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The Interior Error Of Van Cittert Deconvolution Of Differential Filters Is Optimal*

William Layton†

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Abstract

We reconsider the error in van Cittert deconvolution. We show that without any extra boundary conditions on higher derivatives of u, away from the boundary the error in van Cittert deconvolution attains the high order of accuracy seen in the periodic problem. This error result is important for differential filters and approximate deconvolution models of turbulence.

1 Introduction

In multiscale modeling and simulation, one recurring problem is to estimate the effects of the unresolved fluctuations $u' := u - \overline{u}$ on the means $\overline{u}$. This is equivalent to the (ill posed) problem of given $\overline{u}$, construct an approximation to u that can be used for the same purpose. One early method of doing so is the van Cittert approximation. Because it is inexpensive in both computational effort and programmer time, van Cittert has been used as a basis of large eddy simulation turbulence modeling. We consider the error in van Cittert deconvolution of a differential filter on a bounded domain under non-periodic boundary conditions. We show that without any extra boundary compatibility conditions on higher derivatives of u, away from the boundary the error in van Cittert deconvolution attains the high order of accuracy seen in the periodic problem.

The filtering problem is: given a function $u(x)$ defined on a domain Ω, compute an approximation $Gu = \overline{u}(x)$ to $u(x)$ which faithfully represents the behavior of u on scales above some, user selected, filter length (denoted ε), and which truncates scales smaller then $O(\varepsilon)$. The deconvolution or de-filtering problem is: given $\overline{u}$ find an accurate reconstruction of u. When the filter $G : L^2(\Omega) \to L^2(\Omega)$ is smoothing, G is compact and the exact deconvolution problem is ill-posed. One early method of approximate deconvolution is the 1931 van Cittert [1] algorithm:

ALGORITHM 1. (van Cittert approximate deconvolution) Set $u_0 = \overline{u}$. Fix N (moderate). For $n = 1, 2, ..., N-1$, perform $u_{n+1} = u_n + \{\overline{u} - Gu_n\}$.

Define $D_N\overline{u} := u_N$.

Van Cittert deconvolution requires only a few steps of repeated filtering. It is thus both computationally cheap and easy to program, contributing to its popularity in

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†Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, 15260, USA.

various applications, such as turbulence modeling, e.g., [2]. The error in van Cittert deconvolution is given explicitly by (1) below. For convolution filters and in the absence of boundaries, the RHS of (2.1) can be analyzed precisely by Fourier methods, e.g., [3], [4], [5]. With boundaries, there are significant gaps between the improved accuracy seen in computational practice and the pessimistic estimates of its global error obtained in analysis.

The goal of this report is to close this gap somewhat. We use interior regularity results for elliptic-elliptic singular perturbation problems to give error estimates for the van Cittert deconvolution under non-periodic boundary conditions. We take the filter to be a differential filter, [6], specifically the extension of the Pao filter, e.g., [7], to a bounded domain. Let Ω be a bounded, regular, planar domain with smooth boundary and $0 < \varepsilon \leq 1$ a small parameter. Given $u \in H_0^1(\Omega) \cap H^k(\Omega)$, $\overline{u}$ is the unique solution of the elliptic-elliptic singular perturbation problem

$$-\varepsilon^2 \triangle \overline{u} + \overline{u} = u \text{ , in } \Omega, \text{ and } \overline{u} = 0 \text{ , on } \partial\Omega.$$

The error formula (1) reduces the question of convergence rates to regularity. Unfortunately, regularity theory (sharp in 1d examples [8]) predicts no improvement in the rate of convergence in the L^2 norm, denoted $||\cdot||$. We prove the following herein which predicts improvement from higher order deconvolution in negative Sobolev norms and optimal convergence away from the boundary.

THEOREM 1. (Local and global deconvolution error estimates) Suppose $N > 0$ is fixed and $u \in H_0^1(\Omega) \cap H^k(\Omega)$. Then

$$||u - D_0\overline{u}|| = ||u - \overline{u}|| \leq C\varepsilon^2 ||u||_{H^2(\Omega)}.$$

If $N = 1$ we have in L^2 and H^{-2},

$$||u - D_1\overline{u}|| \leq C\varepsilon^2 ||u||_{H^2(\Omega)} \quad \text{and} \quad ||u - D_1\overline{u}||_{H^{-2}(\Omega)} \leq C\varepsilon^4 ||u||_{H^2(\Omega)}.$$

If $N = 1$ and additionally $\triangle u \in H_0^1(\Omega)$,

$$||u - D_1\overline{u}|| \leq C\varepsilon^4 ||u||_{H^2(\Omega)}.$$

If $\triangle u \neq 0$ on $\partial\Omega$ we have for any $N \geq 0$ fixed

$$\begin{aligned} ||u - D_N\overline{u}|| &\leq C\varepsilon^2 ||u||_{H^2(\Omega)}, \\ ||u - D_N\overline{u}||_{H^{-2N}(\Omega)} &\leq C\varepsilon^{2N+2} ||u||_{H^2(\Omega)}. \end{aligned}$$

Let $s \geq 0$ be fixed. Suppose $u \in H_0^1(\Omega) \cap H^{2N+2}(\Omega)$. Let

$$\Omega_{N+1} \subset \Omega_N \subset \cdots \subset \Omega_1 \subset \Omega_0 \subset \Omega_{-1} \equiv \Omega$$

be subdomains with smooth boundaries. For $j = N+1, ..., 0$ suppose

$$\Omega_j \text{ has distance } C_j \varepsilon \ln(1/\varepsilon) \text{ from } \partial\Omega_{j-1},$$

where $C_j = C(s, N, \Omega_j, \Omega_{j-1})$. Then there is a $C = C(N, C_j)$ such that

$$||u - D_N\overline{u}||_{L^2(\Omega_{N+1})} \leq C\varepsilon^{2N+2} \left[||u||_{H^{2N+2}(\Omega_0)} + \varepsilon^s ||u|| \right].$$

REMARK 1. If the differential filter is replaced by a local averaging with radius ε, then the computation on each Ω_j only uses values from $\partial\Omega_{j-1}$, making local error estimates of the above type immediate.

2 Proof of the Deconvolution Error Estimate

The error in van Cittert deconvolution is calculated by summing a geometric series, [3], [4], [5], to be

$$u - D_N\overline{u} = (-1)^{N+1}\varepsilon^{2N+2}(\triangle^{N+1}G^{N+1})u. \tag{1}$$

Thus, accuracy of van Cittert depends on for what norms $|||\cdot|||$ and values on N, the RHS is bounded uniformly in ε:

$$|||\triangle^{N+1}G^{N+1}(u)||| \leq C(u) < \infty \text{ uniformly in } \varepsilon.$$

The proof will follow from the error representation (1) and two regularity results for the elliptic-elliptic singular perturbation problem, Theorems 2.1 and 2.2 below. The global regularity result in Theorem 2.1 was proven in [8], see also [2]. The interior regularity result in Theorem 3 is a special case of Theorem 2.3, page 26 of Nävert [9] (setting the convecting velocity to zero), see also [10]. For related estimates see [11], [12], [13], [14]. We shall first recall these two results, give a preliminary lemma and then give the proof (which is short with this preparation). $H^k(\Omega)$ denotes the Sobolev space of all functions with derivatives of order $\leq k$ in $L^2(\Omega)$. The $L^2(\Omega)$ norm is $||\cdot||$ and $H_0^1(\Omega) := \{v \in H^1 : v = 0 \text{ on } \partial\Omega\}$. For (1) we assume (in particular implying $u = 0$ on $\partial\Omega$)

$$u \in H_0^1(\Omega) \cap H^k(\Omega). \tag{2}$$

This condition precludes simple boundary layers in $\overline{u}$ but does not imply higher derivatives of $\overline{u}$ are free of layers. The shift theorem implies that $\overline{u} \in H_0^1(\Omega) \cap H^{k+2}(\Omega)$. Since traces of $\triangle\overline{u}$ are thus well defined, $-\varepsilon^2\triangle\overline{u} + \overline{u} = u$ implies

$$\overline{u} = 0 \text{ and } \triangle\overline{u} = 0 \text{ on } \partial\Omega.$$

THEOREM 2 (Theorem 1.1 in [8]). Suppose $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Then there is a constant $C > 0$ independent of ε such that

$$||\overline{u}||_{H^l(\Omega)} \leq C||u||_{H^l(\Omega)} \text{ , for } l = 0, 1, 2. \tag{3}$$

If $u \in H_0^1(\Omega) \cap H^4(\Omega)$, $\triangle u \in H_0^1(\Omega)$. Then

$$||\overline{u}||_{H^l(\Omega)} \leq C||u||_{H^l(\Omega)} \text{ , for } l = 0, 1, 2, 3, 4. \tag{4}$$

In general, suppose $u \in H^{2k}(\Omega) \bigcap H_0^1(\Omega), \triangle^j u \in H_0^1(\Omega), j = 1, ..., k-1$. Then for $l = 1, ..., 2k$,

$$||\overline{u}||_{H^l(\Omega)} \leq C||u||_{H^l(\Omega)}. \tag{5}$$

Examples in [8] show that the limit of $l \leq 2$ in (3) is sharp unless higher derivatives of u are zero on $\partial\Omega$, as in (4).

THEOREM 3 (Special case of Theorem 2.3 in Nävert [9]). For $u \in H^k(\Omega) \cap H_0^1(\Omega)$ consider

$$-\varepsilon^2\triangle\overline{u} + \overline{u} = u \text{ , in } \Omega, \text{ and } \overline{u} = 0 \text{ , on } \partial\Omega. \tag{6}$$

Let $m \geq 0, s \geq 0$. Let $\Omega' \subset \Omega'' \subset \Omega$ be subdomains with smooth boundaries with Ω' has distance $C_1\varepsilon\ln(1/\varepsilon)$ from $\partial\Omega''$, Ω'' has distance $C_2\varepsilon\ln(1/\varepsilon)$ from $\partial\Omega$, where $C_i = C_i(s, m, \Omega', \Omega'')$. Then the solution to (6) satisfies

$$||\overline{u}||_{H^m(\Omega')} \leq C\left(||u||_{H^m(\Omega'')} + \varepsilon^s||u||\right).$$

Since $u|_{\partial\Omega} = 0$ implies $\overline{u}|_{\partial\Omega} = 0$ and $\triangle\overline{u}|_{\partial\Omega} = 0$ the second order problem for $\overline{u}$ can be converted into a fourth order problem for $\overline{u}$ by taking Laplacian of the equation. Theorem 2.2 also follows from, for example, a small modification of the proof of Lemma 2.2 in [15]. First we calculate the global regularity of repeated filtering.

PROPOSITION 1. Let $u \in H^k(\Omega) \bigcap H_0^1(\Omega)$. We have for $J \geq 1$

$$||G^J u||_{H^k(\Omega)} \leq C||G^{J-1}u||_{H^k(\Omega)}, k = 0, 1, ..., 2J.$$

PROOF. For $n = 1$ Theorem 3 implies

$$||\overline{u}||_{H^k(\Omega)} \leq C||u||_{H^k(\Omega)} \text{ , for } k = 0, 1, 2 \text{ and } \triangle\overline{u} = 0 \text{ on } \partial\Omega.$$

Since $\triangle\overline{u} = 0$ on $\partial\Omega$, we repeat. Indeed, $G^2u = G\overline{u} = \overline{\overline{u}}$ so that

$$||\overline{\overline{u}}||_{H^k(\Omega)} \leq C||\overline{u}||_{H^k(\Omega)}, \text{ for } k = 0, 1, 2, 3, 4$$

and that

$$\triangle\overline{\overline{u}} = \triangle\overline{u} = 0 \text{ on } \partial\Omega.$$

Taking the Laplacian of the equation for $\overline{\overline{u}}$ gives $-\delta^2\triangle^2\overline{\overline{u}} + \triangle\overline{\overline{u}} = \triangle\overline{u}$, in Ω. Now, let $x \to \partial\Omega$ and use $\triangle\overline{\overline{u}} = \triangle\overline{u} = 0$ on $\partial\Omega$. This implies $\triangle^2\overline{\overline{u}} = \triangle\overline{\overline{u}} = \overline{\overline{u}} = 0$ on $\partial\Omega$ so that for $\overline{\overline{\overline{u}}}$ we have

$$||\overline{\overline{\overline{u}}}||_{H^k(\Omega)} \leq C||\overline{\overline{u}}||_{H^k(\Omega)} \text{ , for } k = 0, 1, 2, 3, 4, 5, 6.$$

The proof continues by induction.

We can now prove the deconvolution error estimate in Theorem 1.

PROOF. (Proof of Theorem 1) We consider $\triangle^{N+1}G^{N+1}(u)$ and use Theorem 1.1 in [8] repeatedly. For $N = 0$ this is $||\triangle(-\varepsilon^2\triangle + 1)^{-1}u||$:

$$||u - D_0\overline{u}|| = ||u - \overline{u}|| = \varepsilon^2||\triangle(-\varepsilon^2\triangle + 1)^{-1}u||.$$

The first estimate follows since

$$||\triangle(-\varepsilon^2\triangle + 1)^{-1}u|| = ||\triangle\overline{u}|| \leq C||\overline{u}||_2 \leq C||u||_2.$$

For $N = 1$ and under $\triangle u \in H_0^1(\Omega)$ we have similarly that $||\overline{\overline{u}}||_4 \leq C||u||_4$. Thus

$$||u - D_1\overline{u}|| = \varepsilon^4||\triangle^2\overline{\overline{u}}|| \leq C\varepsilon^4||\overline{\overline{u}}||_4 \leq C\varepsilon^4||u||_4.$$

For the H^{-2} estimate we use that $\triangle^2\overline{\overline{u}} = \triangle\left(\overline{\triangle\overline{u}}\right)$. Step by step, using $\triangle\overline{u} = 0$ on $\partial\Omega$ we find $||\triangle^2\overline{\overline{u}}||_{-2} \leq C||\overline{\triangle\overline{u}}|| \leq C||\triangle\overline{u}|| \leq C||u||_2$, completing the proof. The case of $N > 1$ follows the same way.

For the interior estimates we use Theorem 2.2 as follows.

$$\begin{aligned} ||u - D_N\overline{u}||_{L^2(\Omega_{N+1})} &= \varepsilon^{2N+2}||(\triangle^{N+1}G^{N+1})u||_{L^2(\Omega_{N+1})} \leq \\ &\leq C\varepsilon^{2N+2}||G^{N+1}u||_{H^{2N+2}(\Omega_{N+1})}. \end{aligned}$$

Note that $||\overline{\phi}|| \leq ||\phi||$ so that $||G^j u|| \leq ||u||$ for all j. Now $G^{N+1}u = \overline{\phi}, \phi = G^N u$. Thus, for any $s > 0$

$$\begin{aligned} ||G^{N+1}u||_{H^{2N+2}(\Omega_{N+1})} &\leq C\left(||G^N u||_{H^{2N+2}(\Omega_N)} + \varepsilon^s||G^N u||\right) \\ &\leq C\left(||G^N u||_{H^{2N+2}(\Omega_N)} + \varepsilon^s||u||\right). \end{aligned}$$

We repeat this argument. Indeed, $G^N u = \overline{\phi}, \phi = G^{N-1}u$. Thus, for any $s > 0$

$$||G^N u||_{H^{2N+2}(\Omega_N)} \leq C\left(||G^{N-1}u||_{H^{2N+2}(\Omega_{N-1})} + \varepsilon^s||u||\right).$$

At the last step we have, for any $s > 0$

$$||G^1 u||_{H^{2N+2}(\Omega_1)} \leq C\left(||u||_{H^{2N+2}(\Omega_0)} + \varepsilon^s||u||\right).$$

Thus (recalling that N is fixed and C can depend on N) we have

$$||u - D_N\overline{u}||_{L^2(\Omega_{N+1})} \leq C\varepsilon^{2N+2}\left[||u||_{H^{2N+2}(\Omega_0)} + \varepsilon^s||u||\right].$$

3 Remarks

The error in deconvolution in the non-periodic case is of high accuracy, away from boundaries, like that of the periodic case. It is an interesting analytic open question, relevant to inverse or approximate deconvolution models of turbulence [16], [17], [18], to establish if a similar result holds for the Stokes differential filter. It is an important algorithmic open question to alter the van Cittert procedure near boundaries to obtain a high order accurate reconstruction of the unknown function up to the boundary.

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