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Reversible Splitting Of Matrix And Its Convergence*

Zhi Ming Yang†

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Abstract

Matrix splittings of the form $A = M - N$ can transform the problem of solving linear systems of the form $Ax = b$ into solving iterative problems of the form $x^{(n+1)} = M^{-1}Nx^{(n)} + M^{-1}b$. In this paper, we introduce a new splitting which is called reversible. Then we discuss the convergence of regular reversible splittings and nonnegative reversible splittings.

1 Introduction

Among iterative methods for solving systems of linear equations of the form

$$Ax = b$$

where $A \in R^{n\times n}$ is a nonsingular matrix and $x, b \in R^n$, one class can be formulated by means of the splitting

$$A = M - N \quad \text{with} \quad M \quad \text{nonsingular} \tag{1}$$

and the approximate solution $x^{(n+1)}$ is generated as follows

$$Mx^{(n+1)} = Nx^{(n)} + b\,, \quad n \geq 0\,,$$

or equivalently,

$$x^{(n+1)} = M^{-1}Nx^{(n)} + M^{-1}b\,, \quad n \geq 0\,, \tag{2}$$

where the initial vector $x^{(0)}$ is given.

The above iterative method is convergent to the unique solution $x = A^{-1}b$ for each $x^{(0)}$ if and only if $\rho(M^{-1}N) < 1$, which means that the splitting of $A = M - N$ is convergent. The convergence analysis of the above method is based on the spectral radius of the iteration matrix $\rho(M^{-1}N)$. As is well known, the smaller the spectral radius $\rho(M^{-1}N)$, the faster is the convergence (see [1]).

Apart from being a useful tool of convergence analysis for the iterative scheme (2), the theory of matrix splitting also provides some extremely interesting results. However, different authors have given different concept of the splittings, such as Varga [1], J. J. Climent and C. Perea [2] and Woźnicki [3]. Here, we recall some useful concepts and notations, and then introduce the concept of splittings defined by Woźnicki.

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†Teachers College , Gansu Lianhe University, Lanzhou, Gansu 730000, P. R. China

DEFINITION 1 ([1]). Let $A = (a_{ij}) \in R^{n\times n}, B = (b_{ij}) \in R^{n\times n}$. Then $A \geq B$ ($A > B$) if $a_{ij} \geq b_{ij}$ ($a_{ij} > b_{ij}$) for all $1 \leq i, j \leq n$. If O is the null matrix and $A \geq O$, we say that A is a nonnegative matrix.

DEFINITION 2 ([1]). Let A be an $n \times n$ complex matrix with eigenvalues λ_i ($i = 1, 2, ..., n$). $\rho(A) = \max_{1\leq i\leq n} |\lambda_i|$ is the spectral radius of A. $\sigma(A) = \min_{1\leq i\leq n} |\lambda_i|$, and I denotes the identity matrix , $\lambda_i(A)$ and $|A|$ denote the i-th eigenvalue and the determinant of matrix A respectively.

DEFINITION 3 ([3]). Let $M, N \in R^{n\times n}$. Then the decomposition $A = M - N$ is called

(a) a regular splitting of A if $M^{-1} \geq 0$ and $N \geq 0$;

(b) a nonnegative splitting of A if $M^{-1} \geq 0$, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$;

(c) a weak nonnegative splitting of A if $M^{-1} \geq 0$ and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type);

(d) a weak splitting of A if M is nonsingular, $M^{-1}N \geq 0$ and $NM^{-1} \geq 0$;

(e) a weaker splitting of A if M is nonsingular and either $M^{-1}N \geq 0$ (the first type) or $NM^{-1} \geq 0$ (the second type);

(f) a convergent splitting of A if $\rho(M^{-1}N) = \rho(NM^{-1}) < 1$.

Different splittings were extensively analyzed by many authors, see, e.g., [4] and [5]. Some splittings, however, are not included in Definition 3. For example, let

$$A = \begin{pmatrix} 4 & 0 & 1 \\ -1 & 0 & 2 \\ 1 & 3 & 3 \end{pmatrix} = M - N\,,$$

where

$$M = \begin{pmatrix} 12 & -36 & 7 \\ -3 & 9 & 14 \\ 3 & -\frac{9}{2} & 21 \end{pmatrix},\ N = \begin{pmatrix} 8 & -36 & 6 \\ -2 & 9 & 12 \\ 2 & -\frac{15}{2} & 18 \end{pmatrix},$$

we can easily see that $N \ngeq 0$ and

$$M^{-1}N = \begin{pmatrix} \frac{2}{3} & -2 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{6}{7} \end{pmatrix} \ngeq 0,\ NM^{-1} = \begin{pmatrix} \frac{410}{189} & \frac{632}{189} & -\frac{8}{3} \\ -\frac{62}{189} & \frac{4}{189} & \frac{2}{3} \\ \frac{13}{21} & \frac{31}{21} & -\frac{1}{3} \end{pmatrix} \ngeq 0.$$

The above example shows that the splitting is not included in Definition 3, so that we cannot discuss its convergence according to the theories established by R. S. Varga, Miller and Neumann, Song, among others. The main purpose of this paper is to present a new splitting which is called reversible and discuss its convergence.

2 Reversible Splitting of a Matrix

We begin with the following

DEFINITION 4. Let $A \in R^{n\times n}$. Then the decomposition $A = M - N$ is called a reversible splitting of matrix A if M, N are nonsingular and $\lambda_i(M^{-1}A) > 0$ for $i = 1, 2, ..., n$.

DEFINITION 5. Let $A \in R^{n\times n}$. Then the reversible splitting $A = M - N$ is called
(a) a regular reversible splitting of A if $M^{-1} \geq 0$ and $A \geq 0$;
(b) a nonnegative reversible splitting of A if $M^{-1} \geq 0$, $M^{-1}A \geq 0$ and $AM^{-1} \geq 0$;
(c) a weak nonnegative reversible splitting of A if $M^{-1} \geq 0$ and either $M^{-1}A \geq 0$ (the first type) or $AM^{-1} \geq 0$ (the second type);
(d) a weak reversible splitting of A if $M^{-1}A \geq 0$ and $AM^{-1} \geq 0$;
(e) a weaker reversible splitting of A if either $M^{-1}A \geq 0$ (the first type) or $AM^{-1} \geq 0$ (the second type).

EXAMPLE 1. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = M - N$, where

$$M = \begin{pmatrix} 6 & -6 \\ -6 & 12 \end{pmatrix}, \quad N = \begin{pmatrix} 4 & -5 \\ -5 & 10 \end{pmatrix}.$$

Evidently, M, N are nonsingular and

$$M^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} \geqslant 0, \quad M^{-1}A = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} \geqslant 0, \quad AM^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} \end{pmatrix} \geqslant 0.$$

From $\lambda_i(M^{-1}A) > 0$ $(i = 1, 2)$ we know that the decomposition $A = M - N$ is a nonnegative reversible splitting.

EXAMPLE 2. Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = M - N$, where

$$M = \begin{pmatrix} 3 & -2 \\ -3 & 4 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

Evidently, M, N are nonsingular and

$$M^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \geq 0, \quad N^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \geq 0.$$

$$M^{-1}A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \geq 0, \quad AM^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \ngeq 0.$$

This is a weak nonnegative reversible splitting of the first type.

EXAMPLE 3. Let $A = \begin{pmatrix} \frac{1}{2} & 2 \\ -\frac{1}{2} & -4 \end{pmatrix} = M - N$, where

$$M = \begin{pmatrix} \frac{3}{2} & 1 \\ -\frac{3}{2} & -2 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Evidently, M, N are nonsingular and

$$M^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ -1 & -1 \end{pmatrix} \ngeq 0, \quad N^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \geq 0.$$

$$M^{-1}A = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix} \geq 0\,, \quad AM^{-1} = \begin{pmatrix} -\frac{4}{3} & -\frac{5}{3} \\ \frac{10}{3} & \frac{11}{3} \end{pmatrix} \ngeq 0.$$

It is a weaker reversible splitting of the first type.

EXAMPLE 4. Consider the splitting given in section 1 where $|M| = -\dfrac{1701}{2} \neq 0$, $|N| = -162 \neq 0$, and

$$M^{-1} = \begin{pmatrix} -\frac{8}{27} & -\frac{23}{27} & \frac{2}{3} \\ -\frac{10}{81} & -\frac{22}{81} & \frac{2}{9} \\ \frac{1}{63} & \frac{4}{63} & 0 \end{pmatrix} \ngeq 0\,,\ M^{-1}A = \begin{pmatrix} \frac{1}{3} & 2 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix} \geq 0.$$

$$AM^{-1} = \begin{pmatrix} -\frac{221}{189} & -\frac{632}{189} & \frac{8}{3} \\ \frac{62}{189} & \frac{185}{189} & -\frac{2}{3} \\ -\frac{13}{21} & -\frac{31}{21} & \frac{4}{3} \end{pmatrix} \ngeq 0\,, \quad \lambda_i(M^{-1}A) > 0,\ i = 1, 2, 3,$$

so the decomposition $A = M - N$ is a weaker reversible splitting of the first type.

3 Convergence of Reversible Splitting

In this section, we discuss the convergence of splittings defined in the first two items of Definition 5.

LEMMA 1 ([1]). If M is an $n \times n$ matrix with $\rho(M) < 1$, then $I - M$ is nonsingular and

$$(I - M)^{-1} = I + M + M^2 + \cdots\,,$$

where the series on the right converging; conversely, if the series on the right converges, then $\rho(M) < 1$.

THEOREM 1. Let $A = M - N$ be a reversible splitting of A and $\rho(M^{-1}A) < 1$, then $\rho(M^{-1}N) < 1$.

PROOF. Since $A = M - N$ is a reversible splitting, so M is nonsingular and $\lambda_i(M^{-1}A) > 0$, $i = 1, 2, ..., n,$ from $N + A = M$, we have

$$M^{-1}N + M^{-1}A = I$$

and

$$\lambda_i(M^{-1}N) + \lambda_i(M^{-1}A) = 1, \quad i = 1, 2, ..., n. \tag{3}$$

On the other hand, $\rho(M^{-1}A) < 1$ means that

$$0 < \lambda_i(M^{-1}A) < 1\,, \quad i = 1, 2, ..., n.$$

From (3) we obtain $\rho(M^{-1}N) = \max\limits_{1 \leq i \leq n} \{\lambda_i(M^{-1}N)\} < 1$.

EXAMPLE 5. Consider the splitting of matrix A in Example 1, where $M^{-1}A = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}$, that is, $\lambda_1(M^{-1}A) = \dfrac{1}{2}\,, \lambda_2(M^{-1}A) = \dfrac{1}{6}$ and

$$M^{-1}N = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{5}{6} \end{pmatrix}$$

means that $\lambda_1(M^{-1}N) = \dfrac{1}{2}\,, \lambda_2(M^{-1}N) = \dfrac{5}{6}$, that is,

$$\lambda_i(M^{-1}N) + \lambda_i(M^{-1}A) = 1, i = 1, 2.$$

For the splittings given in Example 2–4, we can also prove that the equality (3) holds, which means that Theorem 1 holds.

Now according to Theorem 1, we know that the reversible splitting of $A = M - N$ is convergent if $\rho(M^{-1}A) < 1$.

It is evident, from (3), that the following corollary holds.

COROLLARY 1. Let $A = M - N$ be a reversible splitting of A and $\rho(M^{-1}A) < 1$. Then

$$\rho(M^{-1}N) = 1 - \sigma(M^{-1}A).$$

For example, in Example 5, we know that $\rho(M^{-1}A) = \dfrac{1}{2} < 1$ and $\sigma(M^{-1}A) = \dfrac{1}{6}$. According to Corollary 1, we obtain

$$\rho(M^{-1}N) = 1 - \sigma(M^{-1}A) = 1 - \frac{1}{6} = \frac{5}{6}. \tag{4}$$

But for the splitting given in Example 3, it is obvious that

$$1 - \sigma(M^{-1}A) = 1 - \frac{1}{3} = \frac{2}{3},$$

and from $M^{-1}N = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & -1 \end{pmatrix}$ we know that $\rho(M^{-1}N) = 1$, that is,

$$\rho(M^{-1}N) \neq 1 - \sigma(M^{-1}A).$$

It occurs just because the condition $\rho(M^{-1}A) < 1$ does not hold, in fact, $\rho(M^{-1}A) = 2 > 1$ in this example.

THEOREM 2. Let $A = M - N$ be a regular reversible splitting of matrix A. Then the following inequality holds if $N^{-1} \geq 0$:

$$\rho(M^{-1}A) = \frac{\rho(N^{-1}A)}{1 + \rho(N^{-1}A)} < 1\,. \tag{5}$$

Conversely, if $\rho(M^{-1}A) < 1$, then $N^{-1} \geq 0$.

PROOF. From $A = M - N$ we have

$$M = N + A = N(I + N^{-1}A)$$

and

$$M^{-1}A = (I + N^{-1}A)^{-1}N^{-1}A. \tag{6}$$

Since $M^{-1}A \geq 0$, there exists a Perron vector $x \geq 0$ such that

$$M^{-1}Ax = \rho(M^{-1}A)x.$$

Now by equality (6), we have

$$\rho(M^{-1}A)x = (I + N^{-1}A)^{-1}N^{-1}Ax,$$

i.e., $\rho(M^{-1}A)$ is an eigenvalue of $(I + N^{-1}A)^{-1}N^{-1}A$. Hence

$$\rho(M^{-1}A) \leq \frac{\rho(N^{-1}A)}{1 + \rho(N^{-1}A)}. \tag{7}$$

On the other hand, since $N^{-1} \geq 0$ and $A \geq 0$, i.e., $N^{-1}A \geq 0$, there exists a Perron vector $y \geq 0$, such that

$$N^{-1}Ay = \rho(N^{-1}A)y.$$

On account of (6), we have

$$(M^{-1}A)y = \frac{\rho(N^{-1}A)}{1 + \rho(N^{-1}A)}y\,,$$

i.e., $\frac{\rho(N^{-1}A)}{1+\rho(N^{-1}A)}$ is also an eigenvalue of $M^{-1}A$, hence

$$\rho(M^{-1}A) \geq \frac{\rho(N^{-1}A)}{1 + \rho(N^{-1}A)}\,,$$

together with (7) and $\rho(N^{-1}A) > 0$, implies (5).

Conversely, from $A = M - N$ we have

$$N = M - A = M(I - M^{-1}A).$$

If $\rho(M^{-1}A) < 1$, then from Lemma 1 and Definition 5, it follows that

$$N^{-1} = (I - M^{-1}A)^{-1}M^{-1} = (\sum_{i=0}^{\infty}(M^{-1}A)^i)M^{-1} \geq 0.$$

As an immediate consequence of Theorem 1 and Theorem 2, we obtain the following result.

COROLLARY 2. Let $A = M - N$ be a regular reversible splitting of matrix A. If $N^{-1} \geq 0$, then $\rho(M^{-1}N) < 1$.

EXAMPLE 6. Let $A = \begin{pmatrix} 1 & \frac{2}{3} \\ 0 & 1 \end{pmatrix} = M - N$, where

$$M = \begin{pmatrix} 2 & -\frac{4}{3} \\ 0 & 2 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

It is easy to know that the decomposition $A = M - N$ is a regular reversible splitting, and $N^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \geqslant 0$ means that the splitting is convergent. In fact, we have $\rho(M^{-1}N) = \dfrac{1}{2} < 1$.

Similar to the proof of Theorem 2, from

$$N = M - A = M(I - M^{-1}A) = (I - AM^{-1})M\,,$$

i.e.,

$$N^{-1} = (I - M^{-1}A)^{-1}M^{-1} = M^{-1}(I - AM^{-1})^{-1},$$

we can get the following results.

THEOREM 3. If $A = M - N$ is a nonnegative reversible splitting of matrix A, then $\rho(M^{-1}A) < 1$ if and only if $N^{-1} \geqslant 0$.

COROLLARY 3. Let $A = M - N$ be a nonnegative reversible splitting of matrix A. If $N^{-1} \geqslant 0$, then $\rho(M^{-1}N) < 1$.

EXAMPLE 7. Consider the nonnegative splitting given in Example 1, where $N^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{15} \end{pmatrix} \geqslant 0$. By Corollary 3, we know that the splitting is convergent (in fact, we have equality (4)).

4 Remarks

As described by Theorem 2 and Theorem 3, the splittings defined in the first two items of Definition 5 are convergent if $N^{-1} \geqslant 0$, but it is not a sufficient condition to ensure the convergence of weak and weaker reversible splittings of matrix A, moreover, these two types of reversible splittings can also be convergent even when $N^{-1} \ngeqslant 0$. For instance, in Example 3, though $N^{-1} \geqslant 0$, the splitting $A = M - N$ does not converge because we have $\rho(M^{-1}N) = 1$. But in Example 4,

$$N^{-1} = \begin{pmatrix} -\frac{14}{9} & -\frac{67}{18} & 3 \\ -\frac{10}{27} & -\frac{22}{27} & \frac{2}{3} \\ \frac{1}{54} & \frac{2}{27} & 0 \end{pmatrix} \ngeqslant 0\,,$$

the splitting is convergent because

$$\rho(M^{-1}N) = 1 - \sigma(M^{-1}A) = 1 - \frac{1}{7} = \frac{6}{7} < 1.$$

Therefore we need to find other sufficient conditions for a weak (weaker) splitting in the future.

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