Applied Mathematics E-Notes, 14(2014), 1-12 © ISSN 1607-2510
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Implicit Relation And Eldestein-Suzuki Type Fixed Point Results In Cone Metric Spaces*

Fridoun Moradlou[†], Peyman Salimi[‡], Stojan Radenović[§]

Received 15 January 2013

Abstract

In this paper, we prove Eldestein-Suzuki type fixed point results in cone metric spaces by using implicit relation. Our results generalize, extend, unify, enrich and complement many existing results in the literature. Examples are given showing the validity of our results.

1 Introduction

In 1962, M. Edelstein [6] proved another version of Banach Contraction Principle. He assumed a compact metric space (X, d) and a self-mapping T on X such that $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, and he proved T has a unique fixed point. In 2009, T. Suzuki [20] improved the results of Banach and Edelstein. Suzuki replaced the condition $d(Tx, Ty) < d(x, y)$ by

$$\frac{1}{2}d(x, Tx) < d(x, y) \Rightarrow d(Tx, Ty) < d(x, y)$$

for all $x, y \in X$. By this assumption he established T has a unique fixed point. Recently D. Dorić et al. in [5] proved the following theorem and extended the results of Edelstein and Suzuki on compact cone metric spaces.

THEOREM 1. Let (X, d) be a compact cone metric space over a normal and solid cone P and let $T : X \to X$ be given. Assume that

$$d(Tx, Ty) \ll Ad(x, y) + Bd(x, Tx) + Cd(y, Ty) + Dd(x, Ty) + Ed(y, Tx),$$

for all $x, y \in X$, $x \neq y$ and

$$\frac{1}{2}d(x, Tx) - d(x, y) \notin intP.$$

*Mathematics Subject Classifications: 47H10, 54H25, 55M20.

†Department of Mathematics, Sahand University of Technology, Tabriz, Iran.

‡Department of Mathematics, Sahand University of Technology, Tabriz, Iran.

§University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120, Beograd, Serbia.

where $A, B, C, D, E \geq 0$, $A + B + C + 2D = 1$ and $C \neq 1$. Then T has a fixed point in X. If $E \leq B + C + D$, then the fixed point of T is unique.

In 2007, Huang and Zhang [8] introduced cone metric spaces and defined some properties of convergence of sequences and completeness in cone metric spaces, also they proved a fixed point theorem of cone metric spaces. A number of authors were attracted to these results of Huang and Zhang and stimulated to investigate the fixed point theorems in cone metric spaces. During the recent years, cone metric spaces and properties of these spaces have been studied by a number of authors. Also many mathematicians have been extensively investigated fixed point theorems in cone metric spaces (see [16–21]).

Furthermore, many authors considered implicit relation technique to investigation of fixed point theorems in metric spaces (see [2–4, 9, 11–14, 18]).

In this paper, we introduce an implicit relation. This helps us to extend result of D. Dorić et. al. (Theorem 3.8 of [5]).

2 Preliminaries

We begin with the following:

DEFINITION 1. Let E be a real Banach space with norm $\|.\|$ and P be a subset of E. P is called a cone if and only if the following conditions are satisfied:

(i) P is closed, nonempty and $P \neq \{\theta\}$,

(ii) $a, b \geq 0$ and $x, y \in P$ implies $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ implies $x = \theta$.

Let $P \subset E$ be a cone, we define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$ (interior of P). The cone $P \subset E$ is called normal if there exists a positive real number K such that for all $x, y \in E$, $\theta \preceq x \preceq y \Rightarrow \|x\| \leq K\|y\|$. The least positive number satisfying the above inequality is called the normal constant of P. If $K = 1$, then the cone P is called monotone.

DEFINITION 2. A cone metric space is an ordered pair (X, d), where X is any set and $d : X \times X \longrightarrow E$ is a mapping satisfying:

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$, and $d(x, y) = \theta$ if and only if $x = y$,

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Let (X, d) be a cone metric space, P be a normal cone in X with normal constant K, $x \in X$ and $\{x_n\}$ a sequence in X. The sequence $\{x_n\}$ is converges to x if and only if $d(x_n, x) \longrightarrow \theta$. Limit point of every sequence is unique.

It is well known that, there exists a norm $\|.\|_1$ on E, equivalent with the given $\|.\|$, such that the cone P is monotone w.r.t. $\|.\|_1$ (see [1], [22]). By using this fact, from now on, we assume that the cone P is solid and monotone. In this case, we can define a metric on X by $D(x, y) = \|d(x, y)\|$. Furthermore, it is proved that D and d give same topology on X (see [15]).

We will use the following Lemma in the proof of the next theorem.

LEMMA 1. Let (X, d) be a cone metric space. Then

$$\theta \preceq x \ll y \Rightarrow \|x\| < \|y\|.$$

PROOF. According ([22], Proposition (2.2), page 20) $[-(y-x), y-x]$ is neighborhood of θ. Hence, for sufficiently large n, we have $\frac{1}{n}y \in [-(y-x), y-x]$, i.e., $\frac{y}{n} \preceq y-x$. From this it follows that $x \preceq \left(1 - \frac{1}{n}\right) y$, that is $\|x\| \leq \left(1 - \frac{1}{n}\right) \|y\| < \|y\|$.

3 Implicit Relation

In this section, we prove a theorem in the context of compact cone metric spaces over a monotone and solid cone by using implicit relation technique. Our result extends [20, Theorem 8] and [5, Theorem 3.8].

Let $F : P^6 \longrightarrow \mathbb{R}$ be a function satisfies the following conditions:

(M1) $u \preceq v$ implies $F(., ., ., v, ., .) \leq F(., ., ., u, ., .)$,

(M2) $F(u, v, v, u + v, u, \theta) \leq 0$ implies $\|u\| \leq \|v\|$,

(M3) $F(u, v, v, u + v, u, \theta) < 0$ implies $\|u\| < \|v\|$, where $u \succeq \theta$ and $v \gg \theta$,

(M4) $F(u, v, \theta, v, \theta, v) < 0$ implies $\|u\| < \|v\|$, where $u \succeq \theta$ and $v \gg \theta$.

EXAMPLE 1.

(A) $F(p_1, p_2, p_3, p_4, p_5, p_6) = \|p_1\| - \|p_2\|$;

(B) $F(p_1, p_2, p_3, p_4, p_5, p_6) = 2\|p_1\| - \|p_4\|$;

(C) $F(p_1, p_2, p_3, p_4, p_5, p_6) = 2\|p_1\| - (\|p_2\| + \|p_5\|)$;

(D) $F(p_1, p_2, p_3, p_4, p_5, p_6) = 5\|p_1\| - (\|p_2\| + \|p_3\| + \|p_4\| + \|p_5\| + \|p_6\|)$;

(E) $F(p_1, p_2, p_3, p_4, p_5, p_6) = 2\|p_1\| - \max\{\|p_2\|, \|p_3\|, \|p_4\|, \|p_5\|, \|p_6\|\}$;

(F) $F(p_1, p_2, p_3, p_4, p_5, p_6) = 2\|p_1\|^2 - (\|p_2\|^2 + \|p_5\|^2)$.
It is easy to see that, $(M1) - (M4)$ are satisfied for F in (A), (B), (C), (D), (E) and (F).

(G) $F(p_1,p_2,p_3,p_4,p_5,p_6) = \|p_1\| - (a\|p_2\| + b\|p_3\| + c\|p_4\| + d\|p_5\| + e\|p_6\|)$, where a,b,c,d and e are nonnegative numbers, $a+b+2c+d=1$, $d \neq 1$ and $e \leq b+c+d$.

Clearly $(M1)$ holds. Now, let

$$(1-d)\|u\| - \Big((a+b)\|v\| + c\|u+v\|\Big) = F(u,v,v,u+v,u,\theta) \leq 0.$$

Then we see that

$$(1-c-d)\|u\| - (a+b+c)\|v\| \leq F(u,v,v,u+v,u,\theta) \leq 0.$$

So by the assumption $1-c-d = a+b+c$, we observe that $1-c-d \leq 0$ implies $a=b=c=0$. Therefore, $d=1$, which is a contradiction. Hence $1-c-d > 0$. Thus $\|u\| \leq \|v\|$. So $(M2)$ is satisfied. Similar argument shows that $(M3)$ is satisfied. Moreover, if

$$F(u,v,\theta,v,\theta,v) = \|u\| - (a+c+e)\|v\| < 0,$$

then $\|u\| < (a+c+e)\|v\|$. So by the hypothesis, we can write

$$\|u\| < (a+c+e)\|v\| \leq (a+b+2c+d)\|v\| = \|v\|.$$

Therefore, $(M4)$ is satisfied.

(H)

$$\begin{aligned} F(p_1,p_2,p_3,p_4,p_5,p_6) &= \|p_1\| - \big(a\min\{\|p_2\|,\|p_3\|\} + b\min\{\|p_3\|,\|p_4\|\} \\ &\quad + c\min\{\|p_4\|,\|p_5\|\} + \|p_6\|\big), \end{aligned}$$

where a,b and c are nonnegative numbers, $a+b+c=1$ and $c \neq 1$.

Clearly $(M1)$ holds. If $F(u,v,v,u+v,u,\theta) \leq 0$, then

$$\begin{aligned} \|u\| &\leq \Big(a\|v\| + b\min\{\|v\|,\|u+v\|\} + c\min\{\|u+v\|,\|u\|\}\Big) \\ &\leq \Big(a\|v\| + b\min\{\|v\|,\|u\|+\|v\|\} + c\min\{\|u\|+\|v\|,\|u\|\}\Big), \end{aligned}$$

which implies

$$(1-c)\|u\| \leq (a+b)\|v\|.$$

So, by using $a+b+c=1$ and $c \neq 1$, we conclude that $\|u\| \leq \|v\|$. This means $(M2)$ is satisfied. Similarly, we can show that $(M3)$ is satisfied. Also, it is easy to see that $(M4)$ is satisfied.

THEOREM 2. Let (X,d) be a compact cone metric space and T be a self-mapping on X. Suppose that $F : P^6 \longrightarrow \mathbb{R}$ is a continuous function such that $(M1)$-$(M3)$ are satisfied. Assume that

$$F\Big(d(Tx,Ty), d(x,y), d(x,Tx), d(x,Ty), d(y,Ty), d(y,Tx)\Big) < 0, \tag{1}$$

for $x, y \in X$ and

$$\frac{1}{2}D(x, Tx) < D(x, y).$$

Then T has at least one fixed point. Moreover, if F satisfies $(M4)$, then T has a unique fixed point.

PROOF. Let $\alpha = \inf\{D(x, Tx) : x \in X\}$. There exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} D(x_n, Tx_n) = \alpha$. By compactness of X, there exist $w_1, w_2 \in X$ such that $\lim_{n\to\infty} x_n = w_1$ and $\lim_{n\to\infty} Tx_n = w_2$. Hence

$$\lim_{n\to\infty} D(x_n, w_2) = \lim_{n\to\infty} D(x_n, Tx_n) = D(w_1, w_2) = \alpha.$$

Now, we show that α must be equal to 0. If $\alpha > 0$, then there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\frac{2}{3}\alpha < D(x_n, w_2)$ and $D(x_n, Tx_n) < \frac{4}{3}\alpha$. Therefore for all $n \geq N$, $\frac{1}{2}D(x_n, Tx_n) < \frac{2}{3}\alpha < D(x_n, w_2)$. Now, by (1), we have that

$$F\Big(d(Tx_n, Tw_2), d(x_n, w_2), d(x_n, Tx_n), d(x_n, Tw_2), d(w_2, Tw_2), d(w_2, Tx_n)\Big) < 0. \quad (2)$$

By taking the limit as $n \to \infty$ in (2), we get

$$F\Big(d(w_2, Tw_2), d(w_1, w_2), d(w_1, w_2), d(w_1, Tw_2), d(w_2, Tw_2), \theta\Big) \leq 0.$$

By triangle inequality and $(M1)$, we get

$$F\Big(d(w_2, Tw_2), d(w_1, w_2), d(w_1, w_2), d(w_1, w_2) + d(w_2, Tw_2), d(w_2, Tw_2), \theta\Big) \leq 0,$$

so by $(M2)$, we have that $D(w_2, Tw_2) \leq \alpha$. Therefore, $D(w_2, Tw_2) = \alpha > 0$. It follows that $\frac{1}{2}D(w_2, Tw_2) < D(w_2, Tw_2)$. Now by (1), we can obtain that

$$F\Big(d(Tw_2, T^2w_2), d(w_2, Tw_2), d(w_2, Tw_2), d(w_2, T^2w_2), d(Tw_2, T^2w_2), \theta\Big) < 0,$$

which implies that

$$F\Big(d(Tw_2, T^2w_2), d(w_2, Tw_2), d(w_2, Tw_2),$$
$$d(Tw_2, T^2w_2) + d(w_2, Tw_2), d(Tw_2, T^2w_2), \theta\Big) < 0.$$

By $(M3)$, we get $D(Tw_2, T^2w_2) < D(w_2, Tw_2) = \alpha$, which is a contradiction of the definition of α. So $\alpha = 0$, that is, $w_1 = w_2$.

Now, we must show that T has at least one fixed point. Assume towards a contradiction that T does not have a fixed point. Hence,

$$0 < \frac{1}{2}D(x_n, Tx_n) < D(x_n, Tx_n).$$

Then by (1), we have that

$$F\Big(d(Tx_n, T^2x_n), d(x_n, Tx_n), d(x_n, Tx_n), d(x_n, T^2x_n), d(Tx_n, T^2x_n), d(Tx_n, Tx_n)\Big) < 0.$$

By taking the limit as $n \to \infty$ in above inequality, we get

$$F\Big(\lim_{n\to\infty} d(w_1, T^2x_n), \theta, \theta, \lim_{n\to\infty} d(w_1, T^2x_n), \lim_{n\to\infty} d(w_1, T^2x_n), \theta\Big) \leq 0.$$

It follows from $(M2)$ that $\lim_{n\to\infty} D(w_1, T^2x_n) \leq 0$, so $\lim_{n\to\infty} T^2x_n = w_1$. Furthermore, by using (1) and $(M1)$, we obtain that

$$F\Big(d(Tx_n, T^2x_n), d(x_n, Tx_n), d(x_n, Tx_n), d(Tx_n, T^2x_n) + d(x_n, Tx_n), d(Tx_n, T^2x_n), \theta\Big) < 0.$$

Then by $(M3)$ we have that $D(Tx_n, T^2x_n) < D(x_n, Tx_n)$.

Now, suppose that both of the following inequalities hold for some $n \in \mathbb{N}$,

$$\frac{1}{2}D(x_n, Tx_n) \geq D(x_n, w_1) \quad \text{and} \quad \frac{1}{2}D(Tx_n, T^2x_n) \geq D(Tx_n, w_1),$$

then, we have that

$$\begin{aligned} D(x_n, Tx_n) &\leq D(x_n, w_1) + d(w_1, Tx_n) \\ &\leq \frac{1}{2}D(x_n, Tx_n) + \frac{1}{2}D(Tx_n, T^2x_n) \\ &< \frac{1}{2}D(x_n, Tx_n) + \frac{1}{2}D(x_n, Tx_n) = D(x_n, Tx_n), \end{aligned}$$

which is a contradiction. Thus, for each $n \in \mathbb{N}$, either

$$\frac{1}{2}D(x_n, Tx_n) < D(x_n, w_1),$$

or

$$\frac{1}{2}D(Tx_n, T^2x_n) < D(Tx_n, w_1),$$

holds. So by hypotheses, we conclude that one of the following inequalities holds for all n in an infinite subset of $\mathbb{N}$:

$$F\Big(d(Tx_n, Tw_1), d(x_n, w_1), d(x_n, Tx_n), d(x_n, Tw_1), d(w_1, Tw_1), d(w_1, Tx_n)\Big) < 0,$$

or

$$F\Big(d(T^2x_n, Tw_1), d(Tx_n, w_1), d(Tx_n, T^2x_n), d(Tx_n, Tw_1), d(w_1, Tw_1), d(w_1, T^2x_n)\Big) < 0.$$

If we take the limit as $n \to \infty$ in each of these inequalities, then we have that

$$F\Big(d(w_1, Tw_1), \theta, \theta, d(w_1, Tw_1), d(w_1, Tw_1), \theta\Big) \leq 0.$$

So $(M2)$ implies that $D(w_1, Tw_1) \leq 0$, i.e., $w_1 = Tw_1$. Hence, we conclude that w_1 is a fixed point of T.

To prove the uniqueness of w_1, suppose that w_0 is another fixed point of T such that $w_1 \neq w_0$. Hence, $0 = \frac{1}{2}D(w_1, Tw_1) < D(w_1, w_0)$. By (1), we have that

$$F\Big(d(Tw_1, Tw_0), d(w_1, w_0), d(w_1, Tw_1), d(w_1, Tw_0), d(w_0, Tw_0), d(w_0, Tw_1)\Big) < 0.$$

So

$$F\Big(d(w_1, w_0), d(w_1, w_0), \theta, d(w_1, w_0), \theta, d(w_0, w_1)\Big) < 0.$$

Considering $(M4)$, we have that $D(w_1, w_0) < D(w_1, w_0)$, which is a contradiction. Therefore $w_1 = w_0$. Then w_1 is the unique fixed point of T.

THEOREM 3. Let (X, d) be a cone metric space and let G and T be two self-mappings on X such that $TX \subseteq GX$ and GX is compact. Suppose that $F : P^6 \longrightarrow \mathbb{R}$ is a continuous function such that $(M1) - (M3)$ are satisfied. Assume that

$$F\Big(d(Tx, Ty), d(Gx, Gy), d(Gx, Tx), d(Gx, Ty), d(Gy, Ty), d(Gy, Tx)\Big) < 0, \tag{3}$$

for all $x, y \in X$ and

$$\frac{1}{2}D(Gx, Tx) < D(Gx, Gy).$$

Then G and T have at least one point of coincidence. Moreover, if F satisfies $(M4)$ and G and T are weakly compatible, then G and T have a unique common fixed point.

PROOF. Define $H : GX \longrightarrow GX$ by $H(G(w)) = Tw$. Replacing Tx and Ty by $H(Gx)$ and $H(Gy)$, respectively, in (3), we have that, for $Gx, Gy \in GX$,

$$\frac{1}{2}d(Gx, H(Gx)) < d(Gx, Gy),$$

which implies that, for $Gx, Gy \in GX$,

$$F\Big(d(H(Gx), H(Gy)), d(Gx, Gy), d(Gx, H(Gx)),$$
$$d(Gx, H(Gy)), d(Gy, H(Gy)), d(Gy, H(Gx))\Big) < 0.$$

Since GX is compact, by Theorem 2, H has a fixed point, i.e., there exists $z \in X$ such that $Gz = H(Gz) = Tz := u$. Moreover, if F satisfies $(M4)$, then H has a unique fixed point. So we conclude that z is a unique point of coincidence of G and T. Furthermore, if G and T are weakly compatible mappings, we get $GTz = TGz$, so $Gu = Tu$. Therefore $z = u$ and $Gz = Tz = z$. This yields z as the unique common fixed point of G and T.

COROLLARY 1. Let (X, d) be a cone metric space and let G and T be two self-mappings on X such that $TX \subseteq GX$ and GX is compact. Assume that

$$D(Tx, Ty) < aD(Gx, Gy) + bD(Gx, Tx) + cD(Gx, Ty) + dD(Gy, Ty) + eD(Gy, Tx),$$

for all $x, y \in X$ and

$$\frac{1}{2}D(Gx, Tx) < D(Gx, Gy),$$

where $a, b, c, d, e \geq 0$, $a + b + 2c + d = 1$ and $d \neq 1$. Then G and T have at least one point of coincidence. Moreover, if $e \leq b + c + d$ and G and T are weakly compatible, then G and T have a unique common fixed point.

PROOF. The proof follows from Theorem 3 and part (G) of Example 1.

COROLLARY 2. Let (X, d) be a cone metric space and let G and T be two weakly compatible self-mappings on X such that $TX \subseteq GX$ and GX is compact. Assume that

$$\begin{aligned} D(Tx, Ty) \quad < \quad & a \min\{D(Gx, Gy), D(Gx, Tx)\} + b \min\{D(Gx, Tx), D(Gx, Ty)\} \\ & + c \min\{D(Gx, Ty), D(Gy, Ty)\} + D(Gy, Tx), \end{aligned}$$

for all $x, y \in X$ and

$$\frac{1}{2} D(Gx, Tx) < D(Gx, Gy)$$

where $a + b + c = 1$ and $c \neq 1$. Then G and T have a unique common fixed point.

PROOF. The proof follows from Theorem 3 and part (H) of Example 1.

COROLLARY 3. Let (X, d) be a cone metric space and let G and T be two weakly compatible self-mappings on X such that $TX \subseteq GX$ and GX is compact. Assume that $D(Tx, Ty) < D(Gx, Gy)$ for all $x, y \in X$, $x \neq y$, and

$$\frac{1}{2} D(Gx, Tx) < D(Gx, Gy).$$

Then G and T have a unique common fixed point.

PROOF. The proof follows from Theorem 3 and part (A) of Example 1.

REMARK 1. We can obtain some new results by using Theorem 3 and other examples of F.

In the rest of this section, we assume that $\psi_p : P \longrightarrow P$ and $\varphi_p : P^5 \longrightarrow P$ are two mappings satisfying the following conditions:

(P1) $u \preceq v$ implies $\varphi_p(.,.,v,.,.) - \varphi_p(.,.,u,.,.) \in P$,

(P2) $\|\psi_p(u)\| \leq \|\varphi_p(v, v, u + v, u, \theta)\|$ implies $\|u\| \leq \|v\|$,

(P3) $\|\psi_p(u)\| < \|\varphi_p(v, v, u + v, u, \theta)\|$ implies $\|u\| < \|v\|$, where $v \neq \theta$,

(P4) $\|\psi_p(u)\| < \|\varphi_p(v, \theta, v, \theta, v)\|$ implies $\|u\| < \|v\|$, where $v \neq \theta$.

We define $F_p : P^6 \longrightarrow \mathbb{R}$ by

$$F_p(p_1, p_2, p_3, p_4, p_5, p_6) = \|\psi_p(p_1)\| - \|\varphi_p(p_2, p_3, p_4, p_5, p_6)\|.$$

Clearly F_p satisfies $(M1) - (M4)$.

EXAMPLE 2. Suppose that $p, p_1, p_2, p_3, p_4, p_5 \in P$. Let

(A) $\psi_p(p) = p$ and $\varphi_p(p_1, p_2, p_3, p_4, p_5) = p_1$;

(B) $\psi_p(p) = 2p$ and $\varphi_p(p_1, p_2, p_3, p_4, p_5) = p_3$;

(C) $\psi_p(p) = 2p$ and $\varphi_p(p_1, p_2, p_3, p_4, p_5) = p_1 + p_4$;

(D) $\psi_p(p) = 5p$ and $\varphi_p(p_1, p_2, p_3, p_4, p_5) = p_1 + p_2 + p_3 + p_4 + p_5$.
It is easy to show that $(P1) - (P4)$ are satisfied for ψ_p and φ_p in (A), (B), (C) and (D).

(E) $\psi_p(p) = p$ and $\varphi_p(p_1, p_2, p_3, p_4, p_5) = ap_1 + bp_2 + cp_3 + dp_4 + ep_5$, where a, b, c, d and e are nonnegative numbers, $a + b + 2c + d = 1$ and $d \neq 1$. So, $(P1) - (P3)$ are satisfied. Moreover, if $e \leq b + c + d$, then $(P4)$ is satisfied.

THEOREM 4. Let (X, d) be a compact cone metric space and T be a self-mapping on X. Suppose that $\psi_p : P \longrightarrow P$ and $\varphi_p : P^5 \longrightarrow P$ are two continuous mappings such that $(P1) - (P3)$ are satisfied. Assume that

$$\psi_p(d(Tx, Ty)) \ll \varphi_p\Big(d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), (y, Tx)\Big), \tag{4}$$

for all $x, y \in X$, $x \neq y$, and

$$\frac{1}{2}d(x, Tx) - d(x, y) \notin intP.$$

Then T has at least one fixed point. Moreover, if ψ_p and φ_p satisfy $(P4)$, then T has a unique fixed point.

PROOF. Let $\frac{1}{2}D(x, Tx) < D(x, y)$. So $\frac{1}{2}d(x, Tx) - d(x, y) \notin intP$. Therefore by (4), we have that

$$\psi_p(d(Tx, Ty)) \ll \varphi_p\Big(d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\Big).$$

Thus, by Lemma 1, we get

$$\|\psi_p(d(Tx, Ty))\| < \|\varphi_p\Big(d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\Big)\|.$$

That is,

$$F_p\Big(d(Tx, Ty), d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\Big) < 0.$$

Clearly, F_p is continuous. Also, F_p satisfies the conditions $(M1) - (M3)$. Therefore, Theorem 2 implies that T has at least one fixed point. Furthermore, if ψ_p and φ_p satisfy $(P4)$, then F_p satisfies $(M4)$ and so T has a unique fixed point.

THEOREM 5. Let (X, d) be a cone metric space and let G and T be two self-mappings on X such that $TX \subseteq GX$ and GX is compact. Suppose that $\psi_p : P \longrightarrow P$ and $\varphi_p : P^5 \longrightarrow P$ are two continuous mappings satisfying $(P1)$–$(P3)$. Assume that

$$\psi_p(d(Tx, Ty)) \ll \varphi_p\Big(d(Gx, Gy), d(Gx, Tx), d(Gx, Ty), d(Gy, Ty), d(Gy, Tx)\Big),$$

for all $x, y \in X$, $x \neq y$, and

$$\frac{1}{2}d(Gx, Tx) - d(Gx, Gy) \notin intP.$$

Then G and T have at least one point of coincidence. Moreover, if ψ_p and φ_p satisfy $(P4)$ and G and T are weakly compatible, then G and T have a unique common fixed point.

COROLLARY 4. Let (X, d) be a cone metric space and let G and T be two weakly compatible self-mappings on X such that $TX \subseteq GX$ and GX is compact. Assume that $d(Tx, Ty) \ll d(Gx, Gy)$ for all $x, y \in X$, $x \neq y$, and

$$\frac{1}{2}d(Gx, Tx) - d(Gx, Gy) \notin intP.$$

Then G and T have a unique common fixed point.

COROLLARY 5. Let (X, d) be a cone metric space and let G and T be two weakly compatible self-mappings on X such that $TX \subseteq GX$ and GX is compact. Assume that $d(Tx, Ty) \ll \frac{1}{2}d(Gx, Gy)$ for all $x, y \in X$, $x \neq y$, and

$$\frac{1}{2}d(Gx, Tx) - d(Gx, Gy) \notin intP.$$

Then G and T have a unique common fixed point.

COROLLARY 6. Let (X, d) be a cone metric space and let G and T be two weakly compatible self-mappings on X such that $TX \subseteq GX$ and GX is compact. Assume that

$$d(Tx, Ty) \ll \frac{1}{2}\left[d(Gx, Gy) + d(Gx, Tx)\right]$$

for all $x, y \in X$, $x \neq y$, and

$$\frac{1}{2}d(Gx, Tx) - d(Gx, Gy) \notin intP.$$

Then G and T have a unique common fixed point.

COROLLARY 7. Let (X, d) be a cone metric space and let G and T be two weakly compatible self-mappings on X such that $TX \subseteq GX$ and GX is compact. Assume that

$$d(Tx, Ty) \ll \frac{1}{5}[d(Gx, Gy) + d(Gx, Tx) + d(Gx, Ty) + d(Gy, Ty) + d(Gy, Tx)],$$

for all $x, y \in X$, $x \neq y$, and

$$\frac{1}{2}d(Gx, Tx) - d(Gx, Gy) \notin intP.$$

Then G and T have a unique common fixed point.

COROLLARY 8. Let (X, d) be a cone metric space and let G and T be two self-mappings on X such that $TX \subseteq GX$ and GX is compact. Assume that $d(Tx, Ty) \ll M(x, y)$ for all $x, y \in X$, $x \neq y$, and

$$\frac{1}{2}d(Gx, Tx) - d(Gx, Gy) \notin intP,$$

where

$$M(x, y) = Ad(Gx, Gy) + Bd(Gx, Tx) + Cd(Gx, Ty) + Dd(Gy, Ty) + Ed(Gy, Tx),$$

and $A, B, C, D, E \geq 0$, $A + B + 2C + D = 1$ and $D \neq 1$. Then G and T have at least one point of coincidence. Moreover, if $E \leq B + C + D$ and G and T are weakly compatible, then F and T have a unique common fixed point.

Acknowledgment. The Authors would like to thank the referee(s) for a number of valuable suggestions regarding a previous version of this paper.

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