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# Existence of Positive Solutions For A Fourth-Order $p$-Laplacian Boundary Value Problem*

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**Abstract**

This article is concerned with the existence of positive solutions of a fourth-order $p$-Laplacian boundary value problem. Based on a priori estimates achieved by utilizing Jensen's integral inequalities for convex and concave functions, we use fixed point index theory to establish the existence of positive solutions for the above problem.

## 1 Introduction

This article is concerned with the existence of positive solutions for the $p$-Laplacian boundary value problem

$$\begin{cases} (|u''|^{p-1}\, u'')'' = f(t, u, -u''), \\ a_1 u(0) - b_1 u'(0) = c_1 u(1) + d_1 u'(1) = 0, \\ a_2 \left(-u''\right)^p (0) - b_2((-u'')^p)'(0) = c_2 \left(-u''\right)^p (1) + d_2((-u'')^p)'(1) = 0, \end{cases} \tag{1}$$

where $p > 0$, $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R}^+)$, $a_i, b_i, c_i, d_i \geqslant 0$, and $\delta_i = a_i d_i + b_i c_i + a_i c_i > 0$ for $i = 1, 2$.

Fourth order boundary value problems, including those with the $p$-Laplacian operator, have their origin in beam theory, ice formation, fluids on lungs, brain warping, designing special curves on surfaces, etc. In our problem (1), the nonlinearity $f$ involves the second-order derivative $u''$. Such nonlinearity may be encountered in some physical models. For example, the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4} - p\frac{\partial^2 u}{\partial x^2} - a(x)u + b(x)u^3$$

is known in the studies of phase transitions near a Lifschitz point [16].

The $p$-Laplacian boundary value problems arise in non-Newtonian mechanics, nonlinear elasticity, glaciology, population biology, combustion theory, and nonlinear flow

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laws; see [5, 6]. That explains why many authors have extensively studied the existence of positive solutions for $p$-Laplacian boundary value problems, by using topological degree theory, monotone iterative techniques, coincidence degree theory, and the Leggett-Williams fixed point theorem or its variants; see [1, 2, 3, 4, 8, 10, 11, 12, 13, 14, 15] and the references therein.

In [14], by using the method of upper and lower solutions, Zhang and Liu obtained the existence of positive solutions for the fourth-order singular $p$-Laplacian boundary value problem

$$(|u''|^{p-1} u'')'' = f(t, u(t)) \quad \text{for } 0 < t < 1, \tag{2}$$

subject to the boundary conditions

$$u(0) = u(1) - au(\xi) = u''(0) = u''(1) - bu''(\eta) = 0, \tag{3}$$

where $p > 1, 0 < \xi, \eta < 1$, and $f \in C((0,1) \times (0,\infty), (0,\infty))$ may be singular at $t = 0$ and/or at $t = 1$ and $u = 0$.

In [15], Zhang and Liu obtained the existence of positive solutions for (2) with the boundary conditions

$$u(0) - \sum_{i=1}^{m-2} a_i u(\xi_i) = u(1) = u''(0) - \sum_{i=1}^{m-2} b_i u(\eta_i) = u''(1) = 0, \tag{4}$$

where $m \geqslant 3, a_i, b_i, \xi_i, \eta_i \in (0,1) (i = 1, 2, \ldots, m-2)$ are nonnegative constants and $\sum_{i=1}^{m-2} a_i < 1, \sum_{i=1}^{m-2} b_i < 1$, and $f \in C((0,1) \times \mathbb{R}_+, \mathbb{R}_+)$ may be singular at $t = 0$ and/or at $t = 1$. By using the monotone iterative method, they established the existence of positive solutions of *pseudo*-$C^3[0,1]$ for the above problem.

In [8], Guo et al. investigated the existence and multiplicity of positive solutions for the fourth-order $p$-Laplacian boundary value problem

$$(|u''|^{p-2} u'')'' = \lambda g(t) f(u) \quad \text{for } 0 < t < 1, \tag{5}$$

where $\lambda$ is a positive parameter. By using fixed point index theory and the method of upper and lower solutions, they obtained the following result: there exists $\lambda^* < \infty$ such that (5) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, (5) has at least one positive solution for $\lambda = \lambda^*$, and (5) have no positive solution at all for $\lambda > \lambda^*$.

The presence of the second-order derivative $u''$ contributes to the difficulty to obtain a priori estimates of positive solutions for some problems associated with (1). To facilitate the establishment of such estimates, by using the reduction of order, we transform (1) into a boundary value problem for an equivalent second-order integro-differential equation (see the next section for more details). More importantly, we observe that if $p = 1$, then (1) reduces to the semilinear fourth-order boundary value problem

$$\begin{cases} u^{(4)} = f(t, u, -u''), \\ a_1 u(0) - b_1 u'(0) = c_1 u(1) + d_1 u'(1) = 0, \\ a_2 u''(0) - b_2 u'''(0) = c_2 u''(1) + d_2 u'''(1) = 0. \end{cases} \tag{6}$$

Motivated by [11, 12, 13], we regard (6) as a perturbation of (1). In fact, we make repeated use of the Jensen integral inequalities for convex and concave functions in order to derive a priori estimates of positive solutions for some operator equations associated with (1), these estimates based on which we use fixed point index theory to establish the existence of positive solutions for the above problem. Our main results extend the corresponding ones in [11, 12, 13]. Also, some relations between (1) and (6) may be seen from the Jensen inequalities for convex and concave functions.

This article is organized as follows. In Section 2, we provide some preliminary results. Our main results, namely Theorem 3.1 and 3.2, followed by two simple examples, are stated and proved in Section 3.

# 2 Preliminaries

Let

$$E := C[0,1], \|u\| := \max_{0\leqslant t\leqslant 1} |u(t)|, P := \{u \in E : \ u(t) \geqslant 0 \ \text{ for } t \in [0,1]\}. \tag{7}$$

Clearly $(E, \|\cdot\|)$ is a real Banach space and $P$ is a cone in $E$. Define $B_\rho := \{u \in E : \|u\| < \rho\}$ for all $\rho > 0$. Substituting $v := -u''$ into (1), we have

$$\begin{cases} -(|v|^{p-1}v)''(t) = f(t, \int_0^1 k_1(t,s)v(s)ds, v(t)), \\ a_2 v^p(0) - b_2(v^p)'(0) = 0, \\ c_2 v^p(1) + d_2(v^p)'(1) = 0, \end{cases} \tag{8}$$

where

$$k_1(t,s) := \frac{1}{\delta_1}\begin{cases} (b_1 + a_1 s)(c_1(1-t) + d_1), & 0 \leqslant s \leqslant t \leqslant 1, \\ (b_1 + a_1 t)(c_1(1-s) + d_1), & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Moreover, (8) is equivalent to the nonlinear integral equation

$$v(t) = \left(\int_0^1 k_2(t,s) f(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right)^{\frac{1}{p}}, \tag{9}$$

where

$$k_2(t,s) := \frac{1}{\delta_2}\begin{cases} (b_2 + a_2 s)(c_2(1-t) + d_2), & 0 \leqslant s \leqslant t \leqslant 1, \\ (b_2 + a_2 t)(c_2(1-s) + d_2), & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

Define the operator $A : P \longrightarrow P$ by

$$(Av)(t) := \left(\int_0^1 k_2(t,s) f(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right)^{\frac{1}{p}}. \tag{10}$$

Now the condition $f \in C([0,1] \times \mathbb{R}_+^2, \mathbb{R}_+)$ implies that $A : P \to P$ is a completely continuous operator, and the existence of positive solutions for (1) is equivalent to that of positive fixed points of $A$. Let

$$k_3(t,\tau) := \int_0^1 k_2(t,s)k_1(s,\tau)ds.$$

For any given nonnegative constants $\alpha$, $\beta$, let

$$G_{\alpha,\beta}(t,s) := \alpha k_3(t,s) + \beta k_2(t,s) \tag{11}$$

and

$$(L_{\alpha,\beta}v)(t) := \int_0^1 G_{\alpha,\beta}(t,s)v(s)ds. \tag{12}$$

Clearly $L_{\alpha,\beta} : E \to E$ is a completely continuous positive linear operator. If $\alpha+\beta > 0$, then the spectral radius $r(L_{\alpha,\beta})$ is positive. The Krein-Rutmann theorem then implies that there exists $\varphi_{\alpha,\beta} \in P \setminus \{0\}$ such that $r(L_{\alpha,\beta})\varphi_{\alpha,\beta} = L^*_{\alpha,\beta}\varphi_{\alpha,\beta}$, i.e.

$$r(L_{\alpha,\beta})\varphi_{\alpha,\beta}(s) = \int_0^1 G_{\alpha,\beta}(t,s)\varphi_{\alpha,\beta}(t)dt, \tag{13}$$

where $L^*_{\alpha,\beta} : E \to E$ is the dual operator of $A$. Note that we may normalize $\varphi_{\alpha,\beta}$ so that

$$\int_0^1 \varphi_{\alpha,\beta}(t)dt = 1. \tag{14}$$

LEMMA 2.1. For any given nonnegative constants $\alpha$,$\beta$ with $\alpha + \beta > 0$, let

$$\kappa_{\alpha,\beta} := \int_0^{\frac{1}{2}} t\varphi_{\alpha,\beta}(t)dt + \int_{\frac{1}{2}}^1 (1-t)\varphi_{\alpha,\beta}(t)dt,$$

where $\varphi_{\alpha,\beta}$ is given in (13) and (14). Then for every concave function $\phi \in P$, we have

$$\int_0^1 \phi(t)\varphi_{\alpha,\beta}(t)dt \geqslant \kappa_{\alpha,\beta}\|\phi\|.$$

The proof can be carried out as that of Lemma 2.4 in [11]. Thus we omit it.

LEMMA 2.2 (see [9]). Let $a \in \mathbb{R}_+, b \in \mathbb{R}_+$. If $\sigma \in (0,1]$, then

$$(a+b)^\sigma \geqslant 2^{\sigma-1}(a^\sigma + b^\sigma).$$

If $\sigma \in [1,+\infty)$, then

$$(a+b)^\sigma \leqslant 2^{\sigma-1}(a^\sigma + b^\sigma).$$

LEMMA 2.3 (see [9]). Suppose $g \in C[a,b]$ with $I := g([a,b])$ and $h \in C(I)$. If $h$ is convex on $I$, then

$$h\left(\frac{1}{b-a}\int_a^b g(t)dt\right) \leqslant \frac{1}{b-a}\int_a^b h(g(t))dt.$$

If $h$ is concave on $I$, then

$$h\left(\frac{1}{b-a}\int_a^b g(t)dt\right) \geqslant \frac{1}{b-a}\int_a^b h(g(t))dt.$$

LEMMA 2.4. Let $E$ and $P$ be defined in (7). Suppose that $\Omega \subset E$ is a bounded open set and that $T : \overline{\Omega} \cap K \longrightarrow K$ is a completely continuous operator. If there exist $u_0 \in K \backslash \{0\}$ and $\mu > 0$ such that

$$u^\mu - (Tu)^\mu \neq \lambda u_0 \ \text{ for all } \lambda \geqslant 0 \text{ and } u \in \partial\Omega \cap K,$$

then $i(T, \Omega \cap K, K) = 0$ where $i$ indicates the fixed point index on $K$.

PROOF. Note the operator $S_\lambda u := ((Tu)^\mu + \lambda u_0)^{1/\mu} : P \to P$ is a completely continuous operator for all $\lambda \geqslant 0$. If $i(T, \Omega \cap K, K) = i(S_0, \Omega \cap K, K) \neq 0$, then the homotopy invariance implies

$$i(S_\lambda, \Omega \cap K, K) = i(S_0, \Omega \cap K, K) \neq 0$$

for all $\lambda \geqslant 0$, and, in turn, the fixed point equation $u = S_\lambda u$ have at least one solution on $K \cap P$ for all $\lambda \geqslant 0$, contradicting the complete continuity of $T$ and the boundedness of $K$. Thus we have $i(T, \Omega \cap K, K) = 0$, as desired. This completes the proof.

LEMMA 2.5 (see [7]). Let $E$ be a real Banach space and $K$ be a cone in $E$. Suppose that $\Omega \subset E$ is a bounded open set, $0 \in \Omega$, and $T : \overline{\Omega} \cap K \longrightarrow K$ is a completely continuous operator. If

$$u - \lambda Tu \neq 0 \ \text{ for all } \lambda \in [0,1] \text{ and } u \in \partial\Omega \cap K,$$

then $i(T, \Omega \cap K, K) = 1$.

# 3 Main Results

Let $p_* := \min\{1, p\}$, $p^* := \min\{1, p\}$, and $m_i := \max_{t,s\in[0,1]} k_i(t,s)$ for $i = 1, 2, 3$. Now we list our hypotheses on $f$ and $a_i, b_i, c_i, d_i$ for $i = 1, 2$:

(H1) $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R}_+)$.

(H2) $a_i, b_i, c_i, d_i \geqslant 0$ and $\delta_i := a_i d_i + b_i c_i + a_i c_i > 0$ for $i = 1, 2$.

(H3) There are $\alpha_1, \beta_1 > 0$ and $c > 0$, such that $r(L_{n_1,n_2}) > 1$ and

$$f(t,x,y) \geqslant \alpha_1 x^p + \beta_1 y^p - c \ \text{ for all } t \in [0,1] \text{ and } x, y \geqslant 0,$$

where $L_{n_1,n_2}$ is defined as in (11) and (12),

$$n_1 := 2^{\frac{p_*}{p}-1} \alpha_1^{\frac{p_*}{p}} m_1^{p_*-1} m_2^{\frac{p_*}{p}-1} \text{ and } n_2 := 2^{\frac{p_*}{p}-1} \beta_1^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1}.$$

(H4) There are $\alpha_2, \beta_2 > 0$ and $r_1 > 0$ such that $r(L_{n_3,n_4}) < 1$ and

$$f(t,x,y) \leqslant \alpha_2 x^p + \beta_2 y^p \ \text{ for all } t \in [0,1] \text{ and } x, y \in [0, r_1],$$

where $L_{n_3,n_4}$ is defined as in (11) and (12),

$$n_3 := 2^{\frac{p^*}{p}-1} \alpha_2^{\frac{p^*}{p}} m_1^{p^*-1} m_2^{\frac{p^*}{p}-1} \text{ and } n_4 := 2^{\frac{p^*}{p}-1} \beta_2^{\frac{p^*}{p}} m_2^{\frac{p^*}{p}-1}.$$

(H5) There are $\alpha_3, \beta_3 \geqslant 0$ and $r_2 > 0$ such that $r(L_{n_5,n_6}) > 1$ and

$$f(t,x,y) \geqslant \alpha_3 x^p + \beta_3 y^p \text{ for all } t \in [0,1] \text{ and } x, y \in [0, r_2],$$

where $L_{n_5,n_6}$ is defined as in (11) and (12),

$$n_5 := 2^{\frac{p_*}{p}-1}\alpha_3^{\frac{p_*}{p}} m_1^{p_*-1} m_2^{\frac{p_*}{p}-1} \text{ and } n_6 := 2^{\frac{p_*}{p}-1}\beta_3^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1}.$$

(H6) There are $\alpha_4, \beta_4 \geqslant 0$ and $c > 0$ such that $r(L_{n_7,n_8}) < 1$ and

$$f(t,x,y) \leqslant \alpha_4 x^p + \beta_4 y^p + c \text{ for all } t \in [0,1] \text{ and } x, y \geqslant 0,$$

where $L_{n_7,n_8}$ is defined as in (11) and (12),

$$n_7 := 4^{\frac{p^*}{p}-1}\alpha_4^{\frac{p^*}{p}} m_1^{p^*-1} m_2^{\frac{p^*}{p}-1} \text{ and } n_8 := 4^{\frac{p^*}{p}-1}\beta_4^{\frac{p^*}{p}} m_2^{\frac{p^*}{p}-1}.$$

REMARK 3.1. Notice that the expression (10) implies that if $v \in P \setminus \{0\}$ is a fixed point of the operator, then $v(t) > 0$ holds for all $t \in (0,1)$ with $v^p \in P \cap C^2[0,1]$. This, together with the substitution $v := -u''$, in turn, implies that if $u$ is a positive solution of (1), then $(-u'')^p \in (P \setminus \{0\}) \cap C^2[0,1]$ and hence $u \in (P \setminus \{0\}) \cap C^4(0,1)$.

THEOREM 3.1. If (H1)-(H4) hold, then (1) has at least one positive solution $u \in (P\setminus\{0\}) \cap C^4(0,1)$.

PROOF**.** It suffices to prove that $A$ has at least one fixed point $v \in P \setminus \{0\}$. To this end, let

$$\mathscr{M}_1 := \{v \in P : v^{p_*} = (Av)^{p_*} + \lambda, \lambda \geqslant 0\}.$$

We show that $\mathscr{M}_1$ is bounded. Indeed, if $v \in \mathscr{M}_1$, then $v^{p_*}$ is concave on $[0,1]$ and there exists $\lambda \geqslant 0$ such that $v^{p_*} = (Av)^{p_*} + \lambda$. Thus $v^{p_*}(t) \geqslant (Av)^{p_*}(t)$. Note $p_*, p_*/p \in (0,1]$. By (H3) and the Jensen integral inequality for concave functions (Lemma 2.3 ), we have that, for all $v \in \mathscr{M}_1$,

$$\begin{aligned}
v^{p_*}(t) \quad &\geqslant \quad \left(\int_0^1 k_2(t,s) f(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right)^{\frac{p_*}{p}} \\
&\geqslant \quad \int_0^1 k_2^{\frac{p_*}{p}}(t,s) f^{\frac{p_*}{p}}(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau), v(s))ds \\
&\geqslant \quad \int_0^1 k_2(t,s) m_2^{\frac{p_*}{p}-1}\Big\{[\alpha_1 \int_0^1 k_1^p(s,\tau)v^p(\tau)d\tau + \beta_1 v^p(s)]^{\frac{p_*}{p}} - c^{\frac{p_*}{p}}\Big\}ds \\
&\geqslant \quad \int_0^1 k_2(t,s) m_2^{\frac{p_*}{p}-1}\Big\{2^{\frac{p_*}{p}-1}[\alpha_1^{\frac{p_*}{p}} \int_0^1 k_1^{p_*}(s,\tau)v^{p_*}(\tau)d\tau + \beta_1^{\frac{p_*}{p}} v^{p_*}(s)] \\
&\quad\ -c^{\frac{p_*}{p}}\Big\}ds \\
&\geqslant \quad \int_0^1 k_2(t,s) m_2^{\frac{p_*}{p}-1}\Big\{2^{\frac{p_*}{p}-1}[\alpha_1^{\frac{p_*}{p}} m_1^{p_*-1} \int_0^1 k_1(s,\tau)v^{p_*}(\tau)d\tau + \beta_1^{\frac{p_*}{p}} v^{p_*}(s)]
\end{aligned}$$

$$
\begin{aligned}
& \quad -c^{\frac{p_*}{p}}\Big\}ds \\
&= 2^{\frac{p_*}{p}-1}\alpha_1^{\frac{p_*}{p}} m_1^{p_*-1} m_2^{\frac{p_*}{p}-1}\int_0^1\int_0^1 k_2(t,s)k_1(s,\tau)v^{p_*}(\tau)d\tau ds \\
& \quad +2^{\frac{p_*}{p}-1}\beta_1^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1}\int_0^1 k_2(t,s)v^{p_*}(s)ds - c^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1}\int_0^1 k_2(t,s)ds \\
&= 2^{\frac{p_*}{p}-1}\alpha_1^{\frac{p_*}{p}} m_1^{p_*-1} m_2^{\frac{p_*}{p}-1}\int_0^1 k_3(t,s)v^{p_*}(s)ds \\
& \quad +2^{\frac{p_*}{p}-1}\beta_1^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1}\int_0^1 k_2(t,s)v^{p_*}(s)ds - c^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1}\int_0^1 k_2(t,s)ds \\
&= \int_0^1 G_{n_1,n_2}(t,s)v^{p_*}(s)ds - c^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1} m_3.
\end{aligned}
$$

Multiply the above inequality by $\varphi_{n_1,n_2}(t)$ and integrate over $[0,1]$ and use (13) and (14) to obtain

$$
\int_0^1 v^{p_*}(t)\varphi_{n_1,n_2}(t)dt \geqslant r(L_{n_1,n_2})\int_0^1 v^{p_*}(t)\varphi_{n_1,n_2}(t)dt - c^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1} m_3,
$$

so that

$$
\int_0^1 v^{p_*}(t)\varphi_{n_1,n_2}(t)dt \leqslant \frac{c^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1} m_3}{r(L_{n_1,n_2})-1} := N_1 \quad \text{for all } v\in\mathscr{M}_1.
$$

Recall that $v^{p_*}$ is concave on $[0,1]$. By Lemma 2.1, we have

$$
\|v^{p_*}\| \leqslant \frac{\int_0^1 v^{p_*}(t)\varphi_{n_1,n_2}(t)dt}{\kappa_{n_1,n_2}} \leqslant \frac{N_1}{\kappa_{n_1,n_2}}
$$

for all $v\in\mathscr{M}_1$. This proves the boundedness of $\mathscr{M}_1$. Taking $R>\sup\{\|v\| : v\in\mathscr{M}_1\}$, we have

$$
v^{p_*} \neq (Av)^{p_*}+\lambda \quad \text{for } v\in\partial B_R\cap P \text{ and } \lambda\geqslant 0.
$$

Now Lemma 2.4 yields

$$
i(A, B_R\cap P, P) = 0. \tag{15}
$$

Let

$$
\mathscr{M}_2 := \{v\in\overline{B}_{r_1}\cap P : v=\lambda Av, 0\leqslant\lambda\leqslant 1\}.
$$

We claim that $\mathscr{M}_2=\{0\}$. Indeed, if $v\in\mathscr{M}_2$, then there exists $\lambda\in[0,1]$ such that $v(t)=\lambda Av(t)$. Thus we have

$$
v(t) \leqslant (Av)(t) = \left[\int_0^1 k_2(t,s)f(s,\int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right]^{\frac{1}{p}} \quad \text{for all } v\in\overline{B}_{r_1}\cap P.
$$

Note $p^*, p^*/p\geqslant 1$. By (H4) and the Jensen integral inequality for convex functions (Lemma 2.3), we have that, for all $v\in\mathscr{M}_2$,

$$
v^{p^*}(t) \quad\leqslant\quad \left(\int_0^1 k_2(t,s)f(s,\int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right)^{\frac{p^*}{p}}
$$

$$
\begin{aligned}
&\leqslant \int_0^1 k_2^{\frac{p^*}{p}}(t,s) f^{\frac{p^*}{p}}(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds \\
&\leqslant \int_0^1 k_2(t,s) m_2^{\frac{p^*}{p}-1} \left(\alpha_2 \int_0^1 k_1^p(s,\tau)v^p(\tau)d\tau + \beta_2 v^p(s)\right)^{\frac{p^*}{p}} ds \\
&\leqslant \int_0^1 2^{\frac{p^*}{p}-1} k_2(t,s) m_2^{\frac{p^*}{p}-1} \left(\alpha_2^{\frac{p^*}{p}} \int_0^1 k_1^{p^*}(s,\tau)v^{p^*}(\tau)d\tau + \beta_2^{\frac{p^*}{p}} v^{p^*}(s)\right) ds \\
&\leqslant \int_0^1 2^{\frac{p^*}{p}-1} k_2(t,s) m_2^{\frac{p^*}{p}-1} \Big[[\alpha_2^{\frac{p^*}{p}} m_1^{p^*-1} \int_0^1 k_1(s,\tau)v^{p^*}(\tau)d\tau \\
&\quad + \beta_2^{\frac{p^*}{p}} v^{p^*}(s)\Big] ds \\
&\leqslant \int_0^1 2^{\frac{p^*}{p}-1} k_2(t,s) m_2^{\frac{p^*}{p}-1} \Big[\alpha_2^{\frac{p^*}{p}} m_1^{p^*-1} \int_0^1 k_1(s,\tau)v^{p^*}(\tau)d\tau \\
&\quad + \beta_2^{\frac{p^*}{p}} v^{p^*}(s)]\Big] ds \\
&= 2^{\frac{p^*}{p}-1} \alpha_2^{\frac{p^*}{p}} m_1^{p^*-1} m_2^{\frac{p^*}{p}-1} \int_0^1 \int_0^1 k_2(t,s) k_1(s,\tau) v^{p^*}(\tau) d\tau ds \\
&\quad + 2^{\frac{p^*}{p}-1} \beta_2^{\frac{p^*}{p}} m_2^{\frac{p^*}{p}-1} \int_0^1 k_2(t,s) v^{p^*}(s) ds \\
&= 2^{\frac{p^*}{p}-1} \alpha_2^{\frac{p^*}{p}} m_1^{p^*-1} m_2^{\frac{p^*}{p}-1} \int_0^1 k_3(t,s) v^{p^*}(s) ds \\
&\quad + 2^{\frac{p^*}{p}-1} \beta_2^{\frac{p^*}{p}} m_2^{\frac{p^*}{p}-1} \int_0^1 k_2(t,s) v^{p^*}(s) ds \\
&= \int_0^1 G_{n_3,n_4}(t,s) v^{p^*}(s) ds.
\end{aligned}
$$

Multiply the above inequality by $\varphi_{n_3,n_4}(t)$ and integrate over $[0,1]$ and use (13) and (14) to obtain

$$\int_0^1 v^{p^*}(t)\varphi_{n_3,n_4}(t)dt \leqslant r(L_{n_3,n_4}) \int_0^1 v^{p^*}(t)\varphi_{n_3,n_4}(t)dt,$$

so that $\int_0^1 v^{p^*}(t)\varphi_{n_3,n_4}(t)dt = 0$, whence $v^{p^*}(t) \equiv 0$ and $\mathscr{M}_2 = \{0\}$, as claimed. A consequence of that is

$$v \neq \lambda A v \quad \text{for all } v \in \overline{B}_{r_1} \cap P \text{ and } \lambda \in [0,1].$$

Now Lemma 2.5 yields

$$i(A, \overline{B}_{r_1} \cap P, P) = 1. \tag{16}$$

Note that we may assume $R > r_1$. Combining (15) and (16) gives

$$i(A, (B_R \setminus \overline{B}_{r_1}) \cap P, P) = 0 - 1 = -1.$$

Therefore $A$ has at least one fixed point on $(B_R \setminus \overline{B}_{r_1}) \cap P$, and thus (1) has at least one positive solution. This completes the proof.

THEOREM 3.2. If (H1), (H2), (H5) and (H6) hold, then (1) has at least one positive solution $u \in (P\backslash\{0\}) \cap C^4(0,1)$.

PROOF. It suffices to prove that $A$ has at least one fixed point $v \in P \setminus \{0\}$. To this end, let

$$\mathscr{M}_3 := \{v \in \overline{B}_{r_2} \cap P : v^{p_*} = (Av)^{p_*} + \lambda, \lambda \geqslant 0\}.$$

We shall now prove that $\mathscr{M}_3 \subset \{0\}$. Indeed, if $v \in \mathscr{M}_3$, then there exists $\lambda \geqslant 0$ such that $v^{p_*} = (Av)^{p_*} + \lambda$. Thus we have

$$v^{p_*}(t) \geqslant (Av)^{p_*}(t) = \left(\int_0^1 k_2(t,s) f(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right)^{\frac{p_*}{p}} \quad \text{for all } v \in \overline{B}_{r_2} \cap P.$$

Note $p_*, p_*/p \in (0,1]$. By (H5) and the Jensen integral inequality for concave functions (Lemma 2.3), we obtain that, for all $v \in \overline{B}_{r_2} \cap P$,

$$\begin{aligned}
v^{p_*}(t) &\geqslant \left(\int_0^1 k_2(t,s) f(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right)^{\frac{p_*}{p}} \\
&\geqslant \int_0^1 k_2^{\frac{p_*}{p}}(t,s) f^{\frac{p_*}{p}}(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds \\
&\geqslant \int_0^1 k_2(t,s) m_2^{\frac{p_*}{p}-1} \left(\alpha_3 \int_0^1 k_1^p(s,\tau)v^p(\tau)d\tau + \beta_3 v^p(s)\right)^{\frac{p_*}{p}} ds \\
&\geqslant \int_0^1 2^{\frac{p_*}{p}-1} k_2(t,s) m_2^{\frac{p_*}{p}-1} \left(\alpha_3^{\frac{p_*}{p}} \int_0^1 k_1^{p_*}(s,\tau)v^{p_*}(\tau)d\tau + \beta_3^{\frac{p_*}{p}} v^{p_*}(s)\right) ds \\
&\geqslant \int_0^1 2^{\frac{p_*}{p}-1} k_2(t,s) m_2^{\frac{p_*}{p}-1} \left(\alpha_3^{\frac{p_*}{p}} m_1^{p_*-1} \int_0^1 k_1(s,\tau)v^{p_*}(\tau)d\tau + \beta_3^{\frac{p_*}{p}} v^{p_*}(s)\right) ds \\
&= 2^{\frac{p_*}{p}-1} \alpha_3^{\frac{p_*}{p}} m_1^{p_*-1} m_2^{\frac{p_*}{p}-1} \int_0^1 \int_0^1 k_2(t,s) k_1(s,\tau) v^{p_*}(\tau) d\tau ds \\
&\quad + 2^{\frac{p_*}{p}-1} \beta_3^{\frac{p_*}{p}} m_2^{\frac{p_*}{p}-1} \int_0^1 k_2(t,s) v^{p_*}(s) ds \\
&= 2^{\frac{p_*}{p}-1} m_2^{\frac{p_*}{p}-1} \left(\alpha_3^{\frac{p_*}{p}} m_1^{p_*-1} \int_0^1 k_3(t,s) v^{p_*}(s) ds + \beta_3^{\frac{p_*}{p}} \int_0^1 k_2(t,s) v^{p_*}(s) ds\right) \\
&= \int_0^1 G_{n_5,n_6}(t,s) v^{p_*}(s) ds.
\end{aligned}$$

Multiply the above inequality by $\varphi_{n_5,n_6}(t)$ and integrate over $[0,1]$ and use (13) and (14) to obtain

$$\int_0^1 v^{p_*}(t)\varphi_{n_5,n_6}(t)dt \geqslant r(L_{n_5,n_6}) \int_0^1 v^{p_*}(t)\varphi_{n_5,n_6}(t)dt,$$

so that $\int_0^1 v^{p_*}(t)\varphi_{n_5,n_6}(t)dt = 0$, whence $v^{p_*}(t) \equiv 0$ and $\mathscr{M}_3 \subset \{0\}$, as required. As a result of that, we have

$$v^{p_*} \neq (Av)^{p_*} + \lambda \quad \text{for all } v \in \partial B_{r_2} \cap P \text{ and } \lambda \geqslant 0.$$

Now Lemma 2.4 yields
$$i(A, B_{r_2} \cap P, P) = 0. \tag{17}$$
Let
$$\mathscr{M}_4 := \{v \in P : v = \lambda Av, 0 \leqslant \lambda \leqslant 1\}.$$
We are going to prove that $\mathscr{M}_4$ is bounded. Indeed, if $v \in \mathscr{M}_4$,then $v^p$ is concave and

$$v(t) \leqslant (Av)(t) = \left(\int_0^1 k_2(t,s)f(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right)^{\frac{1}{p}} \quad \text{for all } v \in \mathscr{M}_4.$$

Note $p^*, \frac{p^*}{p} \geqslant 1$. By (H6) and the Jensen integral inequality for convex functions (Lemma 2.3), we have

$$\begin{aligned}
v^{p^*}(t) &\leqslant \left(\int_0^1 k_2(t,s)f(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s))ds\right)^{\frac{p^*}{p}} \\
&\leqslant \int_0^1 k_2^{\frac{p^*}{p}}(t,s)f^{\frac{p^*}{p}}\left(s, \int_0^1 k_1(s,\tau)v(\tau)d\tau, v(s)\right)ds \\
&\leqslant \int_0^1 k_2(t,s)m_2^{\frac{p^*}{p}-1}\left(\alpha_4\int_0^1 k_1^p(s,\tau)v^p(\tau)d\tau + \beta_4 v^p(s) + c\right)^{\frac{p^*}{p}} ds \\
&\leqslant \int_0^1 k_2(t,s)m_2^{\frac{p^*}{p}-1}\left\{2^{\frac{p^*}{p}-1}[(\alpha_4\int_0^1 k_1^p(s,\tau)v^p(\tau)d\tau + \beta_4 v^p(s))^{\frac{p^*}{p}} + c^{\frac{p^*}{p}}]\right\}ds \\
&\leqslant \int_0^1 k_2(t,s)m_2^{\frac{p^*}{p}-1}\left\{2^{\frac{p^*}{p}-1}[2^{\frac{p^*}{p}-1}\alpha_4^{\frac{p^*}{p}}\int_0^1 k_1(s,\tau)v^{p^*}(\tau)d\tau\right. \\
&\quad \left. +2^{\frac{p^*}{p}-1}\beta_4^{\frac{p^*}{p}}v^{p^*}(s) + c^{\frac{p^*}{p}}]\right)ds \\
&\leqslant \int_0^1 k_2(t,s)m_2^{\frac{p^*}{p}-1}\left(4^{\frac{p^*}{p}-1}\alpha_4^{\frac{p^*}{p}}m_1^{p^*-1}\int_0^1 k_1(s,\tau)v^{p^*}(\tau)d\tau + 4^{\frac{p^*}{p}-1}\beta_4^{\frac{p^*}{p}}v^{p^*}(s)\right. \\
&\quad \left. +2^{\frac{p^*}{p}-1}c^{\frac{p^*}{p}}\right)ds \\
&= 4^{\frac{p^*}{p}-1}\alpha_4^{\frac{p^*}{p}}m_1^{p^*-1}m_2^{\frac{p^*}{p}-1}\int_0^1\int_0^1 k_3(t,s)v^{p^*}(\tau)d\tau ds \\
&\quad +4^{\frac{p^*}{p}-1}\beta_4^{\frac{p^*}{p}}m_2^{\frac{p^*}{p}-1}\int_0^1 k_2(t,s)v^{p^*}(s)ds + 2^{\frac{p^*}{p}-1}c^{\frac{p^*}{p}}m_2^{\frac{p^*}{p}-1}\int_0^1 k_2(t,s)ds \\
&= \int_0^1 G_{n_7,n_8}(t,s)v^{p_*}(s)ds + 2^{\frac{p^*}{p}-1}c^{\frac{p^*}{p}}m_2^{\frac{p^*}{p}-1}m_3.
\end{aligned}$$

Multiply the above inequality by $\varphi_{n_7,n_8}(t)$ and integrate over $[0,1]$ and use (13) and (14) to obtain

$$\int_0^1 v^{p^*}(t)\varphi_{n_7,n_8}(t) \leqslant r(L_{n_7,n_8})\int_0^1 v^{p_*}(t)\varphi_{n_7,n_8}(t) + 2^{\frac{p^*}{p}-1}c^{\frac{p^*}{p}}m_2^{\frac{p^*}{p}-1}m_3,$$

so that

$$\int_0^1 v^{p^*}(t)\varphi_{n_7,n_8}(t) \leqslant \frac{2^{\frac{p^*}{p}-1}c^{\frac{p^*}{p}}m_2^{\frac{p^*}{p}-1}m_3}{1-r(L_{n_7,n_8})} := N_2.$$

Now $p^*/p \geqslant 1$ and the Jensen integral inequality for convex functions (Lemma 2.3) imply

$$\begin{aligned}\left(\int_0^1 v^p(t)\varphi_{n_7,n_8}(t)dt\right)^{\frac{p^*}{p}} &\leqslant \int_0^1 v^p(t)\varphi_{n_7,n_8}^{\frac{p^*}{p}}(t)dt \\ &\leqslant \|\varphi_{n_7,n_8}\|^{\frac{p^*}{p}-1}\int_0^1 v^{p^*}(t)\varphi_{n_7,n_8}(t)dt \\ &\leqslant N_2\|\varphi_{n_7,n_8}\|^{\frac{p^*}{p}-1}, \end{aligned} \tag{18}$$

so that

$$\int_0^1 v^p(t)\varphi_{n_7,n_8}(t)dt \leqslant N_2^{p_*}\|\varphi_{n_7,n_8}\|^{1-p_*}.$$

Note $v^p$ is concave. By Lemma 2.1, we have

$$\|v^p\| \leqslant \frac{N_2^{p_*}\|\varphi_{n_7,n_8}\|^{1-p_*}}{\kappa_{n_7,n_8}}.$$

This proves the boundedness of $\mathscr{M}_4$. Taking $R >\sup\{\|v\| : v \in \mathscr{M}_4\}$, we have

$$v \neq \lambda Av \quad \text{for all } v \in \partial B_R \cap P \text{ and } \lambda \in [0,1].$$

Now Lemma 2.5 implies

$$i(A, B_R \cap P, P) = 1. \tag{19}$$

Note that we may assume $R > r_2$. Combining (17) and (19) gives

$$i(A,(B_R \setminus \overline{B}_{r_2}) \cap P, P) = 1 - 0 = 1.$$

Therefore the operator $A$ has at least one fixed point on $(B_R \setminus \overline{B}_{r_2}) \cap P$. Thus (1) has at least one positive solution. This completes the proof.

REMARK 3.2**.** (H3) and (H4) describe the $p$-superlinear growth of $f$, as exemplified by $f(t,x,y) := x^{q_1} + y^{q_2}$ with $q_1 > p$ and $q_2 > p$.

REMARK 3.3. (H5) and (H6) describe the $p$-sublinear growth of $f$, as exemplified by $f(t,x,y) := x^{q_3} + y^{q_4}$ with $0 < q_3 < p$, $0 < q_4 < p$.

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