Applied Mathematics E-Notes, 12(2012), 110-117 © ISSN 1607-2510
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Quadratic Harmonic Number Sums*

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Received 12 July 2012

Abstract

In this paper, we obtain some identities for the series $\sum_{n=1}^{\infty} H_n^2/(n(n+k))$ and $\sum_{n=1}^{\infty} H_n^2/\left(n\binom{n+k}{k}\right)$, where $H_n = \sum_{j=1}^{n} j^{-1}$ and k is a positive integer. Then we obtain some series representations for the Apéry's constant, $\zeta(3)$.

1 Introduction

The Riemann zeta function is defined for $s \in \mathbb{C}$ with $\Re(s) > 1$ by $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$. For integer $n \geqslant 1$ we let $\zeta_n(s) = \sum_{j=1}^{n} j^{-s}$. We define the n-th harmonic number by $H_n = \zeta_n(1)$, and the generalized n-th harmonic number by $H_n^{(r)} = \zeta_n(r)$, for any real number r. Moreover, we set $H_0^{(r)} = 0$. Identities for sums involving harmonic numbers, generalized harmonic numbers, and their powers are rare in number in the literature. A classical example is due to L. Euler [3], where for integers $q \geqslant 3$ he proved that

$$2\sum_{n=1}^{\infty} \frac{H_n}{n^q} = (q+2)\zeta(q+1) - \sum_{m=1}^{q-2} \zeta(m+1)\,\zeta(q-m).$$

Some recently obtained identities are $\sum_{n=1}^{\infty} \left(H_n/n\right)^2 = 17\zeta(4)/4$ due to D. Borwein and J. M. Borwein [2], the following one due to A. Sofo [7] which is valid for integers $k \geqslant 2$

$$\sum_{n=1}^{\infty} \frac{H_n^2}{\binom{n+k}{k}} = \frac{k}{k-1}\left(\zeta(2) + \frac{2}{(k-1)^2} - H_{k-1}^{(2)}\right),$$

and $3\sum_{j=1}^{n} \left(H_j^2/j - H_j/j^2\right) = H_n^3 - \zeta_n(3)$ due to M. Hassani [5]. In this paper, we obtain some identities for the series

$$\mathfrak{S}(m) := \sum_{n=1}^{\infty} \frac{H_n^2}{n(n+m)}, \qquad \text{and} \qquad \mathfrak{B}(k) := \sum_{n=1}^{\infty} \frac{H_n^2}{n\binom{n+k}{k}},$$

where m and k are positive integers. More precisely, we show the following.

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*Mathematics Subject Classifications: 05A10, 11B65, 11M06, 33B15, 33D60, 33C20.

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THEOREM 1. Assume that $m \geqslant 1$ is an integer, and let

$$\mathcal{F}(m) = H_{m-1}\zeta(2) + H_{m-1}H_{m-1}^{(2)} - H_{m-1}^{(3)} + H_{m-1}^{3}.$$

Also, for $j \neq m$ let

$$\mathcal{A}(m,j) = \frac{H_{j-1}}{mj^2} + \frac{1}{2j(m-j)}\left(H_{j-1}^2 + H_{j-1}^{(2)}\right).$$

Then, we have

$$\mathfrak{S}(m) = \frac{3\zeta(3) + \mathcal{F}(m)}{m} - \sum_{j=1}^{m-1}\mathcal{A}(m,j). \tag{1}$$

THEOREM 2. Assume that $k \geqslant 1$ is an integer, and let

$$\mathcal{G}(k) = -\zeta(2)H_{k-1} + \frac{H_{k-1}H_{k-1}^{(2)}}{2} + \frac{H_{k-1}^{(3)}}{3} + \frac{H_{k-1}^{3}}{6}.$$

Also, for $j \neq r$ let

$$\mathcal{C}(r) = \frac{H_{r-1}\left(H_{r-1}^2 + H_{r-1}^{(2)}\right)}{r} - \sum_{j=1}^{r-1}\left(\frac{H_{j-1}}{rj^2} + \frac{H_{j-1}^2 + H_{j-1}^{(2)}}{2j(r-j)}\right).$$

Then, we have

$$\mathfrak{B}(k) = 3\zeta(3) + \mathcal{G}(k) + \sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\mathcal{C}(r). \tag{2}$$

Then we obtain some new series representations for $\zeta(3)$, which is known as the Apéry's constant (see [4], pp 40–52). More precisely, applying (1) with $m = 5$, and (2) with $k = 1$ and $k = 4$, respectively, we get the following.

COROLLARY 1. We have

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+5)} = \frac{3\zeta(3)}{5} + \frac{5\zeta(2)}{12} + \frac{8737}{8640}.$$

COROLLARY 2. We have

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n(n+1)} = 3\zeta(3) \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{H_n^2}{n\binom{n+4}{4}} = 3\zeta(3) - \frac{49}{20}\zeta(2) + \frac{128587}{216000}.$$

2 Auxiliary Lemmas

In this section we introduce two auxiliary lemmas, which are the base of proofs of our results. In what follows below, we will use both of notations H_n and $H_n^{(1)}$ for the n-th harmonic number, and we will apply the following known [2] integral representation

$$\frac{H_{n+1}}{n+1} = -\int_0^1 x^n \ln(1-x)\mathrm{d}x. \tag{3}$$

Also, we recall the polylogarithm function defined by $\mathrm{Li}_n : z \mapsto \sum_{j=1}^{\infty} z^j/j^n$ for integral $n \geqslant 2$ and z in the unit disk. The function Li_2 is known as dilogarithm function. We note that $\mathrm{Li}_n(1) = \zeta(n)$. The identity (4) and its proof of the following Lemma is due to Furdui.

LEMMA 1. Assume that $m \geqslant 1$ is an integer, and let

$$\mathcal{H}(m) = \sum_{n=1}^{\infty} \frac{H_n^{(1)} H_{n+m}^{(1)}}{n(n+m)}.$$

Then, we have

$$\mathcal{H}(m) = \frac{2\zeta(3)}{m} + \frac{\zeta(2)H_m^{(1)}}{m} - \frac{1}{m}\sum_{j=1}^{m} \frac{H_j^{(1)}}{j^2} + T(m), \tag{4}$$

where

$$T(m) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \frac{1}{(j+1)^3} \left(\frac{3}{j+1} - \frac{2}{m} \right).$$

Also, we have

$$\begin{aligned} \mathcal{H}(m) = \frac{2\zeta(3)}{m} + \frac{\zeta(2)H_{m-1}^{(1)}}{m} + \frac{\left(H_{m-1}^{(1)}\right)^2}{2m^2} + \frac{\left(H_{m-1}^{(1)}\right)^3}{2m} \\ + \frac{H_{m-1}^{(2)}}{2m^2} + \frac{3H_{m-1}^{(1)}H_{m-1}^{(2)}}{2m} - \sum_{j=1}^{m-1} \frac{H_{j-1}^{(1)}}{mj^2}. \end{aligned} \tag{5}$$

PROOF. For $x \in (0,1)$ we define the function f by

$$f(x) = \ln(x)\ln(1-x) - \frac{1}{2}\ln^2(1-x) + \mathrm{Li}_2(x) + \int_1^{\frac{1}{1-x}} \frac{\ln(u-1)}{u}\mathrm{d}u.$$

Since $f'(x) = 0$ and $\lim_{x\to 0^+} f(x) = 0$, we imply $f(x) = 0$. Thus, for $x \in (0,1)$ we obtain

$$\ln(x)\ln(1-x) - \frac{1}{2}\ln^2(1-x) + \mathrm{Li}_2(x) = -\int_1^{\frac{1}{1-x}} \frac{\ln(u-1)}{u}\mathrm{d}u. \tag{6}$$

By using (3), we get

$$\mathcal{H}(m) = \int_0^1 \int_0^1 x^m \ln(1-x)\ln(1-y)\sum_{n=1}^{\infty}(xy)^{n-1}\mathrm{d}y\mathrm{d}x = \int_0^1 x^m \ln(1-x)\mathcal{I}(x)\mathrm{d}x,$$

where

$$\mathcal{I}(x) = \int_0^1 \frac{\ln(1-y)}{1-xy}\mathrm{d}y.$$

By letting $1 - xy = t$ in $\mathcal{I}(x)$ we obtain

$$\mathcal{I}(x) = \frac{1}{x}\left(\ln(x)\ln(1-x) + \int_{1-x}^1 \frac{\ln(x-1+t)}{t}\mathrm{d}t\right).$$

Then, we substitute $t = u\,(1-x)$, and we combine the result with (6) to get

$$\mathcal{I}(x) = -\frac{1}{x}\left(\frac{1}{2}\ln^2(1-x) + \mathrm{Li}_2(x)\right).$$

Thus, we obtain

$$\mathcal{H}(m) = -\frac{1}{2}\int_0^1 x^{m-1}\ln^3(1-x)\mathrm{d}x - \int_0^1 x^{m-1}\ln(1-x)\mathrm{Li}_2(x)\mathrm{d}x. \tag{7}$$

We have

$$\int_0^1 x^{m-1}\ln^3(1-x)\mathrm{d}x = \sum_{j=0}^{m-1}(-1)^{j+1}\binom{m-1}{j}\frac{6}{(j+1)^4}, \tag{8}$$

and similarly

$$\int_0^1 x^{m-1}\ln^2(1-x)\mathrm{d}x = \sum_{j=0}^{m-1}(-1)^{j}\binom{m-1}{j}\frac{2}{(j+1)^3}. \tag{9}$$

To evaluate the second part of the integral in (7) we use integration by parts by setting $u(x) = \mathrm{Li}_2(x)$ and $v'(x) = x^{m-1}\ln(1-x)$, from which we get $u'(x) = -\frac{\ln(1-x)}{x}$ and $v(x) = \frac{1}{m}\left((x^m-1)\ln(1-x) - \sum_{i=1}^m \frac{x^i}{i}\right)$. Hence, by considering (9) we obtain

$$\begin{aligned}\int_0^1 x^{m-1}\ln(1-x)\mathrm{Li}_2(x)\mathrm{d}x &= \frac{1}{m}\left(\int_0^1 x^{m-1}\ln^2(1-x)\mathrm{d}x\right)\\ &= \frac{1}{m}\left(\sum_{i=1}^m \frac{H_i}{i^2} - 2\zeta(3) - \zeta(2)H_m\right). \end{aligned}\tag{10}$$

Combining (7), (8) and (10) completes the proof of (4).

To prove (5) we consider (4). In the interest of expressing $T(m)$ in terms of harmonic numbers, we note that

$$\begin{aligned}T(m) &= \frac{\left(H_m^{(1)}\right)^3}{2m} + \frac{3H_m^{(1)}H_m^{(2)}}{2m} + \frac{H_m^{(3)}}{m} - \frac{\left(H_m^{(1)}\right)^2}{m^2} - \frac{H_m^{(2)}}{m^2}\\ &= \frac{1}{m^4} + \frac{H_{m-1}^{(1)}}{m^3} + \frac{\left(H_{m-1}^{(1)}\right)^2}{2m^2} + \frac{\left(H_{m-1}^{(1)}\right)^3}{2m} + \frac{H_{m-1}^{(2)}}{2m^2} + \frac{3H_{m-1}^{(1)}H_{m-1}^{(2)}}{2m} + \frac{H_{m-1}^{(3)}}{m}.\end{aligned}$$

Hence, we get

$$\mathcal{H}(m)=\frac{2\zeta\left(3\right)}{m}+\frac{\zeta\left(2\right)}{m}\left(H_{m-1}^{(1)}+\frac{1}{m}\right)-\frac{H_{m-1}^{(1)}}{m^{3}}-\frac{1}{m}\sum_{j=1}^{m-1}\frac{H_{j-1}^{(1)}}{j^{2}}$$
$$+\frac{H_{m-1}^{(1)}}{m^{3}}+\frac{\left(H_{m-1}^{(1)}\right)^{2}}{2m^{2}}+\frac{\left(H_{m-1}^{(1)}\right)^{3}}{2m}+\frac{H_{m-1}^{(2)}}{2m^{2}}+\frac{3H_{m-1}^{(1)}H_{m-1}^{(2)}}{2m},$$

and consequently, we obtain (5). This completes the proof.

Our next lemma, gives identities for $\sum_{n=1}^{\infty}H_n/(n(n+m)(n+r))$, where m and r are positive integers. We distinguish two cases $r\neq m$ and $r=m$, which the last case results in identities involving $\zeta(3)$. During the proof, we need a continuous version of harmonic numbers (to differentiate). Such continuous versions are available by considering the relation of harmonic numbers with digamma (psi) function and polygamma functions of order m, which are defined by $\psi(x):=\mathrm{d}(\log\Gamma(x))/\mathrm{d}x$, and $\psi^{(m)}(x):=\mathrm{d}^m\psi(x)/\mathrm{d}x^m$, respectively. Note that $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\mathrm{d}t$ is the Euler gamma function. Since $\Gamma(x+1)=x\Gamma(x)$, we have $H_n=-\psi(1)+\psi(n+1)$. On the other hand, it is known that $\psi(1)=-\gamma$ (see [1], page 258), where γ is the Euler–Mascheroni constant (see [4], pp 28–40). Thus

$$H_n=\gamma+\psi(n+1). \tag{11}$$

Similar relation for generalized harmonic numbers asserts that

$$H_n^{(r+1)}=\zeta\left(r+1\right)+\frac{(-1)^r}{r!}\psi^{(r)}(n+1), \tag{12}$$

where $r\geqslant 1$ is an integer (see [1], page 260).

LEMMA 2. Assume that m and r are positive integers, and let

$$\mathcal{J}(m,r)=\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n(n+m)(n+r)}.$$

Then, we have

$$\mathcal{J}(m,m)=\frac{\zeta\left(2\right)}{m^{2}}-\frac{\zeta\left(3\right)}{m}-\frac{\zeta\left(2\right)H_{m-1}^{(1)}}{m}$$
$$+\frac{\left(H_{m-1}\right)^{2}}{2m^{2}}+\frac{H_{m-1}^{(2)}}{2m^{2}}+\frac{H_{m-1}H_{m-1}^{(2)}}{m}+\frac{H_{m-1}^{(3)}}{m}. \tag{13}$$

Also, for $r\neq m$ we have

$$\mathcal{J}(m,r)=\frac{\zeta\left(2\right)}{mr}-\frac{\left(H_{m-1}^{(1)}\right)^{2}}{2m\left(m-r\right)}-\frac{H_{m-1}^{(2)}}{2m\left(m-r\right)}+\frac{\left(H_{r-1}^{(1)}\right)^{2}}{2r\left(m-r\right)}+\frac{H_{r-1}^{(2)}}{2r\left(m-r\right)}. \tag{14}$$

PROOF. To prove (13) we start from the known (see [6]) identity

$$2m\sum_{n=1}^{\infty}\frac{\psi(n)}{n(n+m)}=\psi^{2}(m+1)-\gamma^{2}+\zeta(2)-\psi^{(1)}(m+1):=\ell_1(m), \tag{15}$$

say. Differentiating both sides of (15) with respect to m gives us

$$\sum_{n=1}^{\infty}\frac{\psi(n)}{n(n+m)^2}=\frac{\ell_1(m)}{2m^2}-\frac{\ell_2(m)}{2m},$$

where

$$\ell_2(m)=\frac{\mathrm{d}}{\mathrm{d}m}\ell_1(m)=2\psi(m+1)\psi^{(1)}(m+1)-\psi^{(2)}(m+1).$$

By using (11), and considering the known property $\psi(x+1)=\psi(x)+1/x$ (see [1]), we obtain

$$\mathcal{J}(m,m)=\sum_{n=1}^{\infty}\frac{\gamma}{n(n+m)^2}+\sum_{n=1}^{\infty}\frac{1}{n^2(n+m)^2}+\frac{\ell_1(m)}{2m^2}-\frac{\ell_2(m)}{2m}.$$

Following the method used in the paper by A. Sofo [8], we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\gamma}{n(n+m)^2}&=\frac{\gamma}{m}\sum_{n=1}^{\infty}\left(\frac{1}{n(n+m)}-\frac{1}{(n+m)^2}\right)\\&=\frac{\gamma}{m^2}\left(\psi(m+1)+\gamma+\zeta(2)-m\psi^{(1)}(m+1)\right),\end{aligned}\tag{16}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2(n+m)^2}&=\frac{1}{m^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{2}{n(n+m)}+\frac{1}{(n+m)^2}\right)\\&=\frac{1}{m^3}\left(m\zeta(2)+m\psi^{(1)}(m+1)-2\psi(m+1)-2\gamma\right).\end{aligned}\tag{17}$$

Combining (12), (16), (17) and $H_m^{(p)}=H_{m-1}^{(p)}+1/m^p$ for $p=1,2,3$, we get (13).

To prove (14), we assume that $r\neq m$, and we apply (12) and (15) in

$$\mathcal{J}(m,r)=\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n\,(m-r)}\left(\frac{1}{n+r}-\frac{1}{n+m}\right).$$

This gives the identity (14) and completes the proof.

Finally, by using the results and techniques from Wang [9] the following can be shown.

LEMMA 3. Assume that $k\geqslant 1$ is an integer. Then, we have

$$\sum_{r=1}^{k}(-1)^r\binom{k}{r}H_{r-1}^{(1)}=H_{k-1}^{(1)},\tag{18}$$

and

$$\sum_{r=1}^{k}(-1)^r\binom{k}{r}H_{r-1}^{(3)}=\frac{(H_{k-1})^3}{6}+\frac{H_{k-1}^{(1)}H_{k-1}^{(2)}}{2}+\frac{H_{k-1}^{(3)}}{3}.\tag{19}$$

3 Proof of Theorems

We now prove our Theorems.

PROOF of Theorem 1. We consider the following identity

$$\mathcal{H}(m)=\sum_{n=1}^{\infty}\frac{H_n^{(1)}\left(H_n^{(1)}+\sum_{j=1}^{m}\frac{1}{n+j}\right)}{n(n+m)}=\mathfrak{S}(m)+\sum_{j=1}^{m}\mathcal{J}(m,j).$$

Thus $\mathfrak{S}(m)=\mathcal{H}(m)-\sum_{j=1}^{m}\mathcal{J}(m,j)$. By using (5), (13) and (14) in this identity, we obtain (1).

PROOF of Theorem 2. We consider the following expansion

$$\mathfrak{B}(k)=\sum_{n=1}^{\infty}\frac{k!\left(H_n^{(1)}\right)^2}{n\prod\limits_{r=1}^{k}(n+r)}=\sum_{n=1}^{\infty}\frac{k!\left(H_n^{(1)}\right)^2}{n}\sum_{r=1}^{k}\frac{A_r}{n+r},$$

where

$$A_r=\lim_{n\to -r}\left(\frac{(n+r)}{\prod\limits_{r=1}^{k}(n+r)}\right)=(-1)^{r+1}\frac{r}{k!}\binom{k}{r}.$$

Thus, we obtain

$$\mathfrak{B}(k)=\sum_{r=1}^{k}(-1)^{r+1}r\binom{k}{r}\mathfrak{S}(r).$$

Now, we use (1), and then we apply (18) and (19) to get (2). This completes the proof.

Acknowledgment. We thank O. Furdui for the statement of relation (4) and its proof.

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