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Existence Of Positive Periodic Solutions For A Nonlinear Neutral Differential Equation With Variable Delay*

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Abstract

In this paper, we study the existence of positive periodic solutions of the nonlinear neutral differential equation with variable delay

$$\frac{d}{dt}\left(x\left(t\right)-g\left(t,x\left(t-\tau\left(t\right)\right)\right)\right)=r\left(t\right)x\left(t\right)-f\left(t,x\left(t-\tau\left(t\right)\right)\right).$$

The main tool employed here is the Krasnoselskii's hybrid fixed point theorem dealing with a sum of two mappings, one is a contraction and the other is compact. The results obtained here generalize the work of Raffoul [17].

1 Introduction

Due to their importance in numerous applications, for example, physics, population dynamics, industrial robotics, and other areas, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential equations, see the references in this article and the references therein.

In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of delay differential equations. Motivated by the papers [2, 6, 8, 12, 14, 17, 19] and the references therein, we concentrate on the existence of positive periodic solutions for the nonlinear neutral differential equation with variable delay

$$\frac{d}{dt}\left(x\left(t\right)-g\left(t,x\left(t-\tau\left(t\right)\right)\right)\right)=r\left(t\right)x\left(t\right)-f\left(t,x\left(t-\tau\left(t\right)\right)\right), \tag{1}$$

where r is a continuous real-valued function. The functions $g,f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous in their respective arguments. To reach our desired end we have to transform (1) into an integral equation and then use Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions. The obtained integral equation splits in the sum of two mappings, one is a contraction and the other is compact. In the case $g\left(t,x\right)=cx$, Raffoul in [17] shows that (1) has a positive periodic solutions by using Krasnoselskii's fixed point theorem.

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The organization of this paper is as follows. In Section 2, we present the inversion of (1) and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [18]. In Section 3, we present our main results on existence of positive periodic solutions of (1). The results presented in this paper generalize the main results in [17].

2 Preliminaries

For $T > 0$, let P_T be the set of all continuous scalar functions x, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

Since we are searching for the existence of periodic solutions for equation (1), it is natural to assume that

$$r(t+T) = r(t), \ \tau(t+T) = \tau(t), \tag{2}$$

with τ being scalar function, continuous, and $\tau(t) \geq \tau^* > 0$. Also, we assume

$$\int_0^T r(s)\,ds > 0. \tag{3}$$

We also assume that the functions $g(t,x)$ and $f(t,x)$ are periodic in t with period T, that is,

$$g(t+T,x) = g(t,x), \ f(t+T,x) = f(t,x). \tag{4}$$

The following lemma is fundamental to our results.

LEMMA 2.1. Suppose (2)-(4) hold. If $x \in P_T$, then x is a solution of equation (1) if and only if

$$\begin{aligned} x(t) &= g(t, x(t-\tau(t))) \\ &\quad + \int_t^{t+T} G(t,s)\left[f(s, x(s-\tau(s))) - r(s)\, g(s, x(s-\tau(s)))\right] ds, \end{aligned} \tag{5}$$

where

$$G(t,s) = \frac{e^{\int_s^t r(u)du}}{1-e^{-\int_0^T r(u)du}}. \tag{6}$$

PROOF. Let $x \in P_T$ be a solution of (1). First we write this equation as

$$\begin{aligned} \frac{d}{dt}\left(x(t) - g(t, x(t-\tau(t)))\right) &= r(t)\left(x(t) - g(t, x(t-\tau(t)))\right) \\ &\quad - f(t, x(t-\tau(t))) + r(t)\, g(t, x(t-\tau(t))). \end{aligned}$$

Multiply both sides of the above equation by $e^{-\int_0^t r(u)du}$ and then integrate from t to $t+T$ to obtain

$$\begin{aligned}&\int_t^{t+T} \frac{d}{ds}\left[\left(x(s)-g(s,x(s-\tau(s)))\right)e^{-\int_0^s r(u)du}\right]ds\\ &=\int_t^{t+T}\left[-f(s,x(s-\tau(s)))+r(s)\,g(s,x(s-\tau(s)))\right]e^{-\int_0^s r(u)du}ds.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&\left(x(t+T)-g(t+T,x(t+T-\tau(t+T)))\right)e^{-\int_0^{t+T} r(u)du}\\ &-\left(x(t)-g(t,x(t-\tau(t)))\right)e^{-\int_0^t r(u)du}\\ &=\int_t^{t+T}\left[-f(s,x(s-\tau(s)))+r(s)\,g(s,x(s-\tau(s)))\right]e^{-\int_0^s r(u)du}ds.\end{aligned}$$

Dividing both sides of the above equation by $e^{-\int_0^t r(u)du}$ and using the fact that $x(t)=x(t+T)$, (2) and (4), we obtain

$$\begin{aligned}&x(t)-g(t,x(t-\tau(t)))\\ &=\int_t^{t+T}\frac{e^{\int_s^t r(u)du}}{1-e^{-\int_0^T r(u)du}}\left[f(s,x(s-\tau(s)))-r(s)\,g(s,x(s-\tau(s)))\right]ds.\end{aligned}$$

This completes the proof.

To simplify notation, we let

$$m=\frac{e^{-\int_0^{2T}|r(u)|du}}{1-e^{-\int_0^T r(u)du}},\ M=\frac{e^{\int_0^{2T}|r(u)|du}}{1-e^{-\int_0^T r(u)du}}.$$

It is easy to see that for all $(t,s)\in[0,2T]\times[0,2T]$,

$$m\le G(t,s)\le M,$$

and for all $t,s\in\mathbb{R}$, we have

$$G(t+T,s+T)=G(t,s).$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1). For its proof we refer the reader to [18].

THEOREM 2.1 (Krasnoselskii). Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B},\|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that

(i) $x,y\in\mathbb{D}$, implies $\mathcal{A}x+\mathcal{B}y\in\mathbb{D}$,

(ii) $\mathcal{A}$ is compact and continuous,

(iii) $\mathcal{B}$ is a contraction mapping.

Then there exists $z\in\mathbb{D}$ with $z=\mathcal{A}z+\mathcal{B}z$.

3 Existence of Positive Periodic Solutions

To apply Theorem 2.1, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : L \leq \varphi \leq K\}$, where L is non-negative constant and K is positive constant. We express equation (5) as

$$\varphi(t) = (\mathcal{B}\varphi)(t) + (\mathcal{A}\varphi)(t) := (H\varphi)(t),$$

where $\mathcal{A}, \mathcal{B} : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}\varphi)(t) = \int_t^{t+T} G(t,s)\left[f(s,\varphi(s-\tau(s))) - r(s)\, g(s,\varphi(s-\tau(s)))\right] ds, \tag{7}$$

and

$$(\mathcal{B}\varphi)(t) = g(t,\varphi(t-\tau(t))). \tag{8}$$

In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases; $g(t,x) \geq 0$ and $g(t,x) \leq 0$ for all $t \in \mathbb{R}$, $x \in \mathbb{D}$. We assume that function $g(t,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|g(t,x) - g(t,y)| \leq k\|x-y\|, \text{ for all } t \in [0,T], \ x,y \in \mathbb{D}. \tag{9}$$

In the case $g(t,x) \geq 0$, we assume that there exist a non-negative constant k_1 and positive constant k_2 such that

$$k_1 x \leq g(t,x) \leq k_2 x, \text{ for all } t \in [0,T], \ x \in \mathbb{D}, \tag{10}$$

$$k_2 < 1, \tag{11}$$

and for all $t \in [0,T]$, $x \in \mathbb{D}$

$$\frac{L(1-k_1)}{mT} \leq f(t,x) - r(t)\, g(t,x) \leq \frac{K(1-k_2)}{MT}. \tag{12}$$

LEMMA 3.1. Suppose that the conditions (2)-(4) and (10)-(12) hold. Then $\mathcal{A} : \mathbb{D} \to \mathbb{B}$ is compact.

PROOF. Let $\mathcal{A}$ be defined by (7). Obviously, $\mathcal{A}\varphi$ is continuous and it is easy to show that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. For $t \in [0,T]$ and for $\varphi \in \mathbb{D}$, we have

$$\begin{aligned} |(\mathcal{A}\varphi)(t)| &\leq \left| \int_t^{t+T} G(t,s)\left[f(s,\varphi(s-\tau(s))) - r(s)\, g(s,\varphi(s-\tau(s)))\right] ds \right| \\ &\leq MT\frac{K(1-k_2)}{MT} = K(1-k_2). \end{aligned}$$

Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we have

$$\|\mathcal{A}\varphi\| \leq K(1-k_2).$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded.

To show that $\mathcal{A}(\mathbb{D})$ is equicontinuous. Let $\varphi_n \in \mathbb{D}$, where n is a positive integer. Next we calculate $\frac{d}{dt}(\mathcal{A}\varphi_n)(t)$ and show that it is uniformly bounded. By making use of (2) and (4) we obtain by taking the derivative in (7) that

$$\begin{aligned}\frac{d}{dt}(\mathcal{A}\varphi_n)(t) &= [G(t,t+T) - G(t,t)][f(t,\varphi_n(t-\tau(t))) - r(t)g(t,\varphi_n(t-\tau(t)))] \\ &\quad + r(t)(\mathcal{A}\varphi_n)(t).\end{aligned}$$

Consequently, by invoking (12), we obtain

$$\left|\frac{d}{dt}(\mathcal{A}\varphi_n)(t)\right| \le \frac{K(1-k_2)}{MT} + \|r\| K(1-k_2) \le D,$$

for some positive constant D. Hence the sequence $(\mathcal{A}\varphi_n)$ is equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $\left(\mathcal{A}\varphi_{n_k}\right)$ of $(\mathcal{A}\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus $\mathcal{A}$ is continuous and $\mathcal{A}(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$.

LEMMA 3.2. Suppose that (9) holds. If $\mathcal{B}$ is given by (8) with

$$k < 1, \tag{13}$$

then $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction.

PROOF. Let $\mathcal{B}$ be defined by (8). Obviously, $\mathcal{B}\varphi$ is continuous and it is easy to show that $(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned}|(\mathcal{B}\varphi)(t) - (\mathcal{B}\psi)(t)| &\le |g(t,\varphi(t-\tau(t))) - g(t,\psi(t-\tau(t)))| \\ &\le k\|\varphi - \psi\|.\end{aligned}$$

Then $\|\mathcal{B}\varphi - \mathcal{B}\psi\| \le k\|\varphi - \psi\|$. Thus $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction by (13).

THEOREM 3.1. Suppose (2)-(4) and (9)-(13) hold. Then equation (1) has a positive T-periodic solution x in the subset $\mathbb{D}$.

PROOF. By Lemma 3.1, the operator $\mathcal{A} : \mathbb{D} \to \mathbb{B}$ is compact and continuous. Also, from Lemma 3.2, the operator $\mathcal{B} : \mathbb{D} \to \mathbb{B}$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned}(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) &= g(t,\psi(t-\tau(t))) \\ &\quad + \int_t^{t+T} G(t,s)[f(s,\varphi(s-\tau(s))) - r(s)g(s,\varphi(s-\tau(s)))]\,ds \\ &\le k_2K + M\int_t^{t+T} [f(s,\varphi(s-\tau(s))) - r(s)g(s,\varphi(s-\tau(s)))]\,ds \\ &\le k_2K + MT\frac{K(1-k_2)}{MT} = K.\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathcal{B}\psi)(t)+(\mathcal{A}\varphi)(t) &= g(t,\psi(t-\tau(t))) \\ &\quad+\int_t^{t+T} G(t,s)\left[f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\right]ds \\ &\geq k_1L+m\int_t^{t+T}\left[f(s,\varphi(s-\tau(s)))-r(s)g(s,\varphi(s-\tau(s)))\right]ds \\ &\geq k_1L+mT\frac{L(1-k_1)}{mT}=L.\end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x\in\mathbb{D}$ such that $x=\mathcal{A}x+\mathcal{B}x$. By Lemma 2.1 this fixed point is a solution of (1) and the proof is complete.

REMARK 3.1. When $g(t,x)=cx$, Theorem 3.1 reduces to Theorem 3.2 of [17].

In the case $g(t,x)\leq 0$, we substitute conditions (10)-(12) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3x\leq g(t,x)\leq k_4x,\ \text{for all } t\in[0,T],\ x\in\mathbb{D}, \tag{14}$$

$$-k_3<1, \tag{15}$$

and for all $t\in[0,T]$, $x\in\mathbb{D}$

$$\frac{L-k_3K}{mT}\leq f(t,x)-r(t)g(t,x)\leq\frac{K-k_4L}{MT}. \tag{16}$$

THEOREM 3.2. Suppose (2)-(4), (9) and (13)-(16) hold. Then equation (1) has a positive T-periodic solution x in the subset $\mathbb{D}$.

The proof follows along the lines of Theorem 3.1, and hence we omit it.

REMARK 3.2. When $g(t,x)=cx$, Theorem 3.2 reduces to Theorem 3.3 of [17].

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References

[1] A. Ardjouni and A. Djoudi, Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale, Commun Nonlinear Sci Numer Simulat., 17(2012), 3061–3069.

[2] A. Ardjouni and A. Djoudi, Periodic solutions for a second-order nonlinear neutral differential equation with variable delay, Electronic Journal of Differential Equations, 2011(2011), No. 128, 1–7.

[3] A. Ardjouni and A. Djoudi, Periodic solutions in totally nonlinear dynamic equations with functional delay on a time scale, Rend. Sem. Mat. Univ. Politec. Torino, 68(4)(2010), 349–359.

[4] T. A. Burton, Liapunov functionals, fixed points and stability by Krasnoselskii's theorem, Nonlinear Stud., 9(2)(2002), 181–190.

[5] T. A. Burton, Stability by Fixed Point Theory for Functional Differential Equations, Dover Publications, New York, 2006.

[6] F. D. Chen, Positive periodic solutions of neutral Lotka-Volterra system with feedback control, Appl. Math. Comput., 162(3)(2005), 1279–1302.

[7] F. D. Chen and J. L. Shi, Periodicity in a nonlinear predator-prey system with state dependent delays, Acta Math. Appl. Sin. Engl. Ser., 21(1)(2005), 49–60.

[8] Y. M. Dib, M. R. Maroun and Y. N. Raffoul, Periodicity and stability in neutral nonlinear differential equations with functional delay, Electronic Journal of Differential Equations, 2005(2005), No. 142, 1–11.

[9] M. Fan and K. Wang, P. J. Y. Wong and R. P. Agarwal, Periodicity and stability in periodic n-species Lotka-Volterra competition system with feedback controls and deviating arguments, Acta Math. Sin. Engl. Ser., 19(4)(2003), 801–822.

[10] E. R. Kaufmann and Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, J. Math. Anal. Appl., 319(1)(2006), 315–325.

[11] E. R. Kaufmann and Y. N. Raffoul, Periodicity and stability in neutral nonlinear dynamic equations with functional delay on a time scale, Electron. J. Differential Equations, 2007(27)(2007), 1–12.

[12] E. R. Kaufmann, A nonlinear neutral periodic differential equation, Electron. J. Differential Equations, 2010(88)(2010), 1–8.

[13] W. G. Li and Z. H. Shen, An constructive proof of the existence Theorem for periodic solutions of Duffing equations, Chinese Sci. Bull., 42(1997), 1591–1595.

[14] Y. Liu and W. Ge, Positive periodic solutions of nonlinear duffing equations with delay and variable coefficients, Tamsui Oxf. J. Math. Sci., 20(2004) 235–255.

[15] Y. N. Raffoul, Periodic solutions for neutral nonlinear differential equations with functional delay, Electron. J. Differential Equations, 2003(102)(2003), 1–7.

[16] Y. N. Raffoul, Stability in neutral nonlinear differential equations with functional delays using fixed-point theory, Math. Comput. Modelling, 40(7-8)(2004), 691–700.

[17] Y. N. Raffoul, Positive periodic solutions in neutral nonlinear differential equations, E. J. Qualitative Theory of Diff. Equ., 2007(16)(2007), 1–10.

[18] D. S. Smart, Fixed point theorems; Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974.

[19] Q. Wang, Positive periodic solutions of neutral delay equations (in Chinese), Acta Math. Sinica (N.S.), 6(1996), 789–795.