

Random Lifts Of Graphs: Network Robustness Based On The Estrada Index*

Yilun Shang[†]

Received 29 July 2011

Abstract

A k -lift of a graph G is a graph with vertex set $V(G) \times [k]$, and for each edge $(i, j) \in E(G)$ there is a perfect matching between fibers $\{i\} \times [k]$ and $\{j\} \times [k]$. If these matchings are chosen independently and uniformly at random, then we say that we have a random k -lift. In this paper, we view random lifts as network growth models and study their robustness based on the logarithm of the Estrada index. It is shown that the robustness of random k -lifts is decreasing with the parameter k for large networks. We compare our results with edge connectivity for random lifts of networks. Interestingly, these two measures indicate different behaviors for robustness. Simulations are provided that demonstrate this discrepancy.

1 Introduction

The lift of a graph is a basic concept that originates from topological graph theory [19] and has applications in distributed computing [1]. It is a simple scheme to expand a graph while keeping the maximum and minimum degrees unvariant. The lift of a graph has some appealing spectral features (see Section 2). In this paper we consider it as a kind of network growth model and explore its robustness by virtue of a certain spectral measure. Many purely graph theoretical results have been known for (random) lifts of graphs such as connectivity [2, 3], chromatic number [4], hamiltonicity [9, 10], spectral distribution [18, 20, 23] and perfect matchings [21]. To the best of our knowledge, this is the first research to treat the robustness issue of random lifts of graphs.

Social organizations, biological organisms and ecological communities are all examples of complex networks, which rely for their function and performance on their robustness [22]. The classical approach for determining robustness of networks entails the use of basic concepts from graph theory. For instance, the connectivity of a graph (c.f. [28, 29, 30, 31]) is a fundamental measure of robustness of a network [8]. Node/edge connectivity, defined as the size of the smallest node/edge cut, describes the robustness of a network to the deletion of nodes/edges.

Recently, the concept of the logarithm of the Estrada index [12], dubbed as “natural connectivity”, is proposed in [26, 35] as a spectral measure of robustness in complex

*Mathematics Subject Classifications: 05C40, 15A18, 05C80.

[†]Institute for Cyber Security, University of Texas at San Antonio, San Antonio, Texas 78249, USA

networks. The natural connectivity is expressed as the weighted sum of closed walks of all lengths. The authors consider the redundancy of walks as the root of robustness of networks, which ensures that the connection between vertices still remains possible in spite of damage to the network. It is shown [26, 35, 36] that the (normalized) natural connectivity has acute discrimination in measuring the robustness of complex networks and can exhibit the variation of robustness sensitively even for disconnected networks. Some perturbation results for the Estrada index are provided in [25] for weighted networks. The robustness of Internet AS-level topology is tested in [37] using normalized natural connectivity.

In this paper, as mentioned above, we study the robustness of random k -lifts of graphs based on the normalized natural connectivity. Our key finding is that the robustness of random k -lifts is decreasing with the parameter k for large networks. We compare our results with edge connectivity for random lifts of networks. Interestingly, these two measures indicate different behaviors for robustness. Simulation results are presented for various networks such as power-law graphs and classical random graphs to illustrate this discrepancy.

The rest of this paper is organized as follows. Section 2 contains preliminaries for (normalized) natural connectivity and random k -lifts. The theoretical results and numerical simulation studies are given in Sections 3 and 4, respectively. We conclude the paper in Section 5.

2 Preliminaries

In this section, we present some necessary preliminaries leading to the (normalized) natural connectivity and random lifts of graphs.

A complex network can be viewed as a graph [8]. Let $G = (V, E)$ be a simple undirected graph with vertex set V and edge set $E \subseteq V \times V$. Let $|V| = n$ be the number of vertices. Let $A(G) = (a_{ij})_{n \times n}$ be the adjacency matrix of G , where $a_{ij} = a_{ji} = 1$ if $(i, j) \in E$, and $a_{ij} = a_{ji} = 0$ otherwise. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of the adjacency matrix A since it is real and symmetric. The set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is called the spectrum of A (or G).

A weighted sum of the number of closed walks is defined in [35] by $S = \sum_{k=0}^{\infty} n_k/k!$, where n_k is the number of closed walks of length k in G . Since $n_k = \sum_{i=1}^n \lambda_i^k$ [6], we have

$$S = \sum_{k=0}^{\infty} \sum_{i=1}^n \frac{\lambda_i^k}{k!} = \sum_{i=1}^n \sum_{k=0}^{\infty} \frac{\lambda_i^k}{k!} = \sum_{i=1}^n e^{\lambda_i}. \quad (1)$$

Note that (1) corresponds to the original Estrada index of the graph [11, 12, 13, 34, 38], which has been developed for the study of bipartivity [16], subgraph centrality [17] and the degree of proteins and other long-chains biopolymers [14, 15]. The natural connectivity of G is then defined as

$$\bar{\lambda}(G) = \ln \left(\frac{S}{n} \right) = \ln \left(\frac{1}{n} \sum_{i=1}^n e^{\lambda_i} \right), \quad (2)$$

which corresponds to a kind of average eigenvalue of A since $\lambda_n \leq \bar{\lambda} \leq \lambda_1$. Moreover, it is shown [35] that the asymptotic bounds of natural connectivity as $n \rightarrow \infty$ is $0 \leq \bar{\lambda} \leq n - \ln n$. Hence, the normalized natural connectivity [37] is defined as

$$\tilde{\lambda}(G) = \frac{\bar{\lambda}(G)}{n - \ln n} \quad (3)$$

eliminating the different network size effect.

Now we introduce the concept of graph lifts. For a natural number $k \in \mathbb{N}$, denote $[k] := \{1, 2, \dots, k\}$. A k -lift of graph G is a graph $G^{(k)}$ with vertex set $V \times [k]$ and edge set $E^{(k)} = \cup_{(i,j) \in E} M_{ij}$, where each M_{ij} is a perfect matching between sets $\{i\} \times [k]$ and $\{j\} \times [k]$. Thus $G^{(k)}$ induced from G has kn vertices and much more complex interaction structures. Suppose the matchings $\{M_{ij}\}$ are chosen independently and each of them is uniformly distributed. Then the random graphs generated in this manner are called random k -lifts of graph G (see e.g. [2, 3]).

The spectrum of $G^{(k)}$ has a remarkable feature. Let $A^{(k)}$ be the adjacency matrix of a k -lift $G^{(k)}$ of graph G . It is shown [18, 20] that the spectrum of $A^{(k)}$ always contains the spectrum of A in the sense of multisets: any eigenvalue of A with multiplicity m is an eigenvalue of $A^{(k)}$ with multiplicity $\geq m$.

3 Main Results

Let $\text{new}(A^{(k)})$ denote the difference between the spectrum of $A^{(k)}$ and the spectrum of A . It is a multiset in that if λ has multiplicity m_1 in the spectrum of $A^{(k)}$ and multiplicity m_2 in the spectrum of A , λ occurs $m_1 - m_2$ times in $\text{new}(A^{(k)})$. Let Δ be the maximum degree of G .

The following lemma is a useful result for the spectrum of $G^{(k)}$. It can be proved by using a concentration inequality [24] for sums of independent random matrices.

LEMMA 1 ([23]). For $k \in \mathbb{N}$, let $G^{(k)}$ be a random k -lift of graph G . With the above notations, for $\varepsilon \in (0, 1)$, we have

$$P\left(\sup_{\lambda \in \text{new}(A^{(k)})} |\lambda| \leq 16\sqrt{\Delta \ln(2kn/\varepsilon)}\right) \geq 1 - \varepsilon. \quad (4)$$

Lemma 1 indicates that, with high probability, all eigenvalues of $G^{(k)}$ that are not eigenvalues of G are of the order $O(\sqrt{\Delta \ln(kn)})$. Our result regarding the normalized natural connectivity of $G^{(k)}$ goes as follows.

THEOREM 1. For $k \in \mathbb{N}$, let $G^{(k)}$ be a random k -lift of graph G . Let $\varepsilon = \varepsilon(n) \rightarrow 0$ sufficiently slowly (e.g. $\varepsilon \gg ne^{-n}$). If $\Delta \gg \ln(n/\varepsilon)$, then

$$\tilde{\lambda}(G^{(k)}) \leq \frac{1}{k} \quad (5)$$

almost surely, as $n \rightarrow \infty$. If $\Delta = O(\ln(n/\varepsilon))$, then

$$\tilde{\lambda}(G^{(k)}) = 0 \quad (6)$$

almost surely, as $n \rightarrow \infty$.

For any graph G with n vertices, the normalized natural connectivity $\tilde{\lambda}(G)$ resides in the interval $[0, 1]$ for large enough n as mentioned above. For the empty graph H and the complete graph K_n , it is known that [35] the normalized natural connectivity $\tilde{\lambda}(H) \equiv 0$ and $\tilde{\lambda}(K_n) \rightarrow 1$, as $n \rightarrow \infty$, giving the weakest and strongest “robustness” of networks, respectively. Intermediately, for a classical random graph [7] $G_{n,p}$, it is shown that (see [27] Theorem 2.1 or [36] Theorem 3.3) the natural connectivity almost surely is

$$\bar{\lambda}(G_{n,p}) = np - \ln n \quad (7)$$

as $n \rightarrow \infty$. Hence, the normalized natural connectivity $\tilde{\lambda}(G_{n,p})$ tends to p , the edge density.

The inequality (5) in Theorem 1 provides an upper bound for the normalized natural connectivity of $G^{(k)}$, which, surprisingly, is a decreasing function of k . Intuitively, the number of edges as well as the intricacy of edge structures of $G^{(k)}$ are increasing with k and thus, the robustness of $G^{(k)}$ may also be enhanced. In fact, it is shown in [3] that if G is a connected graph with minimum degree $\delta(G) = \delta \geq 3$, then $G^{(k)}$ is δ -edge-connected almost surely as $k \rightarrow \infty$. Recall that for any graph G , the edge connectivity is always not bigger than its minimum degree (see e.g. [8]). Since $\delta(G^{(k)}) \equiv \delta(G)$, the random k -lifts $G^{(k)}$ are rather robust as per the measure of edge-connectivity for large enough k . In the next section, we will perform some simulation studies to further illustrate the discrepancy of these two network robustness measures.

Next, notice that a 1-lift of graph G is G itself. Then take $k = 1$ in (6) in Theorem 1 yields the following

COROLLARY 1. Let $\varepsilon = \varepsilon(n) \rightarrow 0$ sufficiently slowly (e.g. $\varepsilon \gg ne^{-n}$). For any undirected graph G with maximum degree $\Delta = O(\ln(n/\varepsilon))$, we have $\tilde{\lambda}(G) = 0$ almost surely, as $n \rightarrow \infty$.

We now present the proof of Theorem 1 employing Lemma 1.

PROOF OF THEOREM 1. Recall that $G^{(k)}$ has kn vertices, whose maximum degree is equal to that of G . By Lemma 1, the definition (3) and the fact that $\lambda_1 \leq \Delta$ (c.f. [6] pp. 43), we obtain

$$\begin{aligned} \tilde{\lambda}(G^{(k)}) &\leq \frac{\ln\left(\frac{ne^\Delta + (k-1)ne^{16\sqrt{\Delta \ln(2kn/\varepsilon)}}}{kn}\right)}{kn - \ln(kn)} \\ &\leq \frac{\ln\left(\frac{e^\Delta}{k} + e^{16\sqrt{\Delta \ln(2kn/\varepsilon)}}\right)}{kn - \ln(kn)} \end{aligned} \quad (8)$$

with probability at least $1 - \varepsilon$.

If $\Delta \gg \ln(n/\varepsilon)$, the right-hand side of (8) is bounded above by

$$\frac{\ln\left(\frac{(1+o(1))e^\Delta}{k}\right)}{kn - \ln(kn)} \leq \frac{n - \ln k + o(1)}{kn - \ln(kn)} \rightarrow \frac{1}{k}$$

as $n \rightarrow \infty$, since the maximum degree $\Delta \leq n$. Now the statement (5) of Theorem 1 follows by setting $\varepsilon \rightarrow 0$.

If $\Delta = O(\ln(n/\varepsilon))$, then the right-hand side of (8) is bounded above by

$$O\left(\frac{\ln\left(\left(1 + \frac{1}{k}\right)e^{16\ln(2kn/\varepsilon)}\right)}{kn - \ln(kn)}\right) = O\left(\frac{\ln(2kn/\varepsilon)}{kn - \ln(kn)}\right)$$

for large enough n . Consequently, we have

$$P\left(0 \leq \tilde{\lambda}(G^{(k)}) \leq O\left(\frac{\ln(2kn/\varepsilon)}{kn - \ln(kn)}\right)\right) \geq 1 - \varepsilon. \quad (9)$$

Hence, (6) follows by letting $\varepsilon \gg ne^{-n}$ and $\varepsilon \rightarrow 0$ in (9). The proof is complete.

4 Simulation Results

In this section, we provide some simulations to explore the normalized natural connectivity of random lifts of graphs and compare it with another robustness measure, edge connectivity.

(i) First, we generate a base network G_1 with a power-law degree distribution using the BA model [5], where $n = 2000$ and average degree $\langle d \rangle = 8$. Fig. 1 shows the normalized natural connectivity for random k -lifts $G_1^{(k)}$ for different values of k . We observe that $\tilde{\lambda}(G_1^{(k)})$ is decreasing which agrees with our above analyses. The inset in Fig. 1 shows the behavior of edge connectivity for $G_1^{(k)}$ as a function of the parameter k . Note that as the values of k become large, the edge connectivity vibrates relatively slightly, indicating a steady degree of robustness of the random lifts in contrast to the outcome of normalized natural connectivity.

(ii) Next, we generate a base network G_2 from an Erdős-Rényi random graph model $G_{n,p}$ [7], with $n = 2000$ and $p = 1/2$. Fig. 2 shows the normalized natural connectivity $\tilde{\lambda}(G_2^{(k)})$ is decreasing with k . We may conclude that, as observed in Fig. 1 and Fig. 2, the normalized natural connectivity will lose discrimination for large enough k . Indeed, this is what Theorem 1 implies. The inset in Fig. 2 shows the behavior of edge connectivity for $G_2^{(k)}$ as a function of k . Similar with the inset of Fig. 1, the edge connectivity here focuses on some fixed value, again suggesting a steady degree of robustness of $G_2^{(k)}$ in contrast to the outcome of $\tilde{\lambda}(G_2^{(k)})$.

5 Conclusion

We have used spectral graph theory to investigate the robustness of random k -lifts of graphs based on the Estrada index (more precisely, the normalized natural connectivity), which is rooted in the inherent structural properties of a network. Lifts of graphs are related with hierarchical, multi-scale and growing descriptions of complex networks. Our result suggests that the robustness of random k -lifts is decreasing with k for large-scale networks. An external measure of network robustness, edge connectivity, is compared with normalized natural connectivity for random lifts of networks. Interestingly, these two measures indicate different behaviors for the evolution of robustness.

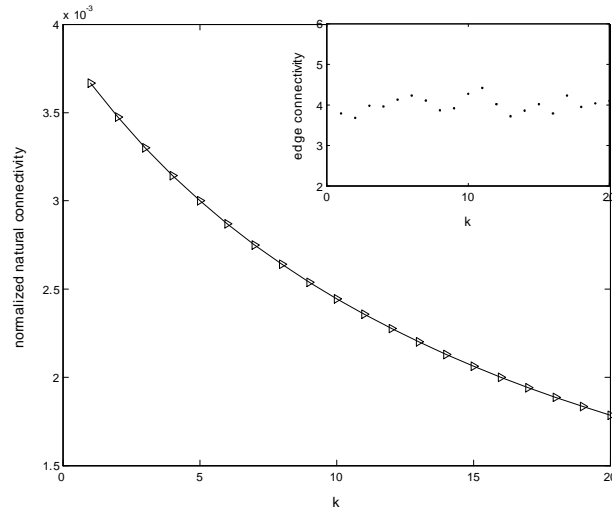


Figure 1: The robustness measured by normalized natural connectivity of $G_1^{(k)}$ versus different k . G_1 is generated using the BA model with $n = 2000$ and $\langle d \rangle \approx 8$. Inset: the robustness measured by the edge connectivity of $G_1^{(k)}$ versus different k .

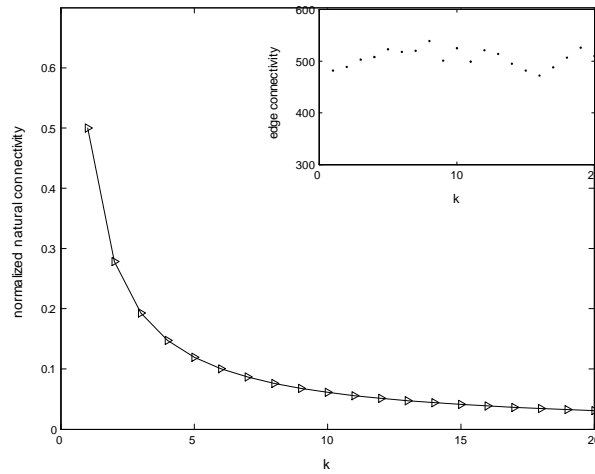


Figure 2: The robustness measured by normalized natural connectivity of $G_2^{(k)}$ versus different k . G_2 is generated from the ER random graph model $G_{2000,1/2}$. Inset: the robustness measured by the edge connectivity of $G_2^{(k)}$ versus different k .

Simulation results are provided to illustrate this discrepancy. For future research, it is desirable to address the random lifts of graphs with some specific structures such as the bipartite graphs, which are networks consisting of two types of nodes with edges running only between nodes of unlike type (see e.g. [32, 33]).

Acknowledgment. I sincerely thank the referees whose comments helped improve the paper.

References

- [1] D. Angluin, Local and global properties in networks of processors, Proc. 12th Annual ACM Symposium on Theory of Computing, Los Angeles, 1980, 82–93.
- [2] A. Amit and N. Linial, Random lifts of graphs: edge expansion, *Combinatorics, Probability and Computing*, 15(2006), 317–322.
- [3] A. Amit and N. Linial, Random graph coverings I: General theory and graph connectivity, *Combinatorica*, 22(2002), 1–18.
- [4] A. Amit, N. Linial and J. Matoušek, Random lifts of graphs: independence and chromatic number, *Random Structure and Algorithms*, 20(2002), 1–22.
- [5] A. L. Barabási and R. Albert, Emergence of scaling in random networks, *Science*, 286(1999), 509–512.
- [6] L. W. Beineke and R. J. Wilson, *Topics in Algebraic Graph Theory*, Cambridge University Press, 2004.
- [7] B. Bollobás, *Random Graphs*, Cambridge University Press, 2001.
- [8] B. Bollobás, *Graph Theory*, Springer-Verlag, New York, 1979.
- [9] K. Burgin, P. Chebolu, C. Cooper and A. M. Frieze, Hamilton cycles in random lifts of graphs, *European Journal of combinatorics*, 27(2006), 1282–1293.
- [10] P. Chebolu and A. Frieze, Hamilton cycles in random lifts of directed graphs, *SIAM J. Discrete Math.*, 22(2008), 520–540.
- [11] K. C. Das and S.-G. Lee, On the Estrada index conjecture, *Lin. Algebra Appl.*, 431(2009), 1351–1359.
- [12] J. A. de la Peña, I. Gutman and J. Rada, Estimating the Estrada index, *Linear Algebra and its Applications*, 427(2007), 70–76.
- [13] E. Estrada, Characterization of 3D molecular structure, *Chem. Phys. Lett.*, 319(2000), 713–718.
- [14] E. Estrada, Characterization of the folding degree of proteins, *Bioinformatics*, 18(2002), 697–704.

- [15] E. Estrada, Characterization of the amino acid contribution to the folding degree of proteins, *Proteins*, 54(2004) 727–737.
- [16] E. Estrada and J. A. Rodríguez-Velázquez, Spectral measures of bipartivity in complex networks, *Phys. Rev. E*, 72(2005), 046105.
- [17] E. Estrada and J. A. Rodríguez-Velázquez, Subgraph centrality in complex networks, *Phys. Rev. E*, 71(2005), 056103.
- [18] J. Friedman, Relative expander or weakly relatively Ramanujan graphs, *Duke Math. J.*, 118(2003), 19–35.
- [19] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience, 1987.
- [20] N. Linial and D. Puder, Words maps and spectra of random graph lifts, *Random Structure and Algorithms*, 37(2010), 100–135.
- [21] N. Linial, E. Rozenman, Random lifts of graphs: perfect matchings, *Combinatorica*, 25(2008), 407–424.
- [22] M. E. J. Newman, The structure and function of complex networks, *SIAM Rev.*, 45(2003), 167–256.
- [23] R. I. Oliveira, The spectrum of random k -lifts of large graphs (with possibly large k), *J. Comb.*, 1(2010), 285–306.
- [24] R. I. Oliveira, Concentration of the adjacency matrix and of the Laplacian in random graphs with independent edges, *arXiv:0911.0600*, 2009.
- [25] Y. Shang, Perturbation results for the Estrada index in weighted networks. *J. Phys. A: Math. Theor.*, 44(2011) 075003.
- [26] Y. Shang, Local natural connectivity in complex networks, *Chin. Phys. Lett.*, 28(2011), 068903.
- [27] Y. Shang, The Estrada index of random graphs, *Sci. Magna*, 7(2011), 79–81.
- [28] Y. Shang, Connectivity in a random interval graph with access points, *Inform. Process. Lett.*, 109(2009), 446–449.
- [29] Y. Shang, A note on the 2-connectivity in one-dimensional ad hoc networks, *Sci. China Inf. Sci.*, 54(2011), 123–128.
- [30] Y. Shang, The rainbow k -connectivity of random graphs, *Sci. Magna*, 6(2010), 108–111.
- [31] Y. Shang, On the isolated vertices and connectivity in random intersection graphs, *Int. J. Comb.*, 2011(2011), 872703.
- [32] Y. Shang, Groupies in random bipartite graphs, *Appl. Anal. Discrete Math.*, 4(2010), 278–283.

- [33] Y. Shang, Estimation of the shortest average distance in bipartite networks with given density, *J. Phys. Soc. Jpn.*, 80(2011), 055001.
- [34] Y. Shang, Lower bounds for the Estrada index of graphs, *Electron. J. Linear Algebra*, 23(2012), 664–668.
- [35] J. Wu, M. Barahona, Y. J. Tan and H. Z. Deng, Natural connectivity of complex networks, *Chin. Phys. Lett.*, 27(2010), 078902.
- [36] J. Wu, M. Barahona, Y. J. Tan and H. Z. Deng, Robustness of random graphs based on natural connectivity, arXiv:1009.3430, 2010.
- [37] J. Wu, H. Z. Deng and Y. J. Tan, Spectral measure of robustness for Internet topology, 3rd IEEE International Conference on Computer Science and Information Technology, Chengdu, China, 2010, 50–54.
- [38] B. Zhou, On Estrada index, *MATCH Commun. Math. Comput. Chem.*, 60(2008), 485–492.