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Convergence Of Solutions Of $x_{n+1} = x_n x_{n-1} - 1$*

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Abstract

Our aim in this paper is to investigate the convergence of solutions of the nonlinear difference equation $x_{n+1} = x_n x_{n-1} - 1,\ n = 1, 2, \ldots$, where the initial conditions x_{-1}, x_0 are in the interval $(-1, 0)$. We show that every solution of this equation converges to the unique negative equilibrium $\bar{x} = (1 - \sqrt{5})/2$.

1 Introduction

Recently there has been some interest in the study of nonlinear and rational difference equations, see, e.g. [3–8]. The results of these equations serve as prototypes in the development of the basic theory of the nonlinear difference equations. Also the techniques and results about these equations are useful to analyze the equations in the mathematical models of biological systems, economics and other applications, see, for instance, [9–11].

In this paper we consider the second-order difference equation

$$x_{n+1} = x_n x_{n-1} - 1,\ n = 1, 2, \ldots, \tag{1}$$

where the initial conditions x_{-1}, x_0 are in the interval $(-1, 0)$.

The long-term behavior of solutions of (1) was systematically investigated in [1]. In particular, the properties of the boundedness, periodic behaviors of solutions, and the dependence on initial conditions are examined. A question is also raised:

Question ([1]). If $-1 < x_{-1}, x_0 < 0$, show whether or not every solution $\{x_n\}_{n=-1}^{\infty}$ of (1) converges to the negative equilibrium $\bar{x} = (1 - \sqrt{5})/2$.

Our goal in this paper is to find the answer to this question. Before stating our result, we list a lemma which is useful to prove our theorem.

LEMMA 1.1 ([1]). If $-1 < x_{-1}, x_0 < 0$, then $-1 < x_n < 0$ for all $n \geq -1$.

We claim that (1) is equivalent to the following difference equation

$$y_{n+1} = 1 - y_n y_{n-1}, \tag{2}$$

where the initial conditions satisfy that $y_{-1} = -x_{-1},\ y_0 = -x_0$. Indeed, it is easy to check that $y_n = -x_n$ and the interval $(0, 1)$ is invariant. Now we only need to investigate the convergence of solutions of (2), with the initial conditions y_{-1}, y_0 in the interval $(0, 1)$.

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2 Convergence of Solutions of (2)

Motivated by the methods used in [2, 6–8], in this section, we verify the convergence of solutions of (2). We have the following result.

THEOREM 2.1. Every solution of (2) with initial conditions in $(0,1)$ converges to the positive equilibrium $\bar{y} = (\sqrt{5}-1)/2$.

PROOF. Let

$$m_0 = \min\{y_{-1},\ y_0,\ y_1\},\ M_0 = \max\{y_{-1},\ y_0,\ y_1\}. \tag{3}$$

Then we shall show that $m_0 \le \bar{y}$, and $M_0 \ge \bar{y}$. For the former case, suppose to the contrary that $m_0 > \bar{y}$. Then $y_1 = 1 - y_0 y_{-1} \le 1 - m_0^2 < 1 - \bar{y}^2 = \bar{y} < m_0$, which contradicts the fact that $y_1 \ge m_0$. Similarly, we can also get $M_0 \ge \bar{y}$.

Now we claim that $m_0 \le y_2 \le M_0$. Firstly, to prove $m_0 \le y_2$, we divide it into three cases:

Case 1. If $m_0 = y_{-1}$, then

$$\begin{aligned} y_2 - m_0 &= y_2 - y_{-1} = 1 - y_1 y_0 - y_{-1} = 1 - y_0(1 - y_0 y_{-1}) - y_{-1} \\ &= (1-y_0)(1 - y_{-1} - y_0 y_{-1}) = (1-y_0)(y_1 - y_{-1}) = (1-y_0)(y_1 - m_0) \\ &\ge 0. \end{aligned} \tag{4}$$

Case 2. If $m_0 = y_0$, then we have that $y_0 \le y_{-1}$, $y_0 \le y_1 = 1 - y_0 y_{-1}$, therefore,

$$y_0 \le \min\left\{y_{-1},\ \frac{1}{1+y_{-1}}\right\}, \tag{5}$$

$$y_2 - m_0 = y_2 - y_0 = 1 - 2y_0 + y_{-1} y_0^2. \tag{6}$$

If $y_{-1} \le \frac{-1+\sqrt{5}}{2} = \bar{y}$, then $y_{-1} \le \frac{1}{1+y_{-1}}$. Therefore, (5) is equivalent to

$$y_0 \le y_{-1} \le \bar{y}.$$

Let

$$g(y_0) = 1 - 2y_0 + y_{-1} y_0^2, \tag{7}$$

then $g(y_0)$ is a quadratic function where y_{-1} is regarded as a parameter. Since its symmetry axis is $\frac{2}{2y_{-1}} = \frac{1}{y_{-1}} > 1$ and the condition (5) holds, we can obtain

$$g(y_0) \ge g(y_{-1}) = 1 - 2y_{-1} + y_{-1}^3. \tag{8}$$

Set $f(y) = 1 - 2y + y^3$ and compute its derivative

$$f'(y) = -2 + 3y^2,$$

for any $y \in [y_{-1}, \bar{y}]$, we have

$$f'(y) = -2 + 3y^2 \le -2 + 3\bar{y}^2 = -2 + 3\left(\frac{\sqrt{5}-1}{2}\right)^2 = \frac{5-3\sqrt{5}}{2} < 0,$$

which means that $f(y)$ is decreasing on the interval $[y_{-1}, \bar{y}]$, thus

$$g(y_0) \geq g(y_{-1}) \geq f(\bar{y}) = 1 - 2\frac{\sqrt{5}-1}{2} + \left(\frac{\sqrt{5}-1}{2}\right)^3 = 0. \tag{9}$$

And by (6) and (7), we have that $y_2 \geq m_0$.

If $y_{-1} \geq \frac{-1+\sqrt{5}}{2} = \bar{y}$, then $y_{-1} \geq \frac{1}{1+y_{-1}}$. In this case, (5) is equivalent to

$$y_0 \leq \frac{1}{1+y_{-1}} \leq \frac{1}{1+\bar{y}} = \bar{y}. \tag{10}$$

Similarly, we can also have that $y_2 \geq m_0$. Therefore, if $m_0 = y_0$, then $y_2 \geq m_0$.

Case 3. If $m_0 = y_1$,

$$y_2 - m_0 = y_2 - y_1 = 1 - y_1 y_0 - (1 - y_0 y_{-1}) = y_0(y_{-1} - y_1) \geq 0. \tag{11}$$

Hence, we have proved that $m_0 \leq y_2$.

Then, to prove $y_2 \leq M_0$, we also divide it into three cases in a similar way as above.

Case 1. If $M_0 = y_{-1}$, then

$$\begin{aligned} y_2 - M_0 &= y_2 - y_{-1} = 1 - y_1 y_0 - y_{-1} = 1 - y_0(1 - y_0 y_{-1}) - y_{-1} \\ &= (1 - y_0)(1 - y_{-1} - y_0 y_{-1}) = (1 - y_0)(y_1 - y_{-1}) = (1 - y_0)(y_1 - M_0) \\ &\leq 0. \end{aligned} \tag{12}$$

Case 2. If $M_0 = y_0$, we have that $y_{-1} \leq y_0$, and $y_1 = 1 - y_0 y_{-1} \leq y_0$, then

$$y_0 \geq \max\left\{y_{-1}, \ \frac{1}{1+y_{-1}}\right\}. \tag{13}$$

Note that $\bar{y}$ is the only positive solution of equation $y_{-1} = \frac{1}{1+y_{-1}}$. Since the function $f_1(x) = x$ is increasing and $f_2(x) = \frac{1}{1+x}$ is decreasing on the interval $(0, 1)$, we get that

$$\max\left\{y_{-1}, \ \frac{1}{1+y_{-1}}\right\} \geq \bar{y}. \tag{14}$$

Thus, $\bar{y} \leq y_0 < 1$. If $y_{-1} \leq \bar{y}$, then using the property of the quadratic equation, we have

$$y_2 - M_0 = y_2 - y_0 = 1 - 2y_0 + y_{-1}y_0^2 \leq 1 - 2y_0 + \bar{y}y_0^2 \leq 1 - 2\bar{y} + \bar{y}^3 = 0. \tag{15}$$

If $y_{-1} \geq \bar{y}$, then $y_0 \geq y_{-1} \geq \bar{y}$. Let

$$\overline{g}(y_0) = y_2 - M_0 = 1 - 2y_0 + y_{-1}y_0^2, \tag{16}$$

where y_{-1} is a parameter.

The symmetry axis of $\overline{g}(y_0)$ is $y_0 = \frac{1}{y_{-1}} > 1$. For $\bar{y} \leq y_0 \leq 1$, we have that

$$\overline{g}(y_0) \leq 1 - 2\bar{y} + y_{-1}\bar{y}^2 = y_{-1}(\frac{1-2\bar{y}}{y_{-1}} + \bar{y}^2) \leq y_{-1}(\frac{1-2\bar{y}}{\bar{y}} + \bar{y}^2) = 0. \tag{17}$$

Case 3. If $M_0 = y_1$,

$$y_2 - M_0 = y_2 - y_1 = 1 - y_1 y_0 - (1 - y_0 y_{-1}) = y_0(y_{-1} - y_1) \leq 0. \tag{18}$$

Clearly, for any of the above three cases, we all have that $y_2 \leq M_0$. Therefore,we have proved that $m_0 \leq y_2 \leq M_0$.

Let

$$m_i = \min\{y_{i-1},\ y_i,\ y_{i+1}\}, M_i = \max\{y_{i-1},\ y_i,\ y_{i+1}\}, \tag{19}$$

for $i = 1, 2, \cdots$.

From the above proof, one can observe that

$$m_0 \leq m_1 \leq \cdots \leq m_i \leq \cdots \leq \cdots \leq M_i \leq \cdots \leq M_1 \leq M_0, \tag{20}$$

and

$$m_i \leq y_{i+2} \leq M_i, \tag{21}$$

for $i = 0, 1, 2, \cdots$.

Since $\{m_i\}_{i=1}^{\infty}$ is a monotonically increasing sequence with an upper bound, it converges to some real number. Similarly, the sequence $\{M_i\}_{i=1}^{\infty}$ converges, for it is monotonically decreasing and has a lower bound.

Let

$$m = \lim_{i \leftarrow \infty} m_i,\ M = \lim_{i \leftarrow \infty} M_i. \tag{22}$$

We claim that

$$m = M. \tag{23}$$

Suppose not, then there must exist a period-two or period-three solution of (2). When (2) has a period-two solution, we have that

$$\begin{cases} m = 1 - Mm \\ M = 1 - mM \end{cases} \tag{24}$$

When (2) has a period-three solution, then there must be some c with $m \leq c \leq M$, thus we have that

$$\begin{cases} m = 1 - Mc \\ c = 1 - Mm \\ M = 1 - mc \end{cases} \tag{25}$$

For the two cases, we both have that $m = M = (\sqrt{5} - 1)/2$.

Therefore, with the initial conditions y_{-1}, y_0 in the interval $(0, 1)$, every solution $\{y_i\}_{i=-1}^{\infty}$ of (2) converges to $\overline{y}$. This completes the proof of the theorem.

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