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Note On A Supported Beam Problem*

Yun-Rui Yang†, Le An‡

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Abstract

In this paper, an existence criterion for triple positive solutions of the supported beam problem

$$\begin{cases} u^{(4)} + \beta u'' - \alpha u = f(t, u(t)), & t \in (0,1) \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

is established by using the Leggett-Williams fixed point theorem, where $f : [0,1] \times R^+ \to R^+$ is continuous, $\alpha, \beta \in R$ and satisfy $\beta < 2\pi^2, \alpha \geq -\frac{\beta^2}{4}, \frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1$. An example is also included to demonstrate the result we obtained.

1 Introduction

There are good reasons for studying fourth-order ordinary differential equations. One reason is that a horizontally supported beam of finite length under loading can be modeled by such equations. One particular equation studied in several recent papers is of the form

$$u^{(4)} + \beta u'' - \alpha u = f(t, u(t)), \quad t \in (0,1), \tag{1}$$

where α, β are real parameters, and $u(t)$ stands for the displacement of the beam from the equilibrium position. In real situations, beams are subject to various side conditions. In one situation, both ends of the finite beam are supported. Then the corresponding boundary value conditions are

$$u(0) = u(1) = u''(0) = u''(1) = 0. \tag{2}$$

There are now several studies which provide existence of nontrivial solutions of the supported beam problem stated above. In particular, in [1], by transforming the beam problem into a fixed point problem, the author uses the fixed point index theory to obtain the existence of positive solutions. In [2], the authors use the critical point theory to establish the existence of at least two nontrivial solutions.

However, there are other basic fixed point theorems which may provide existence of multiple solutions as well. One such theorem is the Leggett-Williams fixed-point

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†School of Mathematics and Physics, Lanzhou Jiaotong University, lanzhou, Gansu 730070, P. R. China

‡College of Mathematics and Statics, Tianshui Normal University, Tianshui, Gansu 741001, P. R. China

theorem [3] which has been used quite extensively in recent studies related to boundary value problems [4, 5]. Therefore, a short note using such a result may provide additional insight and information to the solution space of our boundary value problem.

For the sake of completeness, we first recall some basic definitions and results. Let K be a cone in a real Banach space E with norm $\|\cdot\|$. A map α is said to be a nonnegative continuous concave functional on K if α maps K into $[0,+\infty)$, is continuous and satisfies

$$\alpha(tx+(1-t)y) \geq t\alpha(x)+(1-t)\alpha(y); \quad x,y \in K, t \in [0,1].$$

Let α be a nonnegative continuous concave functional on K. For fixed numbers a, b such that $0 < a < b$, the sets $K_a = \{x \in K : \|x\| < a\}$ and $K(\alpha, a, b) = \{x \in K : a \leq \alpha(x), \|x\| \leq b\}$ are both convex sets.

THEOREM 1.1 (Leggett-Williams, see e.g. [3]). Let $A : \bar{K_c} \to \bar{K_c}$ be a completely continuous operator and α is a nonnegative continuous concave functional on K such that $\alpha(x) \leq \|x\|$ for all $x \in \bar{K_c}$. If there exist positive numbers a, b, c, d such that $0 < d < a < b \leq c$ and

(i) $\{x \in K(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in K(\alpha, a, b)$;

(ii) $\|Ax\| < d$ for $\|x\| \leq d$;

(iii) $\alpha(Ax) > a$ for $x \in K(\alpha, a, c)$ with $\|Ax\| > b$.

Then A has at least three fixed points x_1, x_2, x_3 in $\bar{K_c}$ satisfying $\|x_1\| < d$, $a < \alpha(x_2)$, $\alpha(x_3) < a$ and $\|x_3\| > d$.

As noted in [1] , we may transform our supported beam problem into a fixed point problem. Let λ_1, λ_2 be the roots of the polynomial $P(\lambda) = \lambda^2 + \beta\lambda - \alpha$. If $\alpha, \beta \in R$ satisfy $\beta < 2\pi^2, \alpha \geq -\frac{\beta^2}{4}$ and $\frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1$, then $\lambda_1 = \frac{-\beta+\sqrt{\beta^2+4\alpha}}{2}$ $\lambda_2 = \frac{-\beta-\sqrt{\beta^2+4\alpha}}{2}$ satisfy $\lambda_1 \geq \lambda_2 > -\pi^2$. Furthermore, let $G_i(t,s)$, $i = 1, 2$, be the Green's function of the homogeneous boundary-value problem

$$-u''(t) + \lambda_i u(t) = 0, \quad t \in (0,1), \quad u(0) = u(1) = 0.$$

Then (i) $G_i(t,s) > 0$ for $t, s \in (0,1)$; (ii) $G_i(t,s) \leq D_i G_i(s,s)$ for $t, s \in [0,1]$; and (iii) $G_i(t,s) \geq \delta_i G_i(t,t) G_i(s,s)$ for $t, s \in [0,1]$; where $D_i = 1$ and $\delta_i = \frac{\omega_i}{\sinh \omega_i}$ if $\lambda_i > 0$; $D_i = 1$ and $\delta_i = 1$ if $\lambda_i = 0$; $D_i = \frac{1}{\sin \omega_i}$ and $\delta_i = \omega_i \sin \omega_i$ if $-\pi^2 < \lambda_i < 0$; and $\omega_i = \sqrt{|\lambda_i|}$. In addition, if

$$\begin{aligned} N_i &= \max_{s \in [0,1]} G_i(s,s), \\ n_i &= \min_{s \in \left[\frac{1}{4}, \frac{3}{4}\right]} G_i(s,s), \end{aligned}$$

and

$$D_0 = \int_0^1 G_1(\tau,\tau) G_2(\tau,\tau) d\tau.$$

then $N_i, n_i, D_0 > 0$.

Let $f:[0,1]\times(0,\infty)\to(0,\infty)$ be a continuous function. Then our beam problem (1)-(2), by means of the Green's functions G_1 and G_2, is transformed into the integral equation (see [6, 7])

$$u(t)=\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))ds,\quad t\in[0,1].$$

Let $E=C[0,1]$ be endowed with the usual maximum norm $\|u\|=\max_{t\in[0,1]}|u(t)|$ and let $C^+[0,1]$ be the cone of all nonnegative functions in $C[0,1]$.

LEMMA 1.2. If $h\in C^+[0,1]$, then the function

$$v(t)=\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)h(s)ds,\quad t\in[0,1],$$

satisfies

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} v(t)\geq\sigma\|v\|,$$

where

$$\sigma=\frac{\delta_1\delta_2D_0n_1}{D_1D_2N_1}\in(0,1).$$

PROOF. In view of the property (ii) of the Green's function G_i, it is not difficult to see that $v(t)\leq D_1D_2N_1\int_0^1 G_2(s,s)h(s)ds$ for $t\in[0,1]$. Therefore, $\|v\|\leq D_1D_2N_1\int_0^1 G_2(s,s)h(s)ds$. By property (iii) of G_i, we have

$$\begin{aligned}
v(t) &\geq \delta_1\delta_2\int_0^1\int_0^1 G_1(t,t)G_1(\tau,\tau)G_2(\tau,\tau)G_2(s,s)h(s)dsd\tau\\
&= \delta_1\delta_2D_0G_1(t,t)\int_0^1 G_2(s,s)h(s)ds\\
&\geq \frac{\delta_1\delta_2D_0}{D_1D_2N_1}G_1(t,t)\|v\|
\end{aligned}$$

for $t\in[0,1]$. Therefore,

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} v(t)\geq\frac{\delta_1\delta_2D_0}{D_1D_2N_1}\min_{t\in[\frac{1}{4},\frac{3}{4}]}G_1(t,t)\|v\|=\sigma\|v\|.$$

By means of the definitions of $\delta_1,\delta_2,D_0,D_1,D_2$ and n_1, we may easily check that $\sigma\in(0,1)$. The proof is complete.

Define a mapping $A:C^+[0,1]\to C^+[0,1]$ by

$$Au(t)=\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau.$$

It is easy to check that $A:C^+[0,1]\to C^+[0,1]$, under the continuity condition on f, is completely continuous, and its fixed points are (four times continuously differentiable) solutions of our supported beam problem. To find its fixed points, we let K be a subset of $C^+[0,1]$ defined by

$$K = \left\{ u \in C^+[0,1] : \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u(t) \geq \sigma \|u\| \right\}$$

Then we may check that K is a cone in E. For $u \in K$, we define $\alpha(u) = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u(t)$. It is easy to check that α is a nonnegative continuous concave functional on K with $\alpha(u) \leq \|u\|$ for $u \in K$.

2 Main Result

In this section, in order to state and prove the main result of this paper, we impose growth conditions on f which allow us to apply Theorem 1.1 to establish the existence criterion of triple positive solutions of (1)-(2).

THEOREM 2.1. Suppose $\alpha, \beta \in R$ satisfy $\beta < 2\pi^2, \alpha \geq -\frac{\beta^2}{4}$ and $\frac{\alpha}{\pi^4} + \frac{\beta}{\pi^2} < 1$ and $f : [0,1] \times R^+ \to R^+$ is continuous. Suppose further that there exist positive numbers a, b, c such that $0 < a < b \leq \sigma c$ and

$$f(t,u) < \frac{c}{M_0}, \quad (t,u) \in [0,1] \times [0,c]; \tag{3}$$

$$f(t,u) < \frac{a}{M_0}, \ (t,u) \in [0,1] \times [0,a]; \tag{4}$$

$$f(t,u) > \frac{b}{m_0}, \ (t,u) \in [1/4, 3/4] \times [b, b/\sigma], \tag{5}$$

where

$$M_0 = \max_{t \in [0,1]} \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) ds d\tau \text{ and } m_0 = \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_0^1 G_1(t,\tau) G_2(\tau,s) ds d\tau.$$

Then A has at least three fixed points u_1, u_2 and u_3 satisfying $\|u_1\| < a,\ b < \alpha(u_2),\ \alpha(u_3) < b$ and $\|u_3\| > a$.

PROOF. If $u \in K$, from the positivity of G_i, we know $Au \geq 0$. By Lemma 1.2, it is easy to check that $\min_{t \in [\frac{1}{4}, \frac{3}{4}]} Au(t) \geq \sigma \|Au\|$. Therefore $A(K) \subseteq K$. The complete continuity of A is explained before. Next, we need to check the conditions in Theorem 1.1. First, if $u \in \bar{K}_c$, then $\|u\| \leq c$. By (3), we have

$$\begin{aligned} \|Au\| &= \max_{t \in [0,1]} \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) f(s,u(s)) ds d\tau \\ &< \frac{c}{M_0} \max_{t \in [0,1]} \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) ds d\tau = \frac{c}{M_0} \cdot M_0 = c \end{aligned}$$

Thus, $Au \in K_c$, i.e. $A : \bar{K_c} \to K_c$. On the other hand, if (4) holds, i.e. $0 \le u(t) \le a$ for $t \in [0,1]$, then

$$\begin{aligned} \|Au(t)\| &= \max_{t\in[0,1]} \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau \\ &< \frac{a}{M_0} \max_{t\in[0,1]} \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau = \frac{a}{M_0}\cdot M_0 = a. \end{aligned}$$

Therefore, assumption (ii) of Theorem 1.1 holds.

Second, since $\frac{1}{2}\left(b+\frac{b}{\sigma}\right) \in \{K(\alpha,b,\frac{b}{\sigma})|\alpha(u) > b\}$, so $\{u \in K(\alpha,b,\frac{b}{\sigma})|\alpha(u) > b\} \ne \emptyset$. Moreover, for $u \in K(\alpha,b,b/\sigma)$, we have

$$b \le \alpha(u) = \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} u(t) \le u(t) \le \|u\| \le \frac{b}{\sigma}$$

for all $t \in [\frac{1}{4},\frac{3}{4}]$. Thus, by (5), we see that

$$\begin{aligned} \alpha(Au) &= \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau \\ &> \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,u(s))dsd\tau \\ &> \frac{b}{m_0} \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau = \frac{b}{m_0}\cdot m_0 = b, \end{aligned}$$

i.e. $\alpha(Au) > b$. That is to say, assumption (i) of Theorem 1.1 holds.

Finally, we check that assumption (iii) of Theorem 1.1. If $u \in K(\alpha,b,c)$ with $\|Au\| > d = b/\sigma$, then

$$\alpha(Au) = \min_{t\in[\frac{1}{4},\frac{3}{4}]} Au(t) \ge \sigma\|Au\| > \sigma\cdot\frac{b}{\sigma} = b$$

as required.

As a consequence, our supported beam problem has at least three positive solutions u_1, u_2 and u_3 satisfying $\|u_1\| < a,\ b < \alpha(u_2),\ \alpha(u_3) < b,\ \|u_3\| > a$. The proof is complete.

Next, let us consider an example which shows our result is nonvacuous.

EXAMPLE 2.2. Consider the following boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t,u(t)), & t \in (0,1) \\ u(0) = u(1) = u''(0) = u''(1) = 0 \end{cases}$$

Then

$$G_1(t,s) = G_2(t,s) = G(t,s) = \begin{cases} t(1-s) & 0 \le t \le s \le 1 \\ s(1-t) & 0 \le s \le t \le 1 \end{cases},$$

$\delta_1 = \delta_2 = D_1 = D_2 = 1$, and $D_0 = \int_0^1 G(\tau,\tau)G(\tau,\tau)d\tau = \int_0^1 \tau^2(1-\tau)^2 d\tau = \frac{1}{30}$,

$$n_1 = \min_{s\in\left[\frac{1}{4},\frac{3}{4}\right]} G(s,s) = \min_{s\in\left[\frac{1}{4},\frac{3}{4}\right]} s(1-s) = \frac{3}{16}$$

$$N_1 = \max_{s\in[0,1]} G(s,s) = \max_{s\in[0,1]} s(1-s) = \frac{1}{4}.$$

Thus

$$\sigma = \frac{\delta_1\delta_2 D_0 n_1}{D_1 D_2 N_1} = \frac{D_0 n_1}{N_1} = \frac{3}{4}D_0 = \frac{1}{40}.$$

We may also calculate M_0 and m_0 as follows. First,

$$\int_0^1 G(\tau,s)ds = \int_0^\tau s(1-\tau)ds + \int_\tau^1 \tau(1-s)ds = \frac{(1-\tau)\tau^2}{2} + \frac{\tau(1-\tau)^2}{2} = \frac{\tau(1-\tau)}{2}.$$

Hence

$$\int_0^1\int_0^1 G(t,\tau)G(\tau,s)dsd\tau = \int_0^1 G(t,\tau)\frac{\tau(1-\tau)}{2}d\tau = \frac{1}{24}t^4 - \frac{1}{12}t^3 + \frac{1}{24}t,$$

$$\int_{1/4}^{3/4}\int_0^1 G(t,\tau)G(\tau,s)dsd\tau = \int_{1/4}^{3/4} G(t,\tau)\frac{\tau(1-\tau)}{2}d\tau = \frac{1}{24}t^4 - \frac{1}{12}t^3 - \frac{13}{6144} + \frac{1}{24}t.$$

As a consequence, we see that

$$M_0 = \max_{t\in[0,1]}\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau = \max_{t\in[0,1]}\left(\frac{1}{24}t^4 - \frac{1}{12}t^3 + \frac{1}{24}t\right) = \frac{5}{384},$$

$$m_0 = \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^1 G_1(t,\tau)G_2(\tau,s)dsd\tau = \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\left(\frac{1}{24}t^4 - \frac{1}{12}t^3 - \frac{13}{6144} + \frac{1}{24}t\right) = \frac{11}{1536},$$

Therefore, if we take positive numbers a, b, c such that $0 < a < b$, $c \geq 40b$ and

$$\frac{384}{5}c > \max\left\{\frac{384}{5}a, \frac{1536}{11}b\right\},$$

then we may easily construct a *piecewise linear,* positive and continuous function $f(t,u)$ such that

$$f(t,u) < \frac{c}{M_0} = \frac{384}{5}c,\ (t,x) \in [0,1]\times[0,c].$$

$$f(t,x) > \frac{a}{M_0} = \frac{384}{5}a,\ (t,x) \in [0,1]\times[0,a],$$

$$f(t,x) < \frac{b}{m_0} = \frac{1536}{11}b,\ (t,x) \in \left[\frac{1}{4},\frac{3}{4}\right]\times[b,40b].$$

For such a function, our supported beam problem has at least three positive solutions u_1, u_2 and u_3 satisfying $\|u_1\| < a,\ b < \alpha(u_2),\ \alpha(u_3) < b$ and $\|u_3\| > a$.

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References

[1] Y. Li, Positive solutions of fourth-order boundary value problems with two parameters, J. Math. Anal. Appl., 281(2003), 477–484.

[2] X. L. Liu AND W. T. Li, Positive solutions of the nonlinear fourth-order beam equation with three parameters, J. Math. Anal. Appl., 303(2005), 150–163.

[3] R. I. Avery, A generalization of the Leggett-Williams fixed point theorem, Math. Sci. Res. Hot-line, 2(1998), 9–14.

[4] J. P. Sun, W. T. Li and Y. H. Zhao, Three positive solutions of a nonlinear three-point boundary value problem, J. Math. Anal. Appl., 288(2003), 708–716.

[5] Y. R. Yang, Triple positive solutions of a class of fourth-order two-point boundary value problems, Appl. Math. Lett., 23(2010), 366–370.

[6] S. S. Cheng, Isoperimetric eigenvalue problem of even order differential equations, Pacific J. Math., 99(2)(1982), 303–315.

[7] S. S. Cheng, Form preserving linear integral operators, Compu. Math. Appl., 31 (8)(1996), 117–134.