, \psi \text{ is nondecreasing},\\ &\psi(t) > 0 \text{ for } t > 0, \psi(0) = 0\}.\end{aligned}$$

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†Department of Mathematics, Andhra University, Visakhapatnam-530 003, India
‡Department of Mathematics, PSCMRCET, Vijayawada-520 001, India
§Department of Mathematics, GIET, Rajahmundry, India

In the metric space setting Dutta and Choudhury [12] introduced (ψ,φ)-weakly contractive maps as follows:

DEFINITION 1 ([12]). Let (X,d) be a metric space. Let $T : X \to X$ be a map. If there exist $\psi,\varphi \in \Psi$ such that

$$\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \varphi(d(x,y))$$

for all $x,y \in X$, then T is said to be a (ψ,φ)-weakly contractive map.

Dutta and Choudhury [12] proved that every (ψ,φ)-weakly contractive map has a unique fixed point in complete metric spaces. On the other hand, Berinde [7] introduced 'weak contractions' as a generalization of contraction maps.

DEFINITION 2 ([7]). Let (X,d) be a metric space. A selfmap $T : X \to X$ is said to be a weak contraction if there exist $\delta \in (0,1)$ and $L \ge 0$ such that for all $x,y \in X$,

$$d(Tx,Ty) \le \delta d(x,y) + Ld(y,Tx).$$

Berinde [7] proved that every weak contraction has a fixed point in complete metric spaces and provided an example to show that this fixed point need not be unique. In order to obtain the uniqueness of fixed point, Berinde [7] used the following condition: there exist $\theta \in (0,1)$ and $L_1 \ge 0$ such that

$$d(Tx,Ty) \le \theta d(x,y) + L_1 d(x,Tx) \text{ for all } x,y \in X \tag{1}$$

and proved that every weak contraction together with (1) has a unique fixed point in complete metric spaces, and further posed the following problem: "Find a contractive type condition different from (1), that ensures the uniqueness of fixed point of weak contractions".

In this context Babu, Sandhya and Kameswari [6] answered the above problem by introducing 'condition (B)' as follows:

DEFINITION 3 ([6]). Let (X,d) be a metric space. A map $T : X \to X$ is said to satisfy condition (B) if there exist $0 < \delta < 1$ and $L \ge 0$ such that for all $x,y \in X$,

$$d(Tx,Ty) \le \delta d(x,y) + L\min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Babu, Sandhya and Kameswari [6] proved that every selfmap T of a complete metric space satisfying condition (B) has a unique fixed point. On the other hand, in the direction of generalization of ambient spaces, in 2005, Mustafa and Sims [15] introduced a new notion namely generalized metric space called G-metric space and studied the existence of fixed points of various types of contraction mappings in G-metric spaces.

DEFINITION 4 ([15]). Let X be a nonempty set and let $G : X^3 \to \mathbb{R}_+$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all variables) and,

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially a G-metric on X, and the pair (X, G) is called a G-metric space.

EXAMPLE 1 ([15]). Let (X, d) be a metric space. The mapping $G_s : X^3 \to \mathbb{R}_+$ defined by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in X$ is a G-metric and so (X, G_s) is a G-metric space.

EXAMPLE 2 ([15]). Let (X, d) be a metric space. The mapping $G_m : X^3 \to \mathbb{R}_+$ defined by

$$G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}$$

for all $x, y, z \in X$ is a G-metric and so (X, G_m) is a G-metric space.

EXAMPLE 3. Let X be a nonempty set. We denote the class of all real valued bounded functions on X by $B(X)$. For $f \in B(X)$, we define

$$\|f\| = \sup\{|f(x)| / x \in X\}.$$

Then $(B(X), ||.||)$ is a normed linear space. We define metric d on $B(X)$ by $d(f, g) = \|f - g\|$ for $f, g \in B(X)$. Now we define generalized metric G on $B(X)$ by

$$G(f, g, h) = \|f - g\| + \|g - h\| + \|h - f\|$$

for all $f, g, h \in B(X)$. Then clearly G is a generalized metric on $B(X)$. The space $(B(X), G)$ is a generalized metric space.

DEFINITION 5 ([15]). Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. We say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$; that is, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geq N$. We refer to x as the limit of the sequence $\{x_n\}$.

PROPOSITION 1 ([15]). Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$ we have that:

(1) if $G(x, y, z) = 0$, then $x = y = z$.

(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$.

(3) $G(x,y,y) \leq 2G(y,x,x)$.

(4) $G(x,y,z) \leq G(x,a,z) + G(a,y,z)$.

(5) $G(x,y,z) \leq \frac{2}{3}\left(G(x,y,a) + G(x,a,z) + G(a,y,z)\right)$.

PROPOSITION 2 ([15]). Let (X,G) be a G-metric space. Then the following statements are equivalent:

(1) $\{x_n\}$ is G-convergent to x.

(2) $G(x_n,x_n,x) \to 0$ as $n \to \infty$.

(3) $G(x_n,x,x) \to 0$ as $n \to \infty$.

DEFINITION 6 ([15]). Let X be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$ for all $n,m,l \geq N$; that is, if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

PROPOSITION 3 ([15]). In a G-metric space X, the following two statements are equivalent:

(1) The sequence $\{x_n\}$ is G-Cauchy.

(2) For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \epsilon$ for all $n,m \geq N$.

DEFINITION 7 ([15]). A G-metric space X is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in X is G-convergent in X.

PROPOSITION 4 ([15]). Let X be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.

PROPOSITION 5 ([15]). Every G-metric space X defines a metric space (X,d_G) by

$$d_G(x,y) = G(x,y,y) + G(y,x,x) \text{ for all } x,y \in X.$$

Mustafa, Obiedat and Awawdeh [14] proved the following result.

THEOREM 1 ([14]). Let (X,G) be a complete G-metric space, and let $T : X \to X$ be a mapping satisfying one of the following conditions:

$$G(Tx,Ty,Tz) \leq aG(x,y,z) + bG(x,Tx,Tx) + cG(y,Ty,Ty) + dG(z,Tz,Tz)$$

or

$$G(Tx,Ty,Tz) \leq aG(x,y,z) + bG(x,x,Tx) + cG(y,y,Ty) + dG(z,z,Tz)$$

for all $x,y,z \in X$ where $0 \leq a+b+c+d < 1$. Then T has a unique fixed point (say u, i.e., $Tu = u$), and T is G-continuous at u.

In 2011, Aage and Salunke [1] introduced weakly contractive maps in G-metric spaces and proved the existence of fixed points in G-metric spaces.

DEFINITION 8. Let (X,G) be a G-metric space. Let $T:X\to X$ be a selfmap of X. T is said to be a weakly contractive map in G if, there exists $\varphi\in\Psi$ such that

$$G(Tx,Ty,Tz)\le G(x,y,z)-\varphi(G(x,y,z)) \text{ for each } x,y,z\in X. \tag{2}$$

THEOREM 2 ([1]). Let (X,G) be a complete G-metric space and let $T:X\to X$ be a weakly contractive map in G. Then T has a unique fixed point in X.

DEFINITION 9 ([10]). Let f and g be two selfmaps on a G-metric space (X,G). The mappings f and g are said to be *compatible* if $\lim_{n\to\infty} G\left(fgx_n, gfx_n, gfx_n\right)=0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=z$ for some $z\in X$.

DEFINITION 10 ([10,13]). Two maps f and g on a G-metric space (X,G) are said to be *weakly compatible* if they commute at their coincidence point.

Here we note that every pair of compatible maps is weakly compatible but its converse need not be true (Example 1.4, [10]). Shatanawi [18] proved the following common fixed point theorem for a pair of weakly compatible maps.

THEOREM 3 ([18]). Let X be a G-metric space. Suppose the maps $f,g:X\to X$ satisfy the following condition: there exists a nondecreasing function $\phi:\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\lim_{n\to\infty}\phi^n(t)=0$ for all $t\in(0,\infty)$ such that either

$$G(fx,fy,fz)\le\phi(\max\{G(gx,gy,gz),G(gx,fx,fx),G(gy,fy,fy),G(gz,fz,fz)\}), \tag{3}$$

or

$$G(fx,fy,fz)\le\phi(\max\{G(gx,gy,gz),G(gx,gx,fx),G(gy,gy,fy),G(gz,gz,fz)\})$$

for all $x,y,z\in X$. If $f(X)\subseteq g(X)$ and $g(X)$ is a G-complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Unfortunately, the example given in support of Theorem 2 by Aage and Salunke (Example 2, [1]) is false in the sense that the maps T and φ defined in this example do not satisfy the inequality (2). For, the example considered by Aage and Salunke is the following.

EXAMPLE 4 ([1]). Let $X=[0,1]$. Define $G:X^3\to\mathbb{R}_+$ by

$$G(x,y,z)=|x-y|+|y-z|+|z-x| \text{ for all } x,y,z\in X.$$

Then (X,G) is a complete G-metric space. The authors defined T on X by $Tx = x - \frac{x^2}{2}$ and $\varphi(t) = \frac{t^2}{2}$, $t \geq 0$. Let us choose $x = 1$, $y = \frac{1}{2}$ and $z = \frac{1}{4}$. Then $G(Tx,Ty,Tz) = \frac{9}{16}$, $G(x,y,z) = \frac{3}{2}$ and $\varphi(G(x,y,z)) = \frac{9}{8}$. Hence

$$\frac{9}{16} = G(Tx,Ty,Tz) \nleq G(x,y,z) - \varphi(G(x,y,z)) = \frac{3}{8}.$$

Also the inequality (2) fails to hold at $x = \frac{1}{2}$, $y = \frac{1}{3}$ and $z = 0$. Hence T and φ do not satisfy the inequality (2) so that T is not a weakly contractive map with this φ, even though T has a fixed point 0.

The following is a suitable example in support of Theorem 2.

EXAMPLE 5. Let $X = [0,1]$. We define $G : X^3 \to \mathbb{R}_+$ by

$$G(x,y,z) = \begin{cases} 0, & \text{if } x = y = z \\ \max\{x,y,z\}, & \text{otherwise.} \end{cases}$$

Then (X,G) is a complete G-metric space. Let $Tx = \frac{x^2}{2}$ and $\varphi(t) = \frac{t^2}{4}$. Without loss of generality, we assume that $x > y > z$. Then

$$G(Tx,Ty,Tz) = \max\{Tx,Ty,Tz\} = \frac{x^2}{2},$$

$$G(x,y,z) = \max\{x,y,z\} = x \text{ and } \varphi(G(x,y,z)) = \frac{x^2}{4}.$$

Now, $G(x,y,z) - \varphi(G(x,y,z)) = x - \frac{x^2}{4}$. Therefore $G(Tx,Ty,Tz) = \frac{x^2}{2} < x - \frac{x^2}{4} = G(x,y,z) - \varphi(G(x,y,z))$. Hence T satisfies the inequality (2) so that T is a weakly contractive map. Thus by Theorem 2, we have T has a unique fixed point and it is 0 in X.

Motivated by the '(ψ,φ)-weakly contractive maps' introduced by Dutta and Choudhury [12], 'almost weak contractions' of Berinde [7, 8] and 'condition (B)' of Babu, Sandhya and Kameswari [6] in metric space setting, in this paper we introduce '(ψ,φ)-almost weakly contractive maps' in G-metric spaces and prove the existence of fixed points in complete G-metric spaces. The importance of the class of (ψ,φ)-almost weakly contractive maps is that this class properly includes the class of weakly contractive maps studied by Aage and Salunke [1] so that the class of (ψ,φ)-almost weakly contractive maps is larger than the class of weakly contractive maps, which is illustrated in Example 6. Hence, the results obtained on the existence of fixed points of (ψ,φ)-almost weakly contractive maps generalize the results of Aage and Salunke [1].

In the following, we introduce (ψ,φ)-almost weakly contractive maps.

DEFINITION 11. Let (X,G) be a G-metric space and let T be a selfmap of X. If there exist ψ and φ in Ψ and $L \geq 0$ such that

$$\psi(G(Tx,Ty,Tz)) \leq \psi(G(x,y,z)) - \varphi(G(x,y,z)) + L\ m(x,y,z) \tag{4}$$

for all $x, y, z \in X$, where

$$m(x,y,z) = \min\{G(Tx,x,x), G(Tx,y,y), G(Tx,z,z), G(Tx,y,z)\},$$

then we call T is a (ψ, φ)-almost weakly contractive map on X.

We observe that if ψ is the identity map and $L = 0$ in (4) then T is a weakly contractive map. Hence the class of all weakly contractive maps is contained in the class of all (ψ, φ)-almost weakly contractive maps. Further, every (ψ, φ)-almost weakly contractive map need not be a weakly contractive map (Example 6).

In Section 2, we prove the existence of fixed points of (ψ, φ)-almost weakly contractive maps in G-metric spaces. Our main result (Theorem 4) generalizes the result of Aage and Salunke (Theorem 2, [1]). We also extend it to a pair of weakly compatible maps and prove the existence of common fixed points. Corollaries and examples in support of our results are provided in Section 3.

2 Main Results

The following is the main result of this paper.

THEOREM 4. Let (X, G) be a complete G-metric space and let T be a (ψ, φ)-almost weakly contractive map. Then T has a unique fixed point in X.

PROOF. Let $x_0 \in X$. We define the sequence $\{x_n\}$ by $x_n = T(x_{n-1})$, $n = 1, 2, \ldots$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then trivially x_n a fixed point of T. Suppose $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. We now consider

$$\begin{aligned} \psi(G(x_n, x_{n+1}, x_{n+1})) &= \psi(G(Tx_{n-1}, Tx_n, Tx_n)) \\ &\leq \psi(G(x_{n-1}, x_n, x_n)) - \varphi(G(x_{n-1}, x_n, x_n)) \\ &\quad + Lm(x_{n-1}, x_n, x_n), \end{aligned}$$

where $m(x_{n-1}, x_n, x_n) = 0$ so that

$$\psi(G(x_n, x_{n+1}, x_{n+1})) \leq \psi(G(x_{n-1}, x_n, x_n)) - \varphi(G(x_{n-1}, x_n, x_n)). \tag{5}$$

By using the property of φ, we have

$$\psi(G(x_n, x_{n+1}, x_{n+1})) < \psi(G(x_{n-1}, x_n, x_n)) \text{ for } n = 1, 2, \ldots. \tag{6}$$

Now, by applying the nondecreasing property of ψ, it follows that

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) \text{ for } n = 1, 2, \ldots.$$

Therefore $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a monotone decreasing sequence of nonnegative reals and hence there exists $r \geq 0$ such that $G(x_n, x_{n+1}, x_{n+1}) \to r$ as $n \to \infty$. Now, on letting $n \to \infty$ in the inequality (5), we have $\psi(r) \leq \psi(r) - \varphi(r)$ so that $\varphi(r) \leq 0$. Since $\varphi(r) \geq 0$, it follows that $\varphi(r) = 0$ so that $r = 0$.

$$\text{i.e., } G(x_n, x_{n+1}, x_{n+1}) \to 0 \text{ as } n \to \infty. \tag{7}$$

We now prove that the sequence $\{x_n\}$ is Cauchy.

On the contrary, if $\{x_n\}$ is not Cauchy, then there exists an $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}, \{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k \geq k$ such that

$$G(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon. \tag{8}$$

Corresponding to each m_k, we can choose n_k such that it is the smallest integer with $n_k > m_k$ and satisfying (8). Then, we have

$$G(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon \text{ and } G(x_{n_k-1}, x_{m_k}, x_{m_k}) < \epsilon. \tag{9}$$

We now prove the following three identities:

(i) $\lim_{k\to\infty} G(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon$.

(ii) $\lim_{k\to\infty} G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) = \epsilon$.

(iii) $\lim_{k\to\infty} G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) = \epsilon$.

From (9), we have $G(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon$ so that

$$\epsilon \leq \lim_{k\to\infty} \inf G(x_{n_k}, x_{m_k}, x_{m_k}). \tag{10}$$

Also,

$$\begin{aligned} G(x_{n_k}, x_{m_k}, x_{m_k}) &\leq G(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + G(x_{n_k-1}, x_{m_k}, x_{m_k}) \\ &< G(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + \epsilon, \end{aligned}$$

and hence

$$\lim_{k\to\infty} \sup G(x_{n_k}, x_{m_k}, x_{m_k}) \leq \epsilon. \tag{11}$$

From (10) and (11), we have

$$\epsilon \leq \lim_{k\to\infty} \inf G(x_{n_k}, x_{m_k}, x_{m_k}) \leq \lim_{k\to\infty} \sup G(x_{n_k}, x_{m_k}, x_{m_k}) \leq \epsilon$$

so that $\lim_{k\to\infty} G(x_{n_k}, x_{m_k}, x_{m_k})$ exists and

$$\epsilon = \lim_{k\to\infty} \inf G(x_{n_k}, x_{m_k}, x_{m_k}) \leq \lim_{k\to\infty} \sup G(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon.$$

Hence

$$\lim_{k\to\infty} G(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon. \tag{12}$$

Therefore (i) holds. Also,

$$\begin{aligned} G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) &\leq G(x_{n_k-1}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_k-1}, x_{m_k-1}) \\ &\leq G(x_{n_k-1}, x_{n_k}, x_{n_k}) + G(x_{n_k}, x_{m_k}, x_{m_k}) \\ &\quad + G(x_{m_k}, x_{m_k-1}, x_{m_k-1}). \end{aligned}$$

On taking limit superior as $k \to \infty$ and using (7) and (12), we get

$$\lim_{k\to\infty} \sup G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) \le \epsilon. \tag{13}$$

Now,

$$\begin{aligned} G(x_{n_k}, x_{m_k}, x_{m_k}) \quad &\le \quad G(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) \\ &\quad +G(x_{m_k-1}, x_{m_k}, x_{m_k}). \end{aligned}$$

Hence, we have that

$$\begin{aligned} G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) \quad &\ge \quad G(x_{n_k}, x_{m_k}, x_{m_k}) - G(x_{n_k}, x_{n_k-1}, x_{n_k-1}) \\ &\quad -G(x_{m_k-1}, x_{m_k}, x_{m_k}). \end{aligned}$$

Now on taking limit inferior both sides, and using (7) and (12), we get

$$\epsilon \le \lim_{k\to\infty} \inf G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}). \tag{14}$$

Thus from (13) and (14), we have

$$\epsilon \le \lim_{k\to\infty} \inf G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) \le \lim_{k\to\infty} \sup G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) \le \epsilon.$$

Hence it follows that

$$\lim_{k\to\infty} G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}) = \epsilon. \tag{15}$$

Therefore (ii) holds. Let us now prove (iii). From (8), we have

$$\epsilon \le G(x_{n_k}, x_{m_k}, x_{m_k}) \le G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) + G(x_{m_k-1}, x_{m_k}, x_{m_k}).$$

This implies that

$$G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \ge \epsilon - G(x_{m_k-1}, x_{m_k}, x_{m_k})$$

and

$$\lim_{k\to\infty} \inf G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \ge \epsilon. \tag{16}$$

Now,

$$G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \le G(x_{n_k}, x_{n_k-1}, x_{n_k-1}) + G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1})$$

and hence using (7) and (ii), we get

$$\lim_{k\to\infty} \sup G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \le \epsilon. \tag{17}$$

Now, from (16) and (17), we have

$$\epsilon \le \lim_{k\to\infty} \inf G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \le \lim_{k\to\infty} \sup G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \le \epsilon$$

so that $G(x_{n_k}, x_{m_k-1}, x_{m_k-1}) \to \epsilon$ as $k \to \infty$. Therefore (iii) holds.

Now

$$\begin{aligned}\psi(G(x_{n_k}, x_{m_k}, x_{m_k})) &= \psi(G(Tx_{n_k-1}, Tx_{m_k-1}, Tx_{m_k-1})) \\ &\leq \psi(G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1})) - \varphi(G(x_{n_k-1}, x_{m_k-1}, x_{m_k-1})) \\ &\quad + L\ m(x_{n_k-1}, x_{m_k-1}, x_{m_k-1}).\end{aligned}$$

On letting $k \to \infty$ and using (i)–(iii) and (7), we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon) < \psi(\epsilon),$$

which is a contradiction. Therefore $\{x_n\}$ is a G-Cauchy sequence. Since X is complete, there exists $p \in X$ such that $\{x_n\}$ is G-convergent to p. We now consider

$$\begin{aligned}\psi(G(x_n, Tp, Tp)) &= \psi(G(Tx_{n-1}, Tp, Tp)) \\ &\leq \psi(G(x_{n-1}, p, p)) - \varphi(G(x_{n-1}, p, p)) + Lm(x_{n-1}, p, p).\end{aligned}$$

On letting $n \to \infty$, we have $\varphi(G(p, Tp, Tp)) \leq 0$ so that we must have $Tp = p$. Therefore p is a fixed point of T in X.

Uniqueness: Suppose T has two fixed points p and q in X with $p \neq q$. Now, we consider

$$\begin{aligned}\psi(G(p, q, q)) &= \psi(G(Tp, Tq, Tq)) \\ &\leq \psi(G(p, q, q)) - \varphi(G(p, q, q)) + Lm(p, q, q) \\ &= \psi(G(p, q, q)) - \varphi(G(p, q, q)) < \psi(G(p, q, q)),\end{aligned}$$

which is a contradiction. Therefore $\psi(G((p, q, q)) = 0$ so that $G(p, q, q) = 0$ and hence that $p = q$. Thus, p is the unique fixed point of T in X. Hence the theorem follows.

We now prove a common fixed point theorem for a pair of weakly compatible maps.

THEOREM 5. Let (X, G) be a complete G-metric space and let T and S be two selfmaps on (X, G). Assume that $T(X) \subseteq S(X)$, S is continuous, and there exist $\psi, \varphi \in \Psi$ and $L \geq 0$ such that

$$\psi(G(Tx, Ty, Tz)) \leq \psi(M(x, y, z)) - \varphi(M(x, y, z)) + Lm(x, y, z), \tag{18}$$

where

$$M(x, y, z) = \max\{G(Sx, Sy, Sz), G(Sx, Tx, Tx), G(Sy, Ty, Ty), G(Sz, Tz, Tz)\}$$

and

$$m(x, y, z) = \min\{G(Tx, Sx, Sx), G(Tx, Sy, Sy), G(Tx, Sz, Sz), G(Tx, Sy, Sz)\}$$

for $x, y, z \in X$. Then T and S have a unique common fixed point in X provided T and S are weakly compatible maps.

PROOF. Let $x_0 \in X$ be arbitrary. Since $T(X) \subseteq S(X)$, we can choose $\{x_n\} \subseteq X$ such that $T(x_n) = S(x_{n+1}) = y_n$ (say), $n = 0, 1, 2, ...$ Let $n \geq 1$ be an integer. Then by using the inequality (18) we have

$$\begin{aligned}\psi(G(y_n, y_{n+1}, y_{n+1})) &= \psi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \\ &\leq \psi(M(x_n, x_{n+1}, x_{n+1})) - \varphi(M(x_n, x_{n+1}, x_{n+1})) \\ &\quad +Lm(x_n, x_{n+1}, x_{n+1}),\end{aligned}$$

where

$$\begin{aligned}M(x_n, x_{n+1}, x_{n+1}) &= \max\{G(Sx_n, Sx_{n+1}, Sx_{n+1}), G(Sx_n, Tx_n, Tx_n), \\ &\quad G(Sx_{n+1}, Tx_{n+1}, Tx_{n+1}), G(Sx_{n+1}, Tx_{n+1}, Tx_{n+1})\} \\ &= G(Sx_n, Sx_{n+1}, Sx_{n+1})\end{aligned}$$

and

$$\begin{aligned}&m(x_n, x_{n+1}, x_{n+1}) \\ =\ &\min\{G(Tx_n, Sx_n, Sx_n), G(Tx_n, Sx_{n+1}, Sx_{n+1}), \\ &G(Tx_n, Sx_{n+1}, Sx_{n+1}), G(Tx_n, Sx_{n+1}, Sx_{n+1})\} \\ =\ &\min\{G(y_n, y_{n-1}, y_{n-1}), G(y_n, y_n, y_n), G(y_n, y_n, y_n), G(y_n, y_n, y_n)\} \\ =\ &0\end{aligned}$$

since $G(y_n, y_n, y_n) = 0$. This implies that

$$\psi(G(y_n, y_{n+1}, y_{n+1})) \leq \psi(G(y_{n-1}, y_n, y_n)) - \varphi(G(y_{n-1}, y_n, y_n)). \tag{19}$$

Hence, from the inequality (18), if $y_m = y_{m+1}$ for some m, then it follows that $y_n = y_m$ for all $n \geq m$ so that $\{y_n\}$ is Cauchy. Therefore, without loss of generality, we assume that $y_n \neq y_{n+1}$ for all $n = 0, 1, 2, \ldots$. Now, from (18), we have

$$\psi(G(y_n, y_{n+1}, y_{n+1})) < \psi(G(y_{n-1}, y_n, y_n)).$$

Hence by the nondecreasing nature of ψ, it follows that

$$G(y_n, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n) \text{ for all } n = 1, 2, \ldots.$$

Therefore $\{G(y_n, y_{n+1}, y_{n+1})\}$ is a monotone decreasing sequence of nonnegative reals. So, there exists $r \geq 0$ such that $G(y_n, y_{n+1}, y_{n+1}) \to r$ as $n \to \infty$. Now, from the inequality (19), we have

$$\psi(G(y_n, y_{n+1}, y_{n+1})) \leq \psi(G(y_{n-1}, y_n, y_n)) - \varphi(G(y_{n-1}, y_n, y_n)).$$

On letting $n \to \infty$, we have $\psi(r) \leq \psi(r) - \varphi(r)$ so that $\varphi(r) \leq 0$. Since $\varphi(r) \geq 0$, it follows that $\varphi(r) = 0$ so that $r = 0$.

$$\text{i.e., } G(y_n, y_{n+1}, y_{n+1}) \to 0 \text{ as } n \to \infty. \tag{20}$$

We now prove that the sequence $\{y_n\}$ is Cauchy. If we suppose that $\{y_n\}$ is not Cauchy, then there exists an $\epsilon > 0$ and there exist subsequences $\{y_{n_k}\}, \{y_{m_k}\}$ of $\{y_n\}$ with $n_k > m_k \geq k$ such that

$$G(y_{n_k}, y_{m_k}, y_{m_k}) \geq \epsilon. \tag{21}$$

Corresponding to each m_k, we can choose n_k such that it is the smallest integer with $n_k > m_k$ and satisfying (21). Then, we have

$$G(y_{n_k}, y_{m_k}, y_{m_k}) \geq \epsilon \text{ and } G(y_{n_k-1}, y_{m_k}, y_{m_k}) < \epsilon. \tag{22}$$

Now the following identities follow as in the proof of Theorem 4.

(i) $\lim_{k\to\infty} G(y_{n_k}, y_{m_k}, y_{m_k}) = \epsilon$.

(ii) $\lim_{k\to\infty} G(y_{n_k-1}, y_{m_k-1}, y_{m_k-1}) = \epsilon$.

(iii) $\lim_{k\to\infty} G(y_{n_k}, y_{m_k-1}, y_{m_k-1}) = \epsilon$.

We now consider

$$\begin{aligned}\psi(G(y_{n_k}, y_{m_k}, y_{m_k})) &= \psi(G(Tx_{n_k}, Tx_{m_k}, Tx_{m_k}))\\ &\leq \psi(M(x_{n_k}, x_{m_k}, x_{m_k})) - \varphi(M(x_{n_k}, x_{m_k}, x_{m_k}))\\ &\quad +Lm(x_{n_k}, x_{m_k}, x_{m_k}),\end{aligned}$$

where

$$\begin{aligned}M(x_{n_k}, x_{m_k}, x_{m_k}) &= \max\{G(Sx_{n_k}, Sx_{m_k}, Sx_{m_k}), G(Sx_{n_k}, Tx_{n_k}, Tx_{n_k}),\\ &\quad G(Sx_{m_k}, Tx_{m_k}, Tx_{m_k}), G(Sx_{m_k}, Tx_{m_k}, Tx_{m_k})\}\\ &= G(Sx_{n_k}, Sx_{m_k}, Sx_{m_k}) = G(y_{n_k-1}, y_{m_k-1}, y_{m_k-1})\end{aligned}$$

and

$$\begin{aligned}m(x_{n_k}, x_{m_k}, x_{m_k}) &= \min\{G(Tx_{n_k}, Sx_{n_k}, Sx_{n_k}), G(Tx_{n_k}Sx_{m_k}, Sx_{m_k}),\\ &\quad G(Tx_{n_k}, Sx_{m_k}, Sx_{m_k}), G(Tx_{n_k}, Sx_{m_k}, Sx_{m_k})\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\psi(G(y_{n_k}, y_{m_k}, y_{m_k}))\\ \leq\ &\psi(G(y_{n_k-1}, y_{m_k-1}, y_{m_k-1})) - \varphi(G(y_{n_k-1}, y_{m_k-1}, y_{m_k-1}))\\ &+L\min\{G(y_{n_k}, y_{n_k-1}, y_{n_k-1}), G(y_{n_k}, y_{m_k-1}, y_{m_k-1}),\\ &G(y_{n_k}, y_{m_k-1}, y_{m_k-1}), G(y_{n_k}, y_{m_k-1}, y_{m_k-1})\}.\end{aligned}$$

On letting $k \to \infty$ and using (i)–(iii) and (20), we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon) < \psi(\epsilon),$$

which is a contradiction. Therefore $\{y_n\}$ is a G-Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_{n+1} = z$. We now prove that z is a common fixed point of T and S. Since S is continuous, we have $\lim_{n\to\infty} STx_n = \lim_{n\to\infty} SSx_n = Sz$. Further, since S and T are weakly compatible, we have $\lim_{n\to\infty} G(TSx_n, STx_n, STx_n) = 0$, which implies $\lim_{n\to\infty} TSx_n = Sz$. Now, from (18), we have

$$\begin{aligned}\psi(G(TSx_n, Tx_n, Tx_n)) &\leq \psi(M(Sx_n, x_n, x_n)) - \varphi(M(Sx_n, x_n, x_n)) \\ &\quad + Lm(Sx_n, x_n, x_n),\end{aligned} \tag{23}$$

where

$$\begin{aligned}M(Sx_n, x_n, x_n) &= \max\{G(SSx_n, Sx_n, Sx_n), G(SSx_n, TSx_n, TSx_n), \\ &\quad G(Sx_n, Tx_n, Tx_n), G(Sx_n, Tx_n, Tx_n)\} \\ &= G(SSx_n, Sx_n, Sx_n)\end{aligned}$$

and

$$\begin{aligned}m(Sx_n, x_n, x_n) &= \min\{G(TSx_n, SSx_n, SSx_n), G(TSx_n, Sx_n, Sx_n), \\ &\quad G(TSx_n, Sx_n, Sx_n), G(TSx_n, Sx_n, Sx_n)\}.\end{aligned}$$

Therefore, from (23), we have

$$\begin{aligned}\psi(G(TSx_n, Tx_n, Tx_n)) &\leq \psi(G(SSx_n, Sx_n, Sx_n)) - \varphi(G(SSx_n, Sx_n, Sx_n)) \\ &\quad + L\min\{G(TSx_n, SSx_n, SSx_n), G(TSx_n, Sx_n, Sx_n), \\ &\quad G(TSx_n, Sx_n, Sx_n), G(TSx_n, Sx_n, Sx_n)\}.\end{aligned}$$

On letting $n \to \infty$, we get

$$\psi(G(Sz, z, z)) \leq \psi(G(Sz, z, z)) - \varphi(G(Sz, z, z)),$$

which implies that $\varphi(G(Sz, z, z)) \leq 0$ so that we must have $Sz = z$. Now, we consider

$$\psi(G(Tx_n, Tz, Tz)) \leq \psi(M(x_n, z, z)) - \varphi(M(x_n, z, z)) + Lm(x_n, z, z), \tag{24}$$

where

$$M(x_n, z, z) = \max\{G(Sx_n, Sz, Sz), G(Sx_n, Tx_n, Tx_n), G(Sz, Tz, Tz), G(Sz, Tz, Tz)\}$$

and

$$m(x_n, z, z) = \min\{G(Tx_n, Sx_n, Sx_n), G(Tx_n, Sz, Sz), G(Tx_n, Sz, Sz), G(Tx_n, Sz, Sz)\}.$$

Also, we have

$$\lim_{n\to\infty} M(x_n, z, z) = G(z, Tz, Tz) \text{ and } \lim_{n\to\infty} m(x_n, z, z) = 0, \tag{25}$$

since $\lim_{n\to\infty} G(TSx_n, SSx_n, SSx_n) = 0$. Now, on letting $n \to \infty$ in (24) and using (25), we get

$$\psi(G(z, Tz, Tz)) \leq \psi(G(z, Tz, Tz)) - \varphi(G(z, Tz, Tz)).$$

Then $\varphi(G(z, Tz, Tz)) \leq 0$. Therefore, $\varphi(G(z, Tz, Tz)) = 0$ so that $G(z, Tz, Tz) = 0$. Therefore, $Tz = z$. Thus z is a common fixed point of T and S. Uniqueness of common fixed point of T and S follows from the inequality (18). This completes the proof of the theorem.

3 Corollaries and Examples

In this section, we draw some corollaries from the main results of Section 2 and provide examples in support of our results. The following is an example in support of Theorem 4.

EXAMPLE 6. Let $X = [0, 1]$. We define $G : X^3 \to \mathbb{R}_+$ by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Then (X, G) is a complete G-metric space. We define $T : X \to X$ by

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x = 0, \\ 2x & \text{if } 0 < x < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

We define ψ and φ on $\mathbb{R}_+$ by $\psi(t) = \frac{t^2}{3}$ and $\varphi(t) = \frac{t^2}{2}$. Then, it is easy to verify that T satisfies the inequality (4) with $L = 1$. i.e., T is a (ψ, φ)-almost weakly contractive map. Thus T satisfies all the hypothesis of Theorem 4 and 1 is the unique fixed point of T. Here we observe that T is not a continuous map.

Further, we observe that T is not a weakly contractive map. For, let us choose $x = \frac{1}{3}$ and $y = z = 0$. Then $G(Tx, Ty, Tz) = \frac{2}{3}$, $G(x, y, z) = \frac{1}{3}$ and $\varphi(G(x, y, z)) = \varphi(\frac{1}{3})$. Hence,

$$\frac{2}{3} = G(Tx, Ty, Tz) \nleq G(x, y, z) - \varphi(G(x, y, z)) = \frac{1}{3} - \varphi(\frac{1}{3}) \text{ for any } \varphi \in \Psi.$$

Hence T does not satisfy the inequality (2) for any $\varphi \in \Psi$ so that T is not a weakly contractive map in G-metric space. Thus Theorem 2 is not applicable.

Further, this example suggests that the class of (ψ, φ)-almost weakly contractive maps is larger than the class of weakly contractive maps in G-metric spaces.

REMARK 1. Theorem 2 follows as a corollary to Theorem 4 by choosing ψ as the identity map and $L = 0$. Hence Example 6 suggests that Theorem 4 is a generalization of Theorem 2.

COROLLARY 1. Let (X, G) be a complete G-metric space and let T and S be two selfmaps on (X, G). Assume that $T(X) \subseteq S(X)$, S is continuous, and there exist $\psi, \varphi \in \Psi$ such that

$$\psi(G(Tx, Ty, Tz)) \leq \psi(M(x, y, z)) - \varphi(M(x, y, z)), \tag{26}$$

where

$$M(x, y, z) = \max\{G(Sx, Sy, Sz), G(Sx, Tx, Tx), G(Sy, Ty, Ty), G(Sz, Tz, Tz)\}$$

for $x, y, z \in X$. Then T and S have a unique common fixed point in X provided T and S are weakly compatible maps.

PROOF. Follows from Theorem 5 by choosing $L = 0$.

The following is an example in support of Corollary 1.

EXAMPLE 7. Let $X = [0, 1]$. We define $G : X^3 \to \mathbb{R}_+$ by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise.} \end{cases}$$

Then (X, G) is a complete G-metric space. We define $T, S : X \to X$ and ψ, φ on $\mathbb{R}_+$by

$$T(x) = \frac{x^2}{2},\ S(x) = \frac{x}{4}(5 - x),\ \psi(t) = \frac{4t^2}{3} \text{ and } \varphi(t) = \frac{t^2}{3}$$

for all $x \in X$ and $t \in \mathbb{R}_+$. Then clearly $T(X) \subseteq S(X)$.Without loss of generality, we assume that $x > y > z$. Then

$$G(Tx, Ty, Tz) = \max\{Tx, Ty, Tz\} = \max\{\frac{x^2}{2}, \frac{y^2}{2}, \frac{z^2}{2}\} = \frac{x^2}{2}.$$

Also

$$G(Sx, Sy, Sz) = \max\{Sx, Sy, Sz\} = \max\{\frac{x}{4}(5 - x), \frac{y}{4}(5 - y), \frac{z}{4}(5 - z)\} = \frac{x}{4}(5 - x).$$

Now, we consider

$$\begin{aligned} \psi(G(Tx, Ty, Tz)) &= \frac{x^4}{3} \leq \frac{x^2}{16}(5 - x)^2 = [G(Sx, Sy, Sz)]^2 \\ &\leq [M(x, y, z)]^2 = \frac{4}{3}[M(x, y, z)]^2 - \frac{1}{3}[M(x, y, z)]^2 \\ &= \psi(M(x, y, z)) - \varphi(M(x, y, z)). \end{aligned}$$

Therefore,

$$\psi(G(Tx, Ty, Tz)) \leq \psi(M(x, y, z)) - \varphi(M(x, y, z))$$

so that the inequality (26) of Corollary 1 holds. Thus, T and S satisfy all the hypotheses of Corollary 1 and 0 is the unique common fixed point of T and S.

REMARK 2. We observe that the φ that is used in Theorem 5 is different from ϕ that is used in Theorem 3.

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