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On A Simple Point Of View For Refining Bounds Of The Logarithmic Mean*

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Abstract

In this paper, we show how many bounds of the logarithmic mean, already stated in the literature, can be obtained in a fast and nice way when simple operations between means are conveniently introduced.

1 Introduction

Throughout this paper, we understand by mean a map m between two positive real numbers such that

$$\forall a, b > 0 \qquad \min(a, b) \leq m(a, b) \leq \max(a, b).$$

From this, it is clear that every mean is with positive values and reflexive that is $m(a, a) = a$ for each $a > 0$. The maps $(a, b) \longmapsto \min(a, b)$ and $(a, b) \longmapsto \max(a, b)$ are (trivial) means which will be denoted by min and max, respectively. Standard examples of means are following (see [1]):

- Arithmetic Mean: $A := A(a, b) = \frac{a+b}{2}$;
- Geometric Mean: $G := G(a, b) = \sqrt{ab}$;
- Harmonic Mean: $H := H(a, b) = \frac{2ab}{a+b}$;
- Contra-harmonic Mean: $C := C(a, b) = \frac{a^2+b^2}{a+b}$;
- Logarithmic Mean: $L := L(a, b) = \frac{b-a}{\ln b - \ln a}$, $L(a, a) = a$;
- Identric Mean $I := I(a, b) = e^{-1}\big(b^b/a^a\big)^{1/(b-a)}$, $I(a, a) = a$.

The set of all means can be equipped with a partial ordering, called point-wise order, defined by: $m_1 \leq m_2$ if and only if $m_1(a, b) \leq m_2(a, b)$ for every $a, b > 0$. We write $m_1 < m_2$ if and only if $m_1(a, b) < m_2(a, b)$ for all $a, b > 0$ with $a \neq b$.

For a given mean m, we set $m^*(a, b) = \Big(m\big(a^{-1}, b^{-1}\big)\Big)^{-1}$, and it is easy to see that m^* is also a mean, called the dual mean of m. Every mean m satisfies $m^{**} := (m^*)^* = m$ and, if m_1 and m_2 are two means such that $m_1 < m_2$ then $m_1^* > m_2^*$. It is easy to see that $\min^* = \max$ and $\max^* = \min$. Further, the arithmetic and harmonic means

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are mutually dual (i.e. $A^*=H,\ H^*=A$) and the geometric mean is self-dual (i.e. $G^*=G$).

The following inequalities are well known in the literature, see [2].

$$\min < C^* < H < I^* < L^* < G < L < I < A < C < \max.$$

Let us denote by $\mathcal{M}$ the convex set of all means with two arguments. A given map $\Phi:\mathcal{M}\longrightarrow\mathcal{M}$ is called point-wise convex (in short p-convex) if the following mean-inequality,

$$\Phi\Big((1-t)m_1+tm_2\Big)\le(1-t)\Phi(m_1)+t\Phi(m_2),$$

with respect to the above point-wise ordering, holds true for every real number $t\in[0,1]$ and all means $m_1,m_2\in\mathcal{M}$. We say that Φ is p-concave if the above inequality is reversed. The p-increase and p-decrease monotonicity of Φ can be stated in a similar manner.

EXAMPLE 1. Let us consider the map $m\longmapsto m^*$, where m^* is the dual of m. Clearly, this map is p-increasing. Further, it is well known [4] that it is p-convex, that is, the mean-inequality

$$\Big((1-t)m_1+tm_2\Big)^*\le(1-t)m_1^*+tm_2^*$$

holds for all $t\in[0,1]$ and all means m_1 and m_2. Furthermore, this mean-inequality is strict whenever $t\in(0,1)$ and $m_1\ne m_2$.

In the section below we will see a lot of p-convex (resp. p-concave,...) mean-maps. Other examples, with some extensions, can be found in [4].

2 Some Operations for Means

As already pointed before, this section is focused to define some operations for means and study their properties. We start with the following simple definition.

DEFINITION 1. Let m_1 and m_2 be two means. For $a,b>0$ define

$$m_1\odot m_2(a,b)=m_1\big(\sqrt{a},\sqrt{b}\big)m_2\big(\sqrt{a},\sqrt{b}\big),\tag{1}$$

which we call the mean-product of m_1 and m_2.

For $\lambda\in[0,1]$, we set

$$m_1\oplus_\lambda m_2:=(1-\lambda)m_1+\lambda m_2.$$

If $\lambda=1/2$, we write $m_1\oplus m_2$ instead of $m_1\oplus_{1/2}m_2$ for the sake of simplicity.

The elementary properties of operations $(m_1,m_2)\longmapsto m_1\odot m_2$ and $(m_1,m_2)\longmapsto m_1\oplus_\lambda m_2$ are summarized in the next result.

PROPOSITION 1. With the above the following assertions are met:

(i) For all means m_1,m_2 and $\lambda\in[0,1]$, $m_1\oplus_\lambda m_2$ and $m_1\odot m_2$ are means,

(ii) $m_1 \oplus_\lambda m_2 = m_2 \oplus_{1-\lambda} m_1$ and $m_1 \odot m_2 = m_2 \odot m_1$, (commutativity axiom),

(iii) $\Big(m_1 < m_3$ and $m_2 < m_4\Big)$ imply $\Big(m_1 \oplus_\lambda m_2 < m_3 \oplus_\lambda m_4$ and $m_1 \odot m_2 < m_3 \odot m_4\Big)$, (monotonicity or compatibility axiom),

(iv) $\Big(m_1 \oplus_\lambda m_2\Big)^* \leq m_1^* \oplus_\lambda m_2^*$ ($*$-sub-additivity axiom for $\oplus_\lambda$),

(v) $\Big(m_1 \odot m_2\Big)^* = m_1^* \odot m_2^*$ (self-duality axiom for $\odot$),

(vi) $m \odot \big(m_1 \oplus_\lambda m_2\big) = \big(m \odot m_1\big) \oplus_\lambda \big(m \odot m_2\big)$ (distributivity of $\odot$ for $\oplus_\lambda$).

The proof is straightforward and does not present any difficulties. We left the detail for the reader.

Before stating some concrete examples illustrating the above, we state the following definition which is naturally derived from the above one.

DEFINITION 2. For all mean m we define

$$m^{\odot 2}\big(a,b\big) = \Big(m\big(\sqrt{a},\sqrt{b}\big)\Big)^2, \tag{2}$$

$$m^{\odot 1/2}\big(a,b\big) = \Big(m\big(a^2,b^2\big)\Big)^{1/2}, \tag{3}$$

which we call the mean-square and the mean-root of m, respectively.

Clearly, $m^{\odot 2}$ and $m^{\odot 1/2}$ are means whenever m is a mean. It is easy to verify that the operations $m \longmapsto m^{\odot 2}$ and $m \longmapsto m^{\odot 1/2}$ are mutually reverse in the sense that, $m_1^{\odot 2} = m_2$ if and only if $m_1 = m_2^{\odot 1/2}$, which justifies the above chosen terminology. Further, combining the two above definitions with Proposition 2 we immediately obtain the following result.

PROPOSITION 2. With the above we have

(i) $m_1 < m_2$ implies $m_1^{\odot 2} < m_2^{\odot 2}$ and $m^{\odot 1/2} < m_2^{\odot 1/2}$ (monotonicity axiom),

(ii) $\big(m^{\odot 2}\big)^* = \big(m^*\big)^{\odot 2}$ and $\big(m^{\odot 1/2}\big)^* = \big(m^*\big)^{\odot 1/2}$ (self-duality axiom).

The following result gives another justification for the above chosen terminology.

PROPOSITION 3. The mean-map $m \longmapsto m^{\odot 2}$ is p-convex and $m \longmapsto m^{\odot 1/2}$ is a p-concave one.

PROOF. Let $\lambda \in [0,1]$ be a real number and m_1, m_2 be two means. By definition we have

$$\Big((1-\lambda)m_1 + \lambda m_2\Big)^{\odot 2}(a,b) = \Big((1-\lambda)m_1\big(\sqrt{a}.\sqrt{b}\big) + \lambda m_2\big(\sqrt{a},\sqrt{b}\big)\Big)^2.$$

By the convexity of the real function $t \longmapsto t^2$ we deduce the desired result. The p-concavity of $m \longmapsto m^{\odot 1/2}$ can be obtained in an analogous way.

Now we are in position to state the following examples.

EXAMPLE 2. It is easy to verify that

1) $\min \oplus \max = A$, $\min \odot \max = G$, $\min^{\odot 2} = \min^{\odot 1/2} = \min$, $\max^{\odot 2} = \max^{\odot 1/2} = \max$.

2) $A \odot L = L$, $A \odot C = A$, $A \odot G = \Big(\frac{AG+G^2}{2}\Big)^{1/2}$.

3) For all mean m one has $m \odot m^* = G$.

EXAMPLE 3. Elementary computations lead to

1) $A^{\odot 2} = \dfrac{A+G}{2} := A \oplus G,\ G^{\odot 2} = G,\ H^{\odot 2} = \dfrac{2G^2}{A+G},\ C^{\odot 2} = \dfrac{2A^2}{A+G} = \dfrac{A^2}{A \oplus G}$

2) $L^{\odot 2} = \dfrac{2L^2}{A+G} = \dfrac{L^2}{A \oplus G},\ I^{\odot 2} = \dfrac{1}{e^2} G \exp \dfrac{A+G}{L}$.

We end this section by stating another result which will be needed later for simplifying some hard computations.

PROPOSITION 4. Let m_1, m_2 be two means and $\lambda \in [0,1]$ be a real number. The following equalities hold true

$$\Big(m_1^{1-\lambda} m_2^{\lambda}\Big)^{\odot 2} = \big(m_1^{\odot 2}\big)^{1-\lambda} \big(m_2^{\odot 2}\big)^{\lambda} \tag{4}$$

$$\Big(m_1^{1-\lambda} m_2^{\lambda}\Big)^{\odot 1/2} = \big(m_1^{\odot 1/2}\big)^{1-\lambda} \big(m_2^{\odot 1/2}\big)^{\lambda}. \tag{5}$$

The proof is very simple and we omit here the details.

3 Applications for Mean-Inequalities

In the present section, we investigate some applications of the above theoretical study for obtaining a lot of mean-inequalities involving the logarithmic mean L. We notice that some of these obtained inequalities are well known in the literature and some other ones appear to us to be new. Our present approach stems its importance in the strange fact that the above introduced elementary operations are good tool for obtaining mean-inequalities in a fast and simple ways while certain of them have been shown by different methods in a more or less long way. Let us observe this latter situation in the next examples.

EXAMPLE 4. Starting from $G < L < A$ we deduce, with Proposition 2 and Example 2,

$$G^{\odot 2} = G < L^{\odot 2} = \frac{2L^2}{A+G} < A^{\odot 2} = \frac{A+G}{2},$$

which, after all reduction with Example 2, yields the known inequalities, [7]

$$G < \Big(\frac{AG+G^2}{2}\Big)^{1/2} < L < \frac{A+G}{2} < A. \tag{6}$$

EXAMPLE 5. We can refine the bounds of L given in (6) by continuing the same procedure. For instance, starting from $L < \frac{A+G}{2} := A \oplus G$ we deduce, with Proposition 2 and Example 2,

$$A \odot L = L < A \odot (A \oplus G) = A^{\odot 2} \oplus (A \odot G),$$

and again by Example 2 and Example 2 we deduce

$$L < \frac{1}{2}\Big(\frac{A+G}{2}\Big) + \frac{1}{2}\Big(\frac{AG+G^2}{2}\Big)^{1/2}.$$

EXAMPLE 6. The following inequality $L < (1/3)A + (2/3)G$ is well-known (see [3]) and it refines $L < A \oplus G$. Using our conventional writing $L < A \oplus_{2/3} G$ we obtain by Proposition 2 and Example 2

$$A \odot L = L < A \odot \big(A \oplus_{2/3} G\big) = A^{\odot 2} \oplus_{2/3} (A \odot G)$$

or again

$$L < \Big(\frac{A+G}{2}\Big) \oplus_{2/3} \Big(\frac{AG+G^2}{2}\Big)^{1/2} = \frac{1}{3}\Big(\frac{A+G}{2}\Big) + \frac{2}{3}\Big(\frac{AG+G^2}{2}\Big)^{1/2},$$

which gives a refinement of the initial inequality already obtained in [5, 7].

EXAMPLE 7. Starting from $G < I < A$ we deduce, with Proposition 2 and Example 2,

$$G < \frac{1}{e^2} G \exp \frac{A+G}{L} < \frac{A+G}{2}. \tag{7}$$

The left-hand side of (7) gives the third inequality of (6) (i.e. $L < (A+G)/2$), while the right-hand side of (7) yields, after all reduction,

$$L > \frac{A+G}{2 - \ln\, 2 + \ln \frac{A+G}{G}}. \tag{8}$$

It is not hard to verify that (8) refines $L > \Big(\frac{AG+G^2}{2}\Big)^{1/2}$.

EXAMPLE 8. Starting from $G < L < I$ we deduce, by Proposition 2 with Example 2,

$$G < L^{\odot 2} = \frac{2L^2}{A+G} < I^{\odot 2} = \frac{1}{e^2} G \exp \frac{A+G}{L}. \tag{9}$$

The left-hand side of (9) does not gives new information while the right-hand side yields, after simple manipulation, the following implicit mean-inequality for L

$$L \ln \frac{eL}{\big(\frac{AG+G^2}{2}\big)^{1/2}} < \frac{A+G}{2}, \tag{10}$$

EXAMPLE 9. Starting from the known inequality $A^{1/3}G^{2/3} < L$, [3], we deduce by Proposition 2

$$\big(A^{\odot 2}\big)^{1/3} G^{2/3} < L^{\odot 2}.$$

This, with Example 2 and a simple reduction, yields

$$L > \Big(\frac{A+G}{2}\Big)^{2/3} G^{1/3}, \tag{11}$$

which is a refinement of $A^{1/3}G^{2/3} < L$ already differently obtained in [7]. If we repeat the same procedure for (11) we obtain (by analogous arguments as in the above)

$$L > G^{1/6}\Big(\frac{A+G}{2}\Big)^{1/2} \bigg(\Big(\frac{A+G}{2}\Big)^{\odot 2}\bigg)^{1/3}.$$

We left to the reader the task for formulating other known mean-inequalities in the aim to obtain related refinements of L via our above approach.

The reader can perhaps remark the following: why the above introduced operations are tool for bounding the logarithmic mean L, but not the identric mean I for example. We can understand this situation after pointing the following.

REMARK 1. We notice that

$$m_1 \odot m_2 = \mathcal{R}\big(m_1, m_2, G\big),$$

where the notation of the right-hand side refers to the resultant mean-map introduced by the author in [4]. The logarithm mean L is (A, G)-stabilizable, that is, $A \odot L = L = \mathcal{R}(A, L, G)$. Following [6], the fact that $A \odot L = L$ means that L is A-decomposable.

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