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Accelerated LMI Solvers For The Maximal Solution To A Set Of Discrete-Tme Algebraic Riccati Equations*

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Abstract

We consider a set of discrete-time coupled algebraic Riccati equations that arise in quadratic optimal control of Markovian jump linear systems. The LMI approach for computing the maximal symmetric solution of this system is studied. The special case of the Riccati equations with applications to financial modeling is commented. We construct two new modifications of the standard LMI approach and we show how to apply these new modifications to the investigated problem. Computer realizations of all modifications are compared. Numerical experiments are given where the new LMI modifications are numerically compared. Based on the experiments the main conclusion is the new LMI modifications are faster than the standard LMI approach.

1 Introduction

In recent years, a special class of linear systems subject to abrupt changes in their structures have been investigated. This is the case of Markovian jump linear systems (MJLS), which comprise an important family of models subject to abrupt variations. There are many examples in the literature showing the importance of the different types of discrete-time Riccati equations involved in the construction of the optimal controls of different problems of robust control (see [2, 6, 8, 12] and the literature therein). The properties and the numerical solutions of different types of discrete-time Riccati equations have been intensively studied in many papers [7, 3, 14, 21].

Consider the optimization problem described by the following more complicated dynamic system (first introduced on 2010 in [9]):

$$x(t+1) = \left[A_0(\eta_t) + \sum_{l=1}^{r} w_l(t) A_l(\eta_t)] x(t) + [B_0(\eta_t) + \sum_{l=1}^{r} w_l(t) B_l(\eta_t) \right] u(t)$$

and the cost functional

$$J = \sum_{t=0}^{\infty} E \left[\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} Q(\eta_t) & L(\eta_t) \\ L^T(\eta_t) & R(\eta_t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \right]$$

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where $\{\eta_t\}_{t\geq 0}$ is a Markov chain taking values in $\{1, 2, \ldots, N\}$, while $\{w(t)\}_{t\geq 0}$ is a sequence of independent random vectors $(w(t) = (w_1(t), \ldots, w_r(t))^T)$, for details see e.g. [9, 10, 11].

In the construction of the optimal control $\tilde{u}$ in the above optimization problem a crucial role is played by the maximal solution of the following system of discrete-time generalized Riccati equations (DTGRE) for $i = 1, \ldots, N$ with unknown matrices $X(1), \ldots, X(N)$:

$$\begin{array}{rl} & X(i) = \mathcal{P}(i, \mathbf{X}) \\ := & \sum_{k=0}^{r} A_k(i)^T \mathcal{E}_i(\mathbf{X}) A_k(i) + Q(i) \\ & -(\sum_{k=0}^{r} A_k(i)^T \mathcal{E}_i(\mathbf{X}) B_k(i) + L(i)) \left(R(i) + \sum_{k=0}^{r} B_k(i)^T \mathcal{E}_i(\mathbf{X}) B_k(i)\right)^{-1} , \\ & \times (\sum_{k=0}^{r} B_k(i)^T \mathcal{E}_i(\mathbf{X}) A_k(i) + L(i)^T) , \qquad i = 1, \ldots, N \end{array} \tag{1}$$

with assumptions that $R(i)+\sum_{k=0}^{r} B_k(i)^T \mathcal{E}_i(\mathbf{X}) B_k(i) > 0$ and $\mathcal{E}(\mathbf{X}) = (\mathcal{E}_1(\mathbf{X}), \ldots, \mathcal{E}_N(\mathbf{X}))$ with $\mathbf{X} = (X(1), \ldots, X(N))$ and

$$\mathcal{E}_i(\mathbf{X}) = \sum_{j=1}^{N} \lambda_{ij} X_j , \qquad X_j \text{ is an } n \times n \text{ Hermitian matrix} ,$$

and $\Lambda = (\lambda_{ij})$ denotes a transition probability matrix. Necessary and sufficient conditions for the existence of the maximal solution and stabilizing solution of this kind of discrete-time nonlinear equations were presented in [9, 10] in terms of the concept of the stabilizability of a sequence of linear and positive operators. A solution $\tilde{\mathbf{X}}$ of (1) is called maximal if $\tilde{\mathbf{X}} \geq \mathbf{X}$ for any solution $\mathbf{X}$. The direct proof of the existence of the maximal solution is given in Theorem 5.11 from [11]. An effective modification of the proposed iterative method from [10] to find the maximal and stabilizing solution of (1) is proposed in [15].

Lately, there exists an increasing interest to consider a computational approach to stochastic algebraic Riccati equations via a semidefinite programming problem over linear matrix inequalities (LMIs). Similar investigations can be found in [17, 18, 19, 22]. The main result from such studies is that the equivalence between the feasibility of the LMIs and the solvability of the corresponding stochastic Riccati equation is proved. Moreover, the maximal solution of a given stochastic algebraic Riccati equation can be obtained by solving a corresponding convex optimization problem (an LMI approach).

Many authors have considered a semidefinite programming problem as an unifying approach to the linear quadratic problems in the absence of the positive definiteness (semidefiniteness) of the cost matrices. In this paper, we develop computational approaches, based on the LMIs, to solve the set of nonlinear equations (1) with possibly indefinite matrices in the cost functional. The weighting matrices $R(i), i = 1, \ldots, N$ are singular or zero ones in very important practical problems, see [4, 5] where the applications in the portfolio optimization are investigated.

The paper is devoted to the LMI approach and its modifications. The LMI approach is very important for practical real world problems. Very often the LMI approach is the only method for solving a given class of problems. We introduce two new sets of nonlinear equations equivalent to the DTGRE (1) which lead us to the new convex optimization problems. The LMI approach applied to these new optimization problems gives on a fast way to find the maximal and stabilizing solution to (1). We will compare the numerical effectiveness of the introduced LMI solvers. Numerical simulations are used to demonstrate the performance of the considered solvers.

The notation $\mathcal{H}^n$ stands for the linear space of symmetric matrices of size n over the field of real numbers. For any $X, Y \in \mathcal{H}^n$, we write $X > Y$ or $X \geq Y$ if $X - Y$ is positive definite or $X - Y$ is positive semidefinite. The linear space $\mathcal{H}^n$ is a Hilbert space with the Frobenius inner product $< X, Y >= trace(XY)$.

2 The Standard LMI Approach

Thus, following the classical linear quadratic theory we know that the following optimization problem is associated with (1), for example see Dragan et al. [11]:

$$\begin{aligned}
&\max \textstyle\sum_{i=1}^{N} \ \langle I, X(i) \rangle \\
&\text{subject to } i = 1, \ldots, N \\
&\begin{pmatrix} \begin{array}{c} -X(i) + Q(i) \\ + \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{X}) A_l(i) \end{array} & \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{X}) B_l(i) + L(i) \\ \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{X}) A_l(i) + L(i)^T & R(i) + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{X}) B_l(i) \end{pmatrix} \geq 0 \qquad (2) \\
&R(i) + \textstyle\sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{X}) B_l(i) \ > \ 0\,, \\
&X(i) = X(i)^T\,.
\end{aligned}$$

The case $r = 0$ is analyzed by Costa and Marques in [7]. The relation between the maximal solution to the set equations (1) ($r = 0$) and the solution of the optimization problem (2) ($r = 0$) is given in Theorem 2 in the same paper.

The relation between the maximal solution to (1) ($r > 0$) and the optimal solution to the related semidefinite programming problem is given in the following theorem:

THEOREM 1. Assume that $(\mathbf{A}, \mathbf{B})$ is stabilizable and there exists a solution to the inequalities $\mathcal{P}(i, \mathbf{X}) - X(i) \geq 0$ for $i = 1, \ldots, N$. Then there exists a maximal solution $\mathbf{X}^+$ of (1) if and only if there exists a solution $\hat{\mathbf{X}}$ for the above convex programming problem (2) with $\mathbf{X}^+ \equiv \hat{\mathbf{X}}$.

Thus, the feasibility of the optimization problem (2) is necessary and sufficient for the solvability of the system (1). In addition, if $\begin{pmatrix} Q(i) & L(i) \\ L^T(i) & R(i) \end{pmatrix} \geq 0$ and $R(i) > 0$, then the maximal solution is positive semidefinite and if all matrices $\begin{pmatrix} Q(i) & L(i) \\ L^T(i) & R(i) \end{pmatrix}$, $R(i)$ are positive definite then the maximal solution is positive definite.

We define the real matrices A_k, B_k such that $A_k = (A_k(1), \ldots, A_k(N))$, $B_k = (B_k(1), \ldots, B_k(N))$ where $A_k(i)$ is an $n \times n$ matrix and $B_k(i)$ is an $n \times m$ matrix $k = 0, 1, \ldots, r$ and $i = 1, \ldots, N$, and $\mathbf{A} = (A_0, A_1, A_2, \ldots, A_r)$ and $\mathbf{B} = (B_0, B_1, B_2, \ldots, B_r)$. We use the following definition [11].

DEFINITION 1. We say that the couple $(\mathbf{A},\mathbf{B})$ is stabilizable if for some $\mathbf{F} = (F(1),\ldots,F(N))$ the closed loop system:

$$x(t+1) = [A_0(\eta_t) + B_0(\eta_t)F(\eta_t) + \sum_{k=0}^{r} w_k(t)(A_k(\eta_t) + B_k(\eta_t)F(\eta_t))]x(t)$$

is exponentially stable in mean square (ESMS).

The matrix $\mathbf{F}$ involved in the above definition is called stabilizing feedback gain.

3 The Modified LMI Approaches

In this paper we investigate the numerical solvability of the semidefinite programming problem (2) for different type of matrix $R(i), i = 1,\ldots,N$. However, the numerical experiments for finding the maximal solution of (1) show that the LMI method (2) is slowly working for different types of matrices $R(i)$ in the case $r = 0$ [16]. Here we introduce a new modification to accelerate the LMI method for solving the optimization problem (2) in special cases of weighting matrices. Our new modification will accelerate the considered optimization problems over LMIs. In many applications of control system theory the following fact is used (see [1]).

It is well known that the control matrix B has full column rank in different considerations of the control theory and then there exists a symmetric matrix Y such that $R + B^T Y B$ is invertible. This conclusion is applied in our consideration. The introduced equation (1) appeared in the portfolio optimization [20] where the matrices $R(i), i = 1,\ldots,N$ are zero. In addition, the matrices $B_0(i),\ldots,B_r(i)$ has the following property: the matrix $\begin{pmatrix} B_0(i) \\ \vdots \\ B_r(i) \end{pmatrix}$ has the full column rank (for instance, see the stochastic models with their realization in the portfolio optimization described in [20]). Thus, we choose symmetric matrices $Z(i), i = 1,\ldots,N$ such that the matrices $R(i)+\sum_{k=0}^r B_k(i)^T\mathcal{E}_i(\mathbf{Z})B_k(i)$ are positive definite. The standard approach is to choose the new matrices $Z(i)$ of the form $Z(i) = \alpha I$, for all values of i. In this case the new matrices $Z(i)$ can be considered as the approximate points to the $X(i)$. We take $\alpha = 0.005$ in the numerical simulations in the paper. Next step is to change the variables $X(i)$. We introduce new variables $Y(i)$ with substitution $X(i) = Z(i) + Y(i), i = 1,\ldots,N$.

Then, we put $\mathbf{X} = \mathbf{Z} + \mathbf{Y}$. From (1) it is obtained the following set of Riccati equations regarding to $\mathbf{Y} = (Y(1),\ldots,Y(N))$:

$$\begin{aligned} Y(i) \;\; = \;\; & \mathcal{T}(i,\mathbf{Y}) := \textstyle\sum_{k=0}^r A_k(i)^T\mathcal{E}_i(\mathbf{Y})A_k(i) + \tilde{Q}(i) \\ & -(\textstyle\sum_{k=0}^r A_k(i)^T\mathcal{E}_i(\mathbf{Y})B_k(i) + \tilde{L}(i)) \\ & \times \left(\tilde{R}(i) + \textstyle\sum_{k=0}^r B_k(i)^T\mathcal{E}_i(\mathbf{Y})B_k(i)\right)^{-1} \\ & \times (\textstyle\sum_{k=0}^r B_k(i)^T\mathcal{E}_i(\mathbf{Y})A_k(i) + \tilde{L}(i)^T) \quad i = 1,\ldots,N \end{aligned} , \qquad (3)$$

where

$$\begin{cases} \tilde{Q}(i) = \sum_{k=0}^{r} A_k(i)^T \mathcal{E}_i(\mathbf{Z}) A_k(i) + Q(i) - Z(i), \\ \tilde{L}(i) = L(i) + \sum_{k=0}^{r} A_k(i)^T \mathcal{E}_i(\mathbf{Z}) B_k(i)\,. \end{cases}$$

Further on, the new optimization problem over the LMIs condition related to (3) is derived:

$$\begin{aligned} & \max \sum_{i=1}^{N} \ \langle I, Y(i) \rangle \\ & \text{subject to } i = 1, \ldots, N \\ & \begin{pmatrix} \begin{array}{l} -Y(i) + \tilde{Q}(i) \\ + \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Y}) A_l(i) \end{array} & \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Y}) B_l(i) + \tilde{L}(i) \\ \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Y}) A_l(i) + \tilde{L}(i)^T & \tilde{R}(i) + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Y}) B_l(i) \end{pmatrix} \geq 0 \qquad (4) \\ & \tilde{R}(i) + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Y}) B_l(i) \ > \ 0 \\ & Y(i) = Y(i)^T\,. \end{aligned}$$

Thus, we could use two semidefinite programming problems for solving the introduced DTGRE (1). In the first problem (2) the cost matrices $R(1), \ldots, R(N)$ may be indefinite, negative definite or singular. However, in the second semidefinite programming problem (4) we choose the symmetric matrices $Z(i), i = 1, \ldots, N$ such that the corresponding matrices $\tilde{R}(1), \ldots, \tilde{R}(N)$ are positive definite. Thus, in order that $\tilde{R}(i) + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Y}) B_l(i) \ > \ 0$ it is enough the inequality $Y(i) > 0$ holds for all $i = 1, \ldots, N$. Then, we construct the next semidefinite programming problem

$$\begin{aligned} & \max \sum_{i=1}^{N} \ \langle I, Y(i) \rangle \\ & \text{subject to } i = 1, \ldots, N \\ & \begin{pmatrix} \begin{array}{l} -Y(i) + \tilde{Q}(i) \\ + \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Y}) A_l(i) \end{array} & \sum_{l=0}^{r} A_l(i)^T \mathcal{E}_i(\mathbf{Y}) B_l(i) + \tilde{L}(i) \\ \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Y}) A_l(i) + \tilde{L}(i)^T & \tilde{R}(i) + \sum_{l=0}^{r} B_l(i)^T \mathcal{E}_i(\mathbf{Y}) B_l(i) \end{pmatrix} \geq 0 \qquad (5) \\ & Y(i) \ > \ 0\,. \end{aligned}$$

Moreover, if the last semidefinite programming problem has no optimal solution (which has to be positive definite) then if the set of DTGRE (1) has the maximal solution, it is not a positive definite one.

We will compare numerically these two semidefinite programming problems (2) and (5) with numerical simulations. Before we do this we will introduce our next modification to (1) which leads us to a new semidefinite programming problem.

Let us consider the given set of nonlinear equations (1):

$$X(i) = \mathcal{P}(i, \mathbf{X}), \quad i = 1, \ldots, N\,.$$

Our new idea is to make the following change of unknowns based on the developments in [13]:

$$\mathbf{Y} = (Y(1), \ldots, Y(N))\,, \quad \text{where } \; Y(i) = \mathcal{E}_i(\mathbf{X}) \; \text{ for } i = 1, \ldots, N\,.$$

Then, we have

$$X(i) = \sum_{j=1}^{N} \mu_{ij}\, Y(j)\,, \quad \text{where } \mathcal{M} = \big(\mu_{ij}\big) = \Lambda^{-1}, \quad \Lambda = (\lambda_{ij})\;.$$

We transform the set of nonlinear equations (1) consequently and using notations

$$\begin{cases} \delta_{ip} = \sum_{s\neq i} \lambda_{is}\mu_{sp}\,, \quad i,p = 1,\ldots,N\,; \\ \tilde{A}_k(i) = \sqrt{\frac{\lambda_{ii}}{1-\delta_{ii}}}\, A_k(i)\,, \qquad \tilde{Q}(i) = \frac{\lambda_{ii}}{1-\delta_{ii}}\, Q(i)\,, \\ \tilde{L}(i) = \sqrt{\frac{\lambda_{ii}}{1-\delta_{ii}}}\, L(i)\,, \quad i = 1,\ldots,N,\; k = 0,\ldots,r\,; \\ \mathcal{G}_i(\mathbf{Y}) = \sum_{p\neq i} \gamma_{ip}\, Y(p)\,, \quad \gamma_{ii} = 0,\; \gamma_{ip} = \frac{\delta_{ip}}{1-\delta_{ii}}, \;\text{ for }\; i \neq p\,. \end{cases}$$

The new set of discrete time algebraic Riccati equations is obtained:

$$\begin{aligned} Y(i) \;\; = \;\; & \textstyle\sum_{l=0}^{r} \tilde{A}_l(i)^T Y(i)\, \tilde{A}_l(i) + \tilde{Q}(i) + \mathcal{G}_i(\mathbf{Y}) \\ & -(\textstyle\sum_{l=0}^{r} \tilde{A}_l(i)^T Y(i) B_l(i) + \tilde{L}(i))\, \big(R(i) + \textstyle\sum_{l=0}^{r} B_l(i)^T Y(i) B_l(i)\big)^{-1} \\ & \times (\textstyle\sum_{l=0}^{r} B_l(i)^T Y(i)\, \tilde{A}_l(i) + \tilde{L}(i)^T)\,, \quad i = 1,\ldots,N\,. \end{aligned} \tag{6}$$

If $(\gamma_{ip})_1^N \geq 0$ and $\frac{\lambda_{ii}}{1-\delta_{ii}}, i = 1,\ldots,N$ are positive numbers the set of nonlinear equations (6) is equivalent to the system (1). Thus the map $\mathcal{G}_i(\mathbf{Y})$ is a positive one, i.e. if $\mathbf{Y} \geq 0$ then $\mathcal{G}_i(\mathbf{Y}) \geq 0$ for $i = 1,\ldots,N$.

We are ready to construct the new semidefinite programming problem related to the above set of Riccati equations (6):

$$\max \textstyle\sum_{i=1}^{N} \; \langle I, Y(i) \rangle$$

$$\text{subject } \; \text{to } i = 1,\ldots,N$$

$$\begin{pmatrix} \begin{array}{l} -Y(i) + \tilde{Q}(i) + \mathcal{G}_i(\mathbf{Y}) \\ + \sum_{l=0}^{r} \tilde{A}_l(i)^T Y(i) \tilde{A}_l(i) \end{array} & \sum_{l=0}^{r} \tilde{A}_l(i)^T Y(i) B_l(i) + \tilde{L}(i) \\ & \\ \sum_{l=0}^{r} B_l(i)^T Y(i) \tilde{A}_l(i) + \tilde{L}(i)^T & R(i) + \sum_{l=0}^{r} B_l(i)^T Y(i) B_l(i) \end{pmatrix} \geq 0 \tag{7}$$

$$R(i) + \textstyle\sum_{l=0}^{r} B_l(i)^T Y(i) B_l(i) \; > \; 0$$

$$Y(i) = Y(i)^T\,.$$

Thus, we have obtained three equivalent semidefinite programming problems (2), (5) and (7). The optimal solution of each of them lead us to the maximal solution to the set of discrete-time generalized Riccati equations (1).

4 Numerical Experiments

We investigate the numerical behavior of the LMI approach applied to the described optimization problems LMI: (2), Im LMI: (5) and LMI(Y): (7) for finding the maximal solution to the set of discrete-time generalized Riccati equations (1). We will carry out some experiments for this purpose.

Our experiments are executed in MATLAB on a 1.7GHz PENTINUM computer. In order to execute our experiments the suitable MATLAB procedures are used. The MATLAB function mincx is applied with the relative accuracy equals to $1 \times e^{-10}$ for solving the corresponding optimization problem.

We consider a family of examples in case $N = 3, r = 2, n = 8, 9, \ldots, 15$, where the coefficient real matrices are given as follows: $A_0(i), A_1(i), A_2(i), B_0(i), B_1(i), B_2(i), L(i)$, $i = 1, 2, 3$ were constructed using the MATLAB notations:

$$\begin{array}{lll} A_0(1) = randn(n,n)/8; & A_0(2) = randn(n,n)/8; & A_0(3) = randn(n,n)/8; \\ A_1(1) = randn(n,n)/8; & A_1(2) = randn(n,n)/8; & A_1(3) = randn(n,n)/8; \\ A_2(1) = randn(n,n)/8; & A_2(2) = randn(n,n)/8; & A_2(3) = randn(n,n)/8; \end{array}$$

$$\begin{array}{ll} B_0(1) = 2 * full(sprand(n, m2, 0.3)); & B_0(2) = 2 * full(sprand(n, m2, 0.3)); \\ B_0(3) = 2 * full(sprand(n, m2, 0.3)); & \\ B_1(1) = 2 * full(sprand(n, m2, 0.3)); & B_1(2) = 2 * full(sprand(n, m2, 0.3)); \\ B_1(3) = 2 * full(sprand(n, m2, 0.3)); & \\ B_2(1) = 2 * full(sprand(n, m2, 0.3)); & B_2(2) = 2 * full(sprand(n, m2, 0.3)); \\ B_2(3) = 2 * full(sprand(n, m2, 0.3)); & \end{array}$$

$$L(1) = L(2) = L(3) = zeros(n, m2);$$

$$\begin{array}{l} Q(1) = \mathrm{diag}[0, 1, \ldots, 1]\,, \quad Q(2) = \mathrm{diag}[1, 0, 1, \ldots, 1]\,, \\ Q(3) = \mathrm{diag}[1, 1, 0, 1, \ldots, 1]\,. \end{array}$$

In our definitions the functions randn(p,k) and sprand(q,m,0.3) return a p-by-k matrix of pseudorandom scalar values and a q-by-m sparse matrix respectively (for more information see the MATLAB description). The following transition probability matrix

$$(\lambda_{ij}) = \begin{pmatrix} 0.67 & 0.17 & 0.16 \\ 0.30 & 0.47 & 0.23 \\ 0.26 & 0.10 & 0.64 \end{pmatrix}$$

is applied for all examples.

In addition, we construct the following five tests of examples for different matrices $R(1), R(2)$ and $R(3)$:

Test 1: $m2 = n, \quad R(1) = R(2) = R(3) = zeros(n, n)$.

Test 2: $m2 = n$.

$$\begin{array}{l} R(1) = \mathrm{diag}[-0.002, 0.25, \ldots, 0.25]\,, \quad R(2) = \mathrm{diag}[-0.001, 0.75, \ldots, 0.75]\,, \\ R(3) = \mathrm{diag}[-0.0025, 0.5, \ldots, 0.5]\,, \end{array}$$

Test 3: $m2 = 3, \quad R(1) = R(2) = R(3) = zeros(3, 3)$.

Test 4:

$$\begin{aligned}
&m2 = 3, \\
&R(1) = \mathrm{diag}[0.26,\ -0.0025,\ 0.45]\,, \quad R(2) = \mathrm{diag}[0.15,\ -0.0012,\ 1.05]\,, \\
&R(3) = \mathrm{diag}[1.25,\ -0.005,\ 0.012]\,,
\end{aligned}$$

Test 5:

$$\begin{aligned}
&m2 = 3, \\
&R(1) = -\mathrm{diag}[0.0026,\ 0.0025,\ 0.0045]\,, \quad R(2) = -\mathrm{diag}[0.0015,\ 0.0012,\ 0.0105]\,, \\
&R(3) = -\mathrm{diag}[0.0125,\ 0.005,\ 0.0012]\,.
\end{aligned}$$

For our purpose we have executed hundred examples of each value of n for all tests. All tables report the maximal number of iterations "m It" and average number of iterations "av It" of each size for all examples needed for achieving the relative accuracy. Results from experiments are given in table 1 with $n = 10$ and $n = 5$ for all tests.

5 Conclusions

We have studied three optimization problems for finding the maximal solution to a set of discrete-time generalized Riccati equations (1).

We have investigated two numerical procedures for solving the new optimization problems (5) and (7). We show how to apply problem (5) in the application for fast solution a portfolio optimization problem because in this problem the weighting matrices $R(i), i = 1\ldots, N$ are zero matrices (see [20]). In addition, we extend the approach based on the substitution $Y(i) = \mathcal{E}_i(\mathbf{X})$ introduced in [13]. This extension (7) is applied to the considered set of generalized Riccati equations and numerically compared. Numerical tests show the efficiency of new optimization problem (7).

We have made numerical experiments for computing this solution and we have compared the numerical results. Our numerical experiments confirm the effectiveness of the proposed new transformations which lead us to the equivalent semidefinite programming problems. We have compared the results from the experiments in regard of the number of iterations and CPU time for executing the above optimization problems for $n = 15$. The solution of the optimization problems achieve the same accuracy for different number of iterations. The executed examples have demonstrated that the LMI problem performance for different optimization problems require very close average numbers of iterations (see the columns "av It" for all tests). However, the CPU time is different for the investigated optimization problems. The new optimization problems Im LMI: (5) and LMI(Y): (7) based on the new transformations are faster than the standard optimization problem LMI: (2). The LMI approach applied to the semidefinite programming problem (5) is approximately twice faster than the (2) while the LMI approach to the semidefinite programming problem (7) is six times faster than the standard optimization problem (2). This conclusion descends from the numerical simulations. The proof will be a subject of the future research.

Table 1: Results from 100 runs for each value of n.

	LMI: (2)		Im LMI: (5)		LMI(Y): (7)	
n	m It	av It	m It	av It	m It	av It
Test 1						
10	37	29.8	26	25.0	44	32.4
15	33	26.8	29	27.0	31	28.8
CPU time for executing for 10 runs						
15	980 s		653 s		201 s	
Test 2						
10	26	24.2	26	24.6	40	25.9
15	33	26.8	29	27.0	31	28.8
CPU time for executing for 10 runs						
15	980 s		653 s		201 s	
Test 3						
10	42	32.4	26	25.2	29	28.0
15	50	38.8	30	28.8	36	32.0
CPU time for executing for 10 runs						
15	1257 s		574 s		159 s	
Test 4						
10	52	31.2	26	25.2	30	27.2
15	45	33.8	30	28.4	34	31.2
CPU time for executing for 10 runs						
15	1258 s		558 s		166 s	
Test 5						
10	59	32.2	28	25.8	29	27.8
15	49	39.2	30	29.0	33	30.8
CPU time for executing for 10 runs						
15	1372 s		569 s		158 s	

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