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A System Of Generalized Burgers' Equations With Boundary Conditions*

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Abstract

We construct explicit solutions for a system of generalized Burgers equation in the quater plane $\{(x,t) : x > 0, t > 0\}$ with initial and boundary conditions when viscosity is present and also when viscosity is zero.

1 Introduction

Initial value problem for the system of first order equation,

$$(u_j)_t + \left(\sum_{k=1}^{n} c_k u_k\right)(u_j)_x = \frac{\epsilon}{2}(u_j)_{xx}, \;\; j = 1, 2, ..., n \tag{1}$$

where c_k are real constants and $\epsilon > 0$, was studied by by Joseph [3]. Using a generalized Hopf-Cole transformation, explicit solution was constructed for each $\epsilon > 0$ and weak solutions were constructed for the generalized Hopf equation

$$(u_j)_t + \left(\sum_{k=1}^{n} c_k u_k\right)(u_j)_x = 0 \tag{2}$$

by passing to the limit as ϵ goes to 0.

There are two important special cases of the system (1). When $n = 1, c_1 = 1$ and with $u_1 = u$, (1) is just the Burgers' equation

$$u_t + uu_x = \frac{\epsilon}{2}u_{xx},$$

which was studied by Hopf [2]. The second important case is when $n = 2, c_1 = 1, c_2 = 0$, setting $u_1 = u$ and $u_2 = v$, the system (1) becomes

$$u_t + uu_x = \frac{\epsilon}{2}u_{xx}, \;\; v_t + uv_x = \frac{\epsilon}{2}v_{xx}$$

and then by taking the derivative of the second equation w.r.t. x and taking $\rho = v_x$ we get the one dimensional adhesion model for large scale structures.

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Our aim in this paper is to get explicit formula for the solution of (1) in the quarter plane $\Omega = [0, \infty) \times [0, \infty)$ with initial condition

$$u_j(x, 0) = u_{0j}(x), \;\; x > 0, \tag{3}$$

and boundary condition

$$u_j(0, t) = u_{Bj}(t), \;\; t > 0. \tag{4}$$

Also we construct explicit solution of initial boundary value problem for (2), for special type of boundary data.

It is well-known that with a strong form of boundary conditions (4), existence of solutions is not guaranteed for the system (2). So the vanishing viscosity limit satisfies boundary conditions only in a weak sense.

In section 2, we construct explicit formula for solution of initial value problem for (1) with initial conditions (3) and boundary conditions (4) with viscous parameter $\epsilon > 0$ when u_{Bj} are constants. In section 3, we construct exact solution of (2) with general initial data and homogeneous boundary data by passing to the limit as ϵ goes to 0. Finally we get an explicit formula for (2) with Riemann type initial and boundary data.

2 Explicit Solution with Viscous Term

In this section we construct explicit formula for the initial boundary value problem for the viscous system (1),(3) and (4). First we note that this problem is equivalent to finding $u_j, j = 1, 2, ..., n$, in a domain $\Omega = [0, \infty) \times [0, \infty)$

$$\begin{aligned} &(u_j)_t + \sigma (u_j)_x = \frac{\epsilon}{2}(u_j)_{xx}, x > 0, \;\; t > 0 \\ &u_j(x, 0) = u_{0j}(x), \;\; x > 0 \\ &u_j(0, t) = u_{Bj}, \;\; t > 0 \end{aligned} \tag{5}$$

where

$$\sigma = \sum_{k=1}^{n} c_k u_k \tag{6}$$

satisfies the Burgers equation with corresponding initial and boundary conditions, namely

$$\begin{aligned} &\sigma_t + \frac{1}{2}(\sigma^2)_x = \frac{\epsilon}{2}\sigma_{xx}, \\ &\sigma(x, 0) = \sigma_0(x), \\ &\sigma(0, t) = \sigma_B(t). \end{aligned} \tag{7}$$

Here σ_0 and σ_B are given by

$$\sigma_0(x) = \sum_{k=1}^{n} c_k u_{0k}(x), \;\; \sigma(0, t) = \sigma_B(t) = \sum_{k=1}^{n} c_k u_{Bk}. \tag{8}$$

For future use, it is convenient to introduce the function

$$w_0(x) = \int_0^x \sigma_0(y)dy. \tag{9}$$

We use a generalized Hopf-Cole transformation to solve the system (5).

THEOREM 1. Let $u_{0j}(x)$ be bounded and measurable, u_{Bj}, $j = 1, 2, ..., n$ are constants, and let a^ϵ and ${a_j}^\epsilon$ be given by

$$\begin{aligned} a^\epsilon(x,t) = {} & \frac{1}{(2\pi t\epsilon)^{1/2}} \int_0^\infty (e^{-\frac{1}{\epsilon}\{\frac{(x-y)^2}{2t}+w_0(y)\}} + e^{-\frac{1}{2\epsilon}\{\frac{(x+y)^2}{2t}+w_0(y)\}}dzdy \\ & + \frac{2\sigma_B/\epsilon}{(2\pi t\epsilon)^{1/2}} \int_0^\infty \int_0^\infty e^{-\frac{1}{\epsilon}\{(\frac{(x+y+z)^2}{2t}-\sigma_B z)+w_0(y)\}}dz \end{aligned} \tag{10}$$

$$\begin{aligned} {a_j}^\epsilon(x,t) = {} & \frac{1}{(2\pi t\epsilon)^{1/2}} \int_0^{+\infty} u_{0j}(y)(e^{\frac{-1}{\epsilon}\{\frac{(x-y)^2}{2t}+\}} - e^{-\frac{1}{\epsilon}\{\frac{(x+y)^2}{2t}+w_0 y\}})dy \\ & + \frac{u_{Bj}x}{(2\pi\epsilon)^{1/2}} \int_0^t \frac{a^\epsilon(0,s)}{(t-s)^{3/2}} e^{-\frac{1}{\epsilon}\frac{x^2}{2(t-s)}} ds. \end{aligned} \tag{11}$$

Then the functions

$$u_j^\epsilon(x,t) = \frac{a_j^\epsilon(x,t)}{a^\epsilon(x,t)}, j = 1, 2, 3, ..., n \tag{12}$$

are infinitely differentiable in the variables (x, t) and is the exact solution of the initial boundary value problem (5).

PROOF. First we note that if $w(x, t)$ is a solution of

$$\begin{aligned} & w_t + \frac{(w_x)^2}{2} = \frac{\epsilon}{2} w_{xx}, \\ & w(x,0) = w_0(x), \\ & w_x(0,t) = \sigma_B, \end{aligned} \tag{13}$$

in $\{(x, t) : x > 0, t > 0\}$, then

$$\sigma(x,t) = w_x(x,t) \tag{14}$$

is a solution of (7). We introduce new unknown variables $a, a_j, j = 1, 2, ..., n$. The unknown a is defined by the usual Hopf-Cole transformation and $a_j, j = 1, 2, 3, ..., n$, by a modified version of it in the following way

$$a = e^{-\frac{w}{\epsilon}}, a_j = u_j e^{-\frac{w}{\epsilon}}, j = 1, 2, 3, ..., n. \tag{15}$$

An easy calculation shows that

$$a_t - \frac{\epsilon}{2} a_{xx} = -\frac{1}{\epsilon}[w_t + \frac{(w_x)^2}{2} - \frac{\epsilon}{2} w_{xx}] e^{-\frac{w}{\epsilon}}, \tag{16}$$

and for $j = 1, 2, ..., n$

$$(a_j)_t - \frac{\epsilon}{2}(a_j)_{xx} = [(u_j)_t + \sigma(u_j)_x - \frac{\epsilon}{2} w_{xx}] e^{-\frac{w}{\epsilon}} - \frac{1}{\epsilon}[w_t + \frac{(w_x)^2}{2} - \frac{\epsilon}{2} w_{xx}] u_j e^{-\frac{w}{\epsilon}}. \tag{17}$$

It follows from (16) and (17) that u_j and w are solutions of (5) and (13) iff a and $a_j, j = 1, 2, ..., n$ are solutions of

$$\begin{aligned} &a_t = \frac{\epsilon}{2} a_{xx}, \\ &a(x,0) = e^{-\frac{1}{\epsilon} w_0(x)}, \\ &\epsilon a_x(0,t) + \sigma_B(t)) a(0,t) = 0 \end{aligned} \tag{18}$$

and

$$\begin{aligned} &(a_j)_t = \frac{\epsilon}{2} (a_j)_{xx}, \\ &a_j(x,0) = u_{0j}(x) e^{-\frac{1}{\epsilon} w_0(x)}, j = 1, 2, , ..., n, \\ &a_j(0,t) = a^\epsilon(0,t) u_{B_j}(t). \end{aligned} \tag{19}$$

When σ_B is a constant we can find explicit formula for the solution of (18) and (19) see [6]. The expressions for a^ϵ and a_j^ϵ are given by (10) and (11). Then substituting them in (15) we get the formula (12) for u_j^ϵ.

3 Explicit Formula for Vanishing Viscosity Limit

In this section we study the limit as ϵ goes to 0 of solutions (12), with homogeneous boundary conditions $u_{Bj}(t) = 0$ and show that this limit satisfies the generalized Hopf equation (2) with the initial conditions (3). The boundary condition (4) may not be satisfied in the strong sense.

THEOREM 2. Let $u_j^\epsilon, j = 1, 2, ..., n$ be the solution given by (12) with Lipschitz continuous initial conditions $u_0(x)$ which are bounded, and boundary data $u_{Bj} = 0$. Then the limit $u_j(x,t) = \lim_{\epsilon \to 0} u_j^\epsilon(x,t)$ exists a.e. $x > 0, t > 0$ and is given by

$$u_j(x,t) = \begin{cases} u_{0j}(y(x,t)), \text{ if } y(x,t) > 0, \\ \frac{u_{0j}(0)}{\sum_{k=1}^n u_{0k(0)}} x/t, \text{ if } y(x,t) = 0 \end{cases} \tag{20}$$

$j = 1, 2, ..., n$, where $y(x,t)$ is a minimizer of

$$\min_{y \geq 0} \left\{ w_0(y) + \frac{(x-y)^2}{2t} \right\}. \tag{21}$$

Further the limit functions $u_j, j = 1, 2, ..., n$ satisfy the equation (2) and the initial conditions (3).

PROOF. When $u_{Bj}(t) = 0$, the formula (10)-(12) becomes

$$u_j^\epsilon(x,t) = \frac{\int_0^\infty u_{0j}(y)(e^{-\frac{1}{\epsilon}\{\frac{(x-y)^2}{2t} + w_0(y)\}} - e^{-\frac{1}{2\epsilon}\{\frac{(x+y)^2}{2t} + w_0(y)\}}) dy}{\int_0^\infty (e^{-\frac{1}{\epsilon}\{\frac{(x-y)^2}{2t} + w_0(y)\}} + e^{-\frac{1}{2\epsilon}\{\frac{(x+y)^2}{2t} + w_0(y)\}}) dy}. \tag{22}$$

Let us introduce the functions $A(x,t)$ and $B(x,t)$ by

$$A(x,t) = \min_{y \geq 0} \left\{ w_0(y) + \frac{(x-y)^2}{2t} \right\}, \quad B(x,t) = \min_{y \geq 0} \left\{ w_0(y) + \frac{(x+y)^2}{2t} \right\}.$$

If the minimizer $y(x,t)$ in (21) is positive, then we must have $A(x,t) < B(x,t)$, so by Laplace asymptotic formula the first terms in the numerator and denominator of the expression (22) dominate and we get

$$u_j^\epsilon(x,t) \approx \frac{\int_0^\infty u_{0j}(y) e^{-\frac{1}{\epsilon}\{\frac{(x-y)^2}{2t}+w_0(y)\}} dy}{\int_0^\infty e^{-\frac{1}{\epsilon}\{\frac{(x-y)^2}{2t}+w_0(y)\}} dy}.$$

as ϵ goes to zero. Thus if $y(x,t) > 0$ we get

$$\lim_{\epsilon \to 0} u_j^\epsilon(x,t) = u_{0j}(y(x,t)). \tag{23}$$

Now consider the case when the minimizer $y(x,t) = 0$. In this case $A(x,t) = B(x,t)$ and $w_0'(0) - x/t \geq 0$. The set on which $w_0'(0) - x/t = 0$ is a line and has measure 0. On the set of points (x,t) with $w_0'(0) - x/t > 0$, use the following asymptotic formulae

$$\int_0^\infty u_{0j}(y) e^{-\frac{1}{\epsilon}\{\frac{(x-y)^2}{2t}+w_0(y)\}} \approx \frac{u_{0j}(0)\epsilon t}{-x + t w_0'(0)} e^{\frac{-1}{\epsilon}(w_0(0)+\frac{x^2}{2t})}$$

$$\int_0^\infty u_{0j}(y) e^{-\frac{1}{\epsilon}\{\frac{(x+y)^2}{2t}+w_0(y)\}} \approx u_{0j}(0) \frac{\epsilon u_{0j}(0) t}{x + t w_0'(0)} e^{\frac{-1}{\epsilon}(w_0(0)+\frac{x^2}{2t})}$$

in (22), to get

$$\lim_{\epsilon \to 0} u_j^\epsilon(x,t) = \frac{u_{0j}(0)}{w_0'(0)} x/t. \tag{24}$$

Since $w_0'(x) = \sum_{k=0}^n c_k u_{0k}(x)$, from (23) and (24) we have the formula (20) almost everywhere.

The fact that limit functions (20) satisfy the equation (2) and the initial conditions (3) follows exactly as in [3] and is omitted.

4 Boundary Riemann Problem

When u_{0j} are all constant then the limit (20) has a simple form. For $\sigma_0 = \sum_{j=0}^n u_{0j} > 0$,

$$u_j(x,t) = \begin{cases} \dfrac{u_{0j}}{\sigma_0} x/t, & if \ \ 0 < x < \sigma_0 t \\ u_{0j}, & if \ \ x > \sigma_0 t \end{cases}$$

and for $\sigma_0 \leq 0$,

$$u_j(x,t) = u_{0j},$$

which clearly does not satisfy the boundary condition $u_j(0,t) = 0$, when $\sigma_0 < 0$. In this section we use a weak formulation of the initial and boundary value problem for (2) and obtain an explicit formula when the data are of Riemann type, namely

$$u_j(x,0) = u_{0j}, \tag{25}$$

and boundary condition

$$u_j(0,t) = u_{Bj}, \tag{26}$$

where u_{0j} and u_{Bj} are constants for $j = 1, 2, ..., n$. For the weak formulation of the boundary condition, we note that the equation (2), can be written as

$$(u_j)_t + \sigma(u_j)x = 0, \;\; x > 0, \;\; t > 0, \tag{27}$$

where the characteristic speed $\sigma = \sum_{k=1}^{n} c_k u_k$ satisfies the inviscid Burgers' equation

$$\sigma_t + \left(\frac{\sigma^2}{2}\right)_x = 0, \;\; x > 0, \;\; t > 0. \tag{28}$$

It is well-known that for (28), a weak form of the boundary condition is required as characteristic speed of the system, σ does not have a definite sign at the boundary $x = 0$. For σ we take boundary conditions in the sense of Bardos, Leroux and Nedelec [1], which for the present case take the form

$$\sigma(0+,t) \in \{\sigma_B^+\} U(-\infty, -\sigma_B^+) \tag{29}$$

and initial condition

$$\sigma(x,0) = \sigma_0 \tag{30}$$

where $\sigma_0 = \sum_{k=1}^{n} c_k u_{0k}$ and $\sigma_B = \sum_{k=1}^{n} c_k u_{Bk}$. Explicit formula for the entropy weak solution σ of (28), (29) and (30) was obtained in [4]. The speed of the system (2) or equivalently (27) is σ and boundary condition $u_j(0+,t)$ is required only when the speed $\sigma(0+,t) > 0$. Thus (26) is replaced by the weak formulation of boundary condition

$$\text{if } \sigma(0+,t) > 0, \text{ then } u_j(0+,t) = u_{Bj}. \tag{31}$$

As earlier $\sigma_0 = \sum_{k=1}^{n} c_k u_{0k}$ and $\sigma_B = \sum_{k=1}^{n} c_k u_{Bk}$.

THEOREM 3. Explicit formula for the solution of (2) with initial conditions (25) and boundary conditions (31) is given by the following.

Case 1 $\sigma_0 = \sigma_r = \sigma_B > 0$**:**

$$u_j(x,t) = \begin{cases} u_{Bj}, \text{ if } x < \sigma_0 t, \\ u_{0j}, \text{ if } x > \sigma_0 t. \end{cases}$$

Case 2 $\sigma_0 = \sigma_B < 0$**:**

$$u_j(x,t) = u_{0j}.$$

Case 3 $0 \leq \sigma_B < \sigma_0$**:**

$$u_j(x,t) = \begin{cases} u_{Bj}, \text{ if } 0 \leq x < \sigma_B t, \\ \frac{u_{0j} - u_{Bj}}{\sigma_0 - \sigma_B} . \frac{x}{t} + \frac{u_{Bj}\sigma_0 - u_{0j}\sigma_B}{\sigma_0 - \sigma_B}, \text{ if } \sigma_B t < x < \sigma_0 t \\ u_{0j}, \text{ if } x > \sigma_0 t. \end{cases}$$

Case 4 $\sigma_B \leq 0 < \sigma_0$**:**

$$u_j(x,t) = \begin{cases} \frac{u_{0j}}{\sigma_0} x/t, \text{ if } 0 < x < \sigma_0 t \\ u_{0j}, \text{ if } x > \sigma_0 t. \end{cases}$$

and
Case 5 $\sigma_0 < \sigma_B$**,** $s = \frac{\sigma_0+\sigma_B}{2} > 0$**:**

$$u_j(x,t) = \begin{cases} u_{Bj}, \text{ if } x < st, \\ u_{0j}, \text{ if } x > st. \end{cases}$$

PROOF. The unique entropy weak solution of (28),(29) and (30) is known [4] and is given by
Case 1: $\sigma_0 = \sigma_B > 0$,

$$\sigma(x,t) = \sigma_0,$$

Case 2: $\sigma_0 = \sigma_B \leq 0$,

$$\sigma(x,t) = \sigma_0,$$

Case 3: $0 < \sigma_B < \sigma_0$,

$$\sigma(x,t) = \begin{cases} \sigma_B, \text{ if } x < \sigma_B t, \\ x/t, \text{ if } \sigma_B t < x < \sigma_0 t \\ \sigma_0, \text{ if } x > \sigma_0 t \end{cases}$$

Case 4: $\sigma_B < 0 < \sigma_0$,

$$\sigma(x,t) = \begin{cases} x/t, \text{ if } 0 < x < \sigma_0 t \\ \sigma_0, \text{ if } x > \sigma_0 t \end{cases}$$

Case 5: $\sigma_B < 0$ and $\sigma_0 \leq 0$,

$$\sigma(x,t) = \sigma_0$$

Case 6: $\sigma_0 < \sigma_B$ and $\sigma_B + \sigma_0 > 0$,

$$\sigma(x,t) = \begin{cases} \sigma_B, \text{ if } x < st, \\ \sigma_0, \text{ if } x > st \end{cases}$$

where $s = \frac{\sigma_0+\sigma_B}{2}$.

Now the equation (27) is a linear system with discontinuous coefficient σ with initial conditions (25) and boundary condition (31). To construct solution we use the method of characteristics and use the fact that solution is constant along characteristics.

In the cases 1 and 2, the equation for u_j is $(u_j)_t + \sigma(u_j)_x = 0$ with σ is constant through out $x > 0, t > 0$ and the formula follows easily.

In the cases, the equation for u_j is $(u_j)_t + \sigma(u_j)_x = 0$ with σ is takes the constant value σ_B in the region $0 < x < \sigma_B t$ and σ_0 in the region $x > \sigma_0 t$. Thus in the region $0 < x \leq \sigma_B t$, the characteristics from (x,t) drawn backward in time hit the boundary point at $(0, t - x/\sigma_B)$ and u_j constant and it is equal to u_{Bj}. Similarly in the region $x > \sigma_0 t$ u_j takes the constant value u_{0j}. Now in the region $\sigma_B t \leq x \leq \sigma_0 t$ the characteristics converge to the origin $(0,0)$. Here we fill the value of u_j smoothly as a rarefaction wave connecting u_{Bj} from left to u_{0j} on the right such that $\sigma = \sum_{k=1}^{n} c_k u_k = x/t$. An easy computation shows that $u_j(x,t) = \frac{u_{0j} - u_B j}{\sigma_0 - \sigma_B} . \frac{x}{t} + \frac{u_{Bj}\sigma_0 - u_{0j}\sigma_B}{\sigma_0 - \sigma_B}$ if $\sigma_{Bj} t < x < \sigma_0 t$.

The case 4 is similar to case 3. In the case 5, the speed of the characteristics σ are constants on either side of the shock $x = \frac{u_l+u_r}{2}t$ and impinge on the shock curve. The formula then follows by the method of characteristics on either side of the shock curve.

The proof that $u_j(x,t)$ satisfies the initial and boundary conditions is clear from the construction. As the functions $u_j(x,t), j = 1, 2, ..., n$ are not smooth, we need to justify the equation (2) in the sense of Volpert [7], which can be done exactly as in [3]. The details are omitted.

References

[1] C. Bardos, A. Y. Leroux and J. C. Nedelec, First order quasi-linear equations with boundary conditions, Comm. Part. Diff. Eqn., 4(1979), 1017–1034.

[2] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math., 13(1950), 201–230.

[3] K. T. Joseph, Exact solution of a system of generalized Hopf equations, J. Anal. Appl., 21(2002), 669–680.

[4] K. T. Joseph and P. L. Sachdev, Initial boundary value problems for scalar and vector Burgers equations, Studies in Appl. Math., 106(2001), 481–505.

[5] K. T. Joseph, Explicit solutions for a system of first-order partial differential equations, Electronic J. Diff. Eqn., 2008(157)(2008), 1–8.

[6] A. D. Polyanin, Hand book of Linear Partial Differential Equations for Engineers and Scientists, Chapman and Hall/CRC, Boca Raton, Florida, 2001.

[7] A. I. Volpert, The space BV and quasi-linear equations, Math USSR Sb, 2(1967), 225–267.