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Nonclassical Symmetry Of A Hyperbolic System And Its Invariant*

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Abstract

Non-classical symmetry $\mathbf{V} = \partial_t + \xi\partial_x + \sum_{i=1}^{n}\phi_i$ admitted by a system of hyperbolic partial differential equations $\mathbf{u}_t + \mathbf{M}\cdot\mathbf{u}_x = 0$ is considered. Non-classical symmetry is divided into two classes: the first is the case in which ξ is an eigenvalue of the matrix $\mathbf{M}$. We understand the first case well. The latter is the case in which ξ is not an eigenvalue. In this paper, we focus on the latter case and present a simple method for constructing an infinitesimal generator for the symmetry and, incidentally, produce its invariant.

1 Introduction

A hyperbolic system of partial differential equations plays an important role in the mathematical physics of wave phenomena. Wave phenomena in compressible, non-viscous fluids are governed by a hyperbolic system which consists of mass and momentum conservation under the adiabatic assumption. The characteristic curve method and the associative invariant, called the Riemann invariant, provide a simple mathematical framework for understanding wave propagation. Generally, the Riemann invariant exists only in the two-component system [1, 2, 3]. Consequently, the framework cannot be applied to multi-component systems. The purpose of the study is to seek an alternative method for constructing an invariant for the hyperbolic system.

Recently, the relationship between the Riemann invariant and non-classical symmetry analysis for the hyperbolic system in a conservation form have been reported [4, 5]. The concept of non-classical symmetry was originally proposed by Blumann and Cole to enlarge the class of Lie point symmetries [6, 7, 8]. The infinitesimal generator characterizes non-classical symmetry; it is determined by solving the determining equations. It is difficult to find the infinitesimal generator because the determining equations are nonlinear. However, looking from the viewpoint of symmetry may enable us to generalize the characteristic curve method for application to the multi-component system. The determining equations can be obtained by a systematic procedure regardless of the number of components.

For a system written in the conservation form ($\mathbf{u}_t + \mathbf{M}(\mathbf{u})\cdot\mathbf{u}_x = \mathbf{0}$) and an infinitesimal generator ($\mathbf{V} = \partial_t + \xi\partial_x + \Sigma\phi_i\partial_{u_i}$), the class of non-classical symmetry can be classified into:

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Case 1: The coefficient ξ is an eigenvalue of the matrix $\mathbf{M}$ and the coefficients $\phi_i = 0$.

Case 2: The coefficient ξ is not an eigenvalue of the matrix $\mathbf{M}$.

Non-classical symmetry analysis naturally comprises the characteristic curve method and leads the Riemann invariant (i.e. the case 1). On the other hand, case 2 indicates an alternative approach of the characteristic curve method and its invariant. In this paper, we focus on the latter case and provide a proof of a theorem which provides a simple method for constructing an infinitesimal generator by matrix calculation.

2 Nonclassical Symmetry of a Hyperbolic System

Let us consider a one-dimensional hyperbolic system in the conservation form:

$$\mathbf{u}_t + \mathbf{M}(\mathbf{u}) \cdot \mathbf{u}_x = \mathbf{0}, \tag{1}$$

where $\mathbf{u} = (u_1, ..., u_n)^T$ and $\mathbf{M}$ is an $n \times n$ regular matrix whose elements depend on the variables $u_1, ..., u_n$.

Non-classical symmetry is characterized by the following infinitesimal generator:

$$\mathbf{V} = \partial_t + \xi(x, t, \mathbf{u})\partial_x + \sum_{i=1}^{n} \phi_i(x, t, \mathbf{u})\partial_{u_i}, \tag{2}$$

and the constraints of Eq.(1) are imposed by an invariant condition which is associated with generator (2)

$$\mathbf{u}_t + \xi \mathbf{u}_x - \boldsymbol{\omega} = \mathbf{0}, \tag{3}$$

where $\boldsymbol{\omega}$ denotes a column vector $\boldsymbol{\omega} = (\phi_1, ..., \phi_n)^T$. As the coefficient ξ is not an eigenvalue of the matrix $\mathbf{M}$, the vector $\boldsymbol{\omega}$ is not a zero vector [5] and the matrix $\mathbf{M} - \xi\mathbf{E}$ is regular, where the matrix $\mathbf{E}$ denotes the unit matrix.

Substituting Eq.(3) into Eq.(1), we obtain

$$\mathbf{u}_x + \mathbf{N} \cdot \boldsymbol{\omega} = \mathbf{0}, \tag{4}$$

or, for α-th component,

$$u_{\alpha x} + \sum_{\beta=1}^{n} N_{\alpha\beta}\phi_\beta = 0, \tag{5}$$

where the matrix $\mathbf{N}$ is the inverse matrix of one of $\mathbf{M} - \xi\mathbf{E}$, and $N_{\alpha\beta}$ is its (α, β) component.

As the determining equations of Eq.(3) are satisfied automatically, we only consider those of Eq.(5)

$$D_x\phi_\alpha + (D_x\xi)\sum_{\beta=1}^{n} N_{\alpha\beta}\phi_\beta + \mathbf{V}\left(\sum_{\beta=1}^{n} N_{\alpha\beta}\phi_\beta\right) = 0. \tag{6}$$

Here, we note that the compatibility condition of Eq.(3) and Eq.(4) are satisfied, due to Eq.(6).

To simplify Eq.(6), we introduce a vector $\tilde{\boldsymbol{\omega}}$ defined as

$$\tilde{\omega}_\alpha = \sum_{\beta=1}^{n} N_{\alpha\beta}\phi_\beta. \tag{7}$$

The vector $\tilde{\boldsymbol{\omega}}$ is not a zero vector because $\boldsymbol{\omega} \neq 0$. We assume that

$$\frac{\partial \xi}{\partial x} = 0, \qquad \frac{\partial \phi_\alpha}{\partial x} = 0, \tag{8}$$

then Eq.(6) becomes

$$\sum_{\beta=1}^{n} \tilde{\omega_\beta}\left(\frac{\partial \phi_\alpha}{\partial u_\beta} + \tilde{\omega_\alpha}\frac{\partial \xi}{\partial u_\beta}\right) = \mathbf{V}\left(\tilde{\omega_\alpha}\right). \tag{9}$$

Let us split Eq.(9) into two equations

$$\sum_{\beta=1}^{n} \tilde{\omega_\beta}\left(\frac{\partial \phi_\alpha}{\partial u_\beta} + \tilde{\omega_\alpha}\frac{\partial \xi}{\partial u_\beta}\right) = 0, \tag{10}$$

and

$$\mathbf{V}\left(\tilde{\omega_\alpha}\right) = 0. \tag{11}$$

We assume that $\boldsymbol{\omega}$ is an eigenvector of the matrix $\mathbf{N}$:

$$\tilde{\boldsymbol{\omega}} = \mathbf{N} \cdot \boldsymbol{\omega} = \lambda\omega, \tag{12}$$

where λ is an eigenvalue of the matrix $\mathbf{N}$. Substituting Eq.(11) into Eq.(10), we get

$$\sum_{\beta=1}^{n} \lambda^2 \phi_\alpha \phi_\beta \frac{\partial}{\partial u_\beta}\left(\frac{1}{\lambda} + \xi\right) = 0. \tag{13}$$

Consequently, it is easy to see the following theorem.

THEOREM. Let us consider the infinitesimal generator

$$\tilde{\mathbf{V}} = \partial_t - \frac{1}{\lambda}\partial_x + \sum_{i=1}^{n} \phi_i(t, \mathbf{u})\partial_{u_i}, \tag{14}$$

The vector $\tilde{\omega}$ is an invariant, then $\tilde{\mathbf{V}}$ is non-classical symmetry admitted by Eq.(1).

3 Example of the Two Component System

Let us consider the two-component system to illustrate our approach

$$\mathbf{u}_t + \mathbf{M} \cdot \mathbf{u}_x = 0, \tag{15}$$

where

$$\mathbf{u} = (\rho, u)^T, \qquad \mathbf{M} = \begin{pmatrix} u & \rho \\ f(\rho) & u \end{pmatrix}. \tag{16}$$

The conventional characteristic curve is $\mathbf{V} = \partial_t + (u \pm \sqrt{\rho f})\partial_x$.

We seek non-classical symmetry based on the theorem

$$\mathbf{V} = \partial_t + \xi\partial_x + \phi(t, \mathbf{u})\partial_\rho + \psi(t, \mathbf{u})\partial_u. \tag{17}$$

The simplest case is that the vector $\tilde{\omega}$ is a constant vector, and then non-classical symmetry is given as

$$\tilde{\mathbf{V}} = \frac{1}{\lambda}\Big(\lambda\partial_t - \partial_x + c_1\partial_\rho + c_2\partial_u\Big). \tag{18}$$

where $\tilde{\omega} = (c_1, c_2)^T$. The eigenvalue λ of the matrix $\mathbf{N}$ is

$$\lambda = \frac{1}{u \pm \sqrt{\rho f}}. \tag{19}$$

Consequently, another characteristic curve and its invariant are $\tilde{\mathbf{V}}$ given by (18) and c_i $(i = 1, 2)$; respectively.

4 Summary

In this paper, we provide a simple theorem to seek an invariant of non-classical symmetry admitted by Eq.(1). The theorem simply requires calculation of an eigenvalue and the associated eigenvector of the matrix $\mathbf{N}$. Therefore, it is not necessary to calculate the determining equations when constructing an invariant. As an example, non-classical symmetry of the two-component hyperbolic system and its invariant are provided based on the theorem.

References

[1] R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience publishers, INC., New York, 1948.

[2] T. Taniuchi and K. Nichihara, Nonlinear Waves, Pitman Advanced Publishing Program, 1983.

[3] A. M. Grandland and L. Martina, On fluid dynamics equations from the point of view of the symmetry group reduction and Riemann invariants methods, Proceedings of the Annual Seminar of the Canadian Mathematical Society on Lie Theory, Differential Equations and Representation Theory, (1989) 181.

[4] M. Souichi, Non-classical symmetry and Riemann invariants, Internat. J. Non-Linear Mech., 41(2006), 242–246.

[5] M. Souichi, Non-classical symmetry analysis for hyperbolic partial differential equation, Communications in Nonlinear Science and Numerical Simulation,, 13(2008), 1472–1474.

[6] G. W. Bluman and J. D. Cole, The general similarity solution of the heat equation, J. Math. Mech., 18(1969), 1025–1042.

[7] P. J. Olver, Application of Lie Groups to Differential Equations, Springer-Verlag, 1986.

[8] G. Baumann, G. Hagger and T. F. Nonnenmacher, Applications of non-classical symmetries, J. Phys. A., 27(1994), 6479–6493.