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Travelling Wave Solutions Of Burgers' Equation For Gee-Lyon Fluid Flows*

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Abstract

In this work we present some analytic and semi-analytic traveling wave solutions of a generalized Burger' equation for isothermal unidirectional flow of viscous non-Newtonian fluids obeying the Gee-Lyon nonlinear rheological equation. The solutions include the corresponding well-known traveling wave solution of the Burgers' equation for Newtonian flow as a special case. We also derive estimates of shock thickness for the non-Newtonian flows.

1 Introduction

In this work we derive a traveling wave solution to the following generalized Burgers' equation

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\phi^{-1}\left(\mu_0\frac{\partial u}{\partial x}\right) \tag{1}$$

where $\phi(t)=(1+ct^2)t,\ 0<c<\infty$. The solution can be written as the following:

$$\frac{\xi}{\mu_0}=\frac{2}{\rho(u_2-u_1)}\ln\left(\frac{u_2-u}{u-u_1}\right)-\frac{4}{\rho(u_2-u_1)}extra\left(b,\frac{2u-(u_2+u_1)}{u_2-u_1}\right) \tag{2}$$

where

$$extra(b,\nu)=\mathrm{Re}\left(\frac{\arctan(\frac{\nu}{\sqrt{-1-i\ \sinh(2b)}})}{\sqrt{-1-i\ \sinh(2b)}}\right) \tag{3}$$

in which the constant b is defined by $\sinh(2b)=\frac{8}{\sqrt{c}\rho(u_2-u_1)}$. It is well-known that for $c=0$, equation (1) is the classical Burgers' equation for Newtonian fluid flows and the traveling wave solution is

$$\frac{\xi}{\mu_0}=\frac{2}{\rho(u_2-u_1)}\ln(\frac{u_2-u}{u-u_1})$$

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satisfying the upstream and downstream boundary conditions

$$\lim_{\xi \to +\infty} u(\xi) = u_1 \,, \qquad \lim_{\xi \to -\infty} u(\xi) = u_2, \qquad \lim_{|\xi| \to +\infty} \frac{du}{d\xi}(\xi) = 0$$

with $\xi = x - \lambda t, \quad \lambda = \frac{u_1 + u_2}{2}$.

It is interesting to note the if the second term in our solution (2) is dropped, the first term coincides with the classical solution. So the solution to the Non-Newtonian flow equals the solution to the Newtonian flow plus an extra term "$extra(b, \nu)$". We also show that using the first order approximation, the thickness δ of the transition layer between upstream and downstream can be given by $\delta = \frac{8\mu_0}{\rho(u_2 - u_1)\left\{1 + c[\frac{\rho}{8}(u_2 - u_1)]^2\right\}}$ which for $c = 0$ gives the corresponding classical estimate $\delta = \frac{8\mu_0}{\rho(u_2 - u_1)}$ for Newtonian fluid flows. Similar results for power-law flows have been established in [13]. Although the profiles of the transition layer for both power-law flows and Gee-Lyon flows look similar, the mathematical solutions describing these profiles are quite different.

2 The Generalized Burgers' Equation

The general Navier-Stokes equation for incompressible viscous flows is given by

$$\rho \frac{Du}{Dt} = \mathrm{div}(\sigma) - \bigtriangledown p + \mathbf{g} \tag{4}$$

where $u = (u_1, u_2, u_3)$ is the fluid velocity,

$$\sigma = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}, \qquad \text{and } Du = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

are the stress tensor and the strain tensor, ρ is the density, $\mathbf{g}$ the external force, p the scalar pressure, and $d_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, $1 \le i, j \le 3$. For unidirectional flows,we assume that $u = (u_1, 0, 0)$, $\tau_{ij} = 0$ for $i \neq 1$ or $j \neq 1$, $\mathbf{g} = (g_1, 0, 0)$, and $\bigtriangledown p = (\frac{\partial p}{\partial x_1}, 0, 0)$. The Navier-Stokes equation (4), in this case, takes the following simple scalar form

$$\frac{Du_1}{Dt} = \frac{d\tau_{11}}{dx_1} - \frac{\partial p}{\partial x_1} + g_1 \tag{5}$$

where $\frac{Du_1}{Dt} = \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1}$. Rheological relationships between σ and Du are frequently used to determine the type of fluids. Polyethylene and polystyrene melts can be described approximately by a rheological equation proposed by Rabinowitch and later generalized by Gee and Lyon [12], taking into account that the viscosity of these fluids depends highly on the temperature and the high stress levels. The rheological equation proposed by Gee and Lyon is given by

$$\mu_0 d_{ij} = (\delta_{ij} + c\,|\tau_{kl}\tau_{lk}|^{\frac{n}{2}})t_{ij}, \; 1 \le i, j \le 3 \tag{6}$$

where μ_0, n, and c are constants, $\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$, see [2], [10] or [12]. The temperature dependence of the viscosity is expressed by $\mu_0 = Ae^{\frac{E}{RT}}$. In this work, we refer to fluid flow satisfying the rheological equation (6) as Gee-Lyon flows.

If $c = 0$, then the fluid is said to be a Newtonian fluid; it is non-Newtonian if $c \neq 0$. For many important industrial polymer fluids, the values of A, E, R, c and n have been experimentally determined. For unidirectional flows, the rheological equation (6) reduces to $\mu_0 d_{11} = (1 + c\,|\tau_{11}|^n\,\tau_{11}$. Let u_1, x_1, g_1 be denoted by u, x, g respectively. Then from (5), let $-\frac{\partial p}{\partial x} + g = 0$, we have the generalized Burgers' equation

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\phi^{-1}\left(\mu_0\frac{\partial u}{\partial x}\right) \tag{7}$$

where $\phi(t) = (1 + c\,|t|^n)t$, $0 < c < \infty$. Equation (7) is referred to as the generalized Burgers' equation for Gee-Lyon flows. For $c = 0$, $\nu = \frac{\mu_0}{\rho}$, (7) reduces to Burgers' equation for Newtonian flows

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}. \tag{8}$$

It is well known that if we impose $\lim_{\xi\to+\infty} u(\xi) = u_1$, $\lim_{\xi\to-\infty} u(\xi) = u_2$, $\lim_{|\xi|\to+\infty} \frac{du}{d\xi}(\xi) = 0$, and $u_1 < u_2$, (8) has the celebrated traveling wave solution $\frac{\xi}{\mu_0} = \frac{2}{\rho(u_2-u_1)}\ln\frac{u_2-u}{u-u_1}$, which is equivalent to

$$u(\xi) = \frac{u_1 + u_2\exp[-\frac{\xi}{2\nu}(u_2 - u_1)]}{1 + \exp[-\frac{\xi}{2\nu}(u_2 - u_1)]} \tag{9}$$

where $\xi = x - \lambda t$, u_1 and u_2 are the downstream and upstream fluid velocities.

It can be shown that there exists a thin transition layer of thickness δ of order $\frac{8\nu}{u_2-u_1}$ for (9). This thickness δ can be referred to as the shock thickness, which tends to zero as $\nu \to 0$, and for fixed ν, $\delta \to \infty$, as $(u_2 - u_1) \to 0$. See, for example, [7] or [10] for a derivation of (9) and analysis of (8). In this work, we find analytic and semi-analytic solutions to (7) for $c \neq 0$, and $n = 2$, and we derive the corresponding order of thickness for the transition layers in non-isothermal flow of viscous non-Newtonian fluids. Applications of these types of flows are abundant in studying flows in drilling fluids, food, oil, polymers, etc; see e.g. [1], [2], and [11]. There are numerous papers devoted to the study of equation (5) in the literature on shock formation and traveling waves in Newtonian flows dating back to the original papers of Burgers, Cole, and Hopf, see [3], [5] and [8]. A generalized Burgers' equation for non-Newtonian flows based on the Maxwell model has recently been studied in [4]. We have not found any paper which deals with Burgers' equation (7) for $c \neq 0$, and $n = 2$.

3 The Integral Equation for the Traveling Waves and the Solution

Let $u(x,t) = u(\xi)$, with $\xi = x - \lambda t$. Then $\frac{\partial u}{\partial t} = \frac{du}{d\xi}\frac{d\xi}{dt} = -\lambda\frac{du}{d\xi}$ and $\frac{\partial u}{\partial x} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{du}{d\xi}$. Substituting $\frac{\partial u}{\partial t} = -\lambda\frac{du}{d\xi}$ and $\frac{\partial u}{\partial x} = \frac{du}{d\xi}$ into equation (7), we get

$$-\lambda\frac{du}{d\xi} + u(\xi)\frac{du}{d\xi} = \frac{1}{\rho}\frac{d}{d\xi}\left[\phi^{-1}\left(\mu_0\frac{du}{d\xi}\right)\right]. \tag{10}$$

Therefore

$$\frac{d}{d\xi}\left[\frac{1}{2}u^2 - \lambda u - \frac{1}{\rho}\phi^{-1}\left(\mu_0\frac{du}{d\xi}\right)\right] = 0,$$

which gives

$$\frac{1}{2}u^2 - \lambda u - \frac{1}{\rho}\phi^{-1}\left(\mu_0\frac{du}{d\xi}\right) = A \tag{11}$$

where A is an arbitrary integration constant. Applying the downstream and upstream boundary conditions: $\lim_{\xi\to+\infty} u(\xi) = u_1$, $\lim_{\xi\to-\infty} u(\xi) = u_2$, and $\lim_{|\xi|\to+\infty}\frac{du}{d\xi}(\xi) = 0$ to equation (11), we get

$$\phi^{-1}\left(\mu_0\frac{du}{d\xi}\right) = \frac{\rho}{2}(u^2 - 2\lambda u - 2A) = \frac{\rho}{2}(u-u_1)(u-u_1) \tag{12}$$

where $\lambda = \frac{1}{2}(u_1+u_2)$ and $A = -\frac{1}{2}u_1u_2$, u_1 and u_2 are the given constants. We have $\mu_0\frac{du}{d\xi} = \phi(\frac{\rho}{2}(u-u_1)(u-u_1))$, which gives

$$\frac{\xi}{\mu_0} = \int\frac{du}{\phi(\frac{\rho}{2}(u-u_1)(u-u_1))}. \tag{13}$$

Without loss of generality, in the following, we assume that $u_1 < u < u_2$. For $c = 0$, (13) gives

$$\frac{1}{2\nu}\xi = \int\frac{du}{(u-u_1)(u-u_1)} = \frac{1}{u_1-u_2}\ln\left|\frac{u-u_1}{u_2-u}\right|$$

where $\nu = \frac{\mu_0}{\rho}$, which gives the classical traveling wave solution

$$u(\xi) = \frac{u_1 + u_2\exp[-\frac{\xi}{2\nu}(u_2-u_1)]}{1+\exp[-\frac{\xi}{2\nu}(u_2-u_1)]}$$

to Burgers' equation for Newtonian flows.

In the following, we are interested in finding solutions to (13) for $c \neq 0$ and $n = 2$. Let $u = \frac{u_2-u_1}{2}\nu + \frac{u_2+u_1}{2}$. Then

$$\phi\left(\frac{\rho}{2}(u-u_1)(u-u_2)\right) = \frac{\rho(u_2-u_1)^2}{8}\left\{1+\frac{c\rho^2(u_2-u_1)^4}{2^6}\left[(\nu+1)(\nu-1)\right]^2\right\}(\nu+1)(\nu-1)$$

and (13) becomes

$$\frac{\xi}{\mu_0} = \frac{8}{\rho(u_2-u_1)^2}\int \frac{d\nu}{\left\{1+\frac{c\rho^2(u_2-u_1)^4}{2^6}\left[(\nu+1)(\nu-1)\right]^2\right\}(\nu+1)(\nu-1)}.$$

Let the constant b be defined by $\sinh 2b = \frac{8}{\sqrt{c}\rho(u_2-u_1)^2}$, and define $\Psi(t,b) = (1+\frac{t^2}{\sinh^2(2b)})t$. We have the decomposition

$$\begin{aligned}\frac{1}{\Psi((\nu+1)(\nu-1),b)} &= \frac{1}{(\nu+1)(\nu-1)} \\ &- \frac{1}{(\nu-\cosh(b)-i\sinh(b))(\nu-\cosh(b)+i\sinh(b))} \\ &\times\frac{\nu^2-1}{(\nu+\cosh(b)-i\sinh(b))(\nu+\cosh(b)+i\sinh(b))}.\end{aligned}$$

By using Mathematica, we find that

$$\begin{aligned}\int\frac{d\nu}{\Psi((\nu+1)(\nu-1),b)} &= \ln(\frac{\nu-1}{\nu+1})+ \\ &\frac{1}{2}\left(\frac{\arctan\left[\frac{\nu}{\sqrt{-1-i\sinh(2b)}}\right]}{\sqrt{-1-i\sinh(2b)}}+\frac{\arctan\left[\frac{\nu}{\sqrt{-1+i\sinh(2b)}}\right]}{\sqrt{-1+i\sinh(2b)}}\right)\end{aligned}$$

Let

$$extra(b,\nu) = \mathrm{Re}\left(\frac{\arctan(\frac{\nu}{\sqrt{-1-i\ \sinh(2b)}})}{\sqrt{-1-i\ \sinh(2b)}}\right).$$

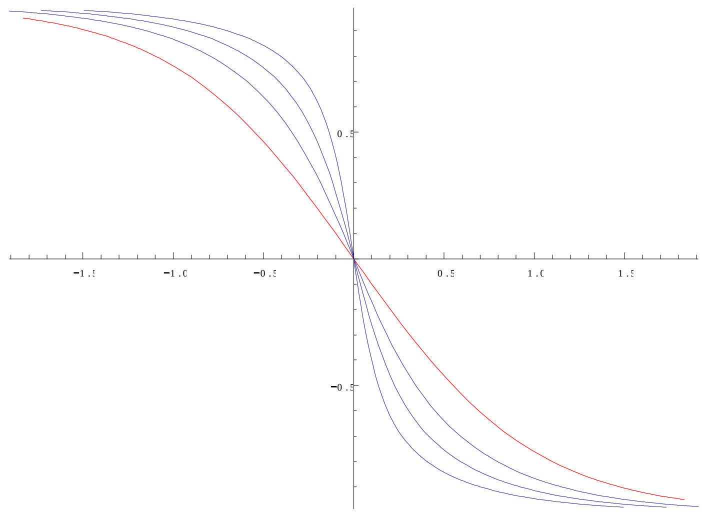
Then we have

$$\frac{\xi}{\mu_0} = \frac{8}{\rho(u_2-u_1)^2}\left[\frac{1}{2}\ln(\frac{\nu-1}{\nu+1}) - \mu_0 extra(b,\nu)\right].$$

Therefore $\frac{\xi}{\mu_0} = \frac{8}{\rho(u_2-u_1)^2}\left[\frac{1}{2}\ln(\frac{u_2-u}{u-u_1}) - extra(b,\frac{2u-(u_2+u_1)}{u_2-u_1})\right]$ and the traveling wave solution of (1) is implicitly defined by

$$\frac{\xi}{\mu_0} = \frac{4}{\rho(u_2-u_1)^2}\ln(\frac{u_2-u}{u-u_1}) - \frac{8}{\rho(u_2-u_1)^2}extra(b,\frac{2u-(u_2+u_1)}{u_2-u_1}).$$

We have omitted the integration constants in the above solutions. For simplicity, we plot the profile of the transition layer of $u = u(\xi)$ and provide the following graphic representation of the profiles of the transition layers. The blue curves correspond to $b = 0.5$, 0.35, and 0.25 respectively and the red curve represents the classical solution corresponding to $b = 0.0$.



4 The Order of Thickness of the Transition Layers

The transition layer thickness or the shock thickness can be estimated by using the first order derivative $\left.\frac{du}{d\xi}\right|_{\xi=0}$. From $\frac{du}{d\xi} = \frac{1}{\mu_0}\phi\left(\frac{\rho}{2}(u-u_1)(u-u_2)\right)$ and $u(0) = \frac{u_1+u_2}{2}$, we get $\left.\frac{du}{d\xi}\right|_{\xi=0} = -\frac{1}{\mu_0}\phi\left(\frac{\rho}{2}(u_2-u_1)^2\right)$. Let δ denote the thickness of the transition layer, using the Taylor expansion, we have

$$u_2 - u_1 = u(-\frac{\delta}{2}) - u(\frac{\delta}{2}) = -\delta \left.\frac{du}{d\xi}\right|_{\xi=0} + O(\delta^2).$$

Therefore we have

$$\delta = \frac{u_2-u_1}{\frac{du}{d\xi}(0)} = \frac{\mu_0(u_2-u_1)}{\phi(\frac{\rho}{8}(u_2-u_1)^2)} = \frac{8\mu_0}{\rho(u_2-u_1)\left\{1+c\left[\frac{\rho}{8}(u_2-u_1)\right]^2\right\}},$$

Which is the first order approximation of the thickness of the transition layer for power-law flows. This estimate, for $c=0$, gives the well-known estimate $\delta = \frac{8\mu_0}{\rho(u_2-u_1)}$ for the thickness of the transition layer of Newtonian flows.

5 Conclusion

In this work, we consider a generalized Burgers' equation for Gee-Lyon fluid flows, and derive a new general traveling wave solution of this equation. As special cases of this solution, we show several analytic solutions and profiles of the thickness of the transition layer of the solution. We defined a first order approximation of the thickness of the transition layer or the thickness of the shock which generalized the known estimate for the shock thickness of the corresponding Burgers' solution for Newtonian flows.

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